Robinson manifolds and Cauchy–Riemann spaces

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Abstract.
A *Robinson manifold* is defined as a Lorentz manifold (M, g) of dimension $2n \geqslant 4$ with a bundle $N \subset \mathbb{C} \otimes TM$ such that the fibres of N are maximal totally null and there holds the integrability condition $[\mathsf{Sec}\, N, \mathsf{Sec}\, N] \subset \mathsf{Sec}\, N$. The real part of $N \cap \bar{N}$ is a bundle of null directions tangent to a congruence of null geodesics. This generalizes the notion of a shear-free congruence of null geodesics (SNG) in dimension 4. Under a natural regularity assumption, the set $\mathcal{M}$ of all these geodesics has the structure of a *Cauchy–Riemann* manifold of dimension $2n - 1$. Conversely, every such CR manifold lifts to many Robinson manifolds. Three definitions of a CR manifold are described here in considerable detail; they are equivalent under the assumption of real analyticity, but not in the smooth category. The distinctions between these definitions have a bearing on the validity of the Robinson theorem on the existence of null Maxwell fields associated with SNGs.

This paper is largely a review intended to recall the major influence that Ivor Robinson exerted on the development of this subject.

1. Introduction

Around 1956, Ivor Robinson started studying null solutions of Maxwell's equations and discovered that, with every such solution, there is associated a shear-free congruence of null geodesics (SNG) [14]. This notion turned out to be of significance in the work on solutions of Einstein's equations. A particularly simple *twisting* SNG, discovered by Robinson around 1963, played a major role in Roger Penrose's work on twistors [10, 12]. Maxwell's equations for a null field associated with this Robinson congruence lead to the celebrated Hans Lewy differential operator. All SNGs on compactified Minkowski space-time are obtained by a twistor construction (the Kerr–Penrose theorem). The notion of an SNG generalizes to Lorentz manifolds of higher even dimension [4, 9] (Section 2). I propose to call them *Robinson manifolds*. There is a further generalization to Riemannian manifolds of arbitrary signature. Every Robinson manifold of dimension $2n$ has an associated Cauchy–Riemann manifold of dimension $2n - 1$. Conversely, such a CR manifold can be 'lifted' to many Robinson manifolds [15, 16] (Section 4).

In Robinson's original paper [14] there is a sketch of the proof of a theorem saying that, with every SNG, there is an associated null and non-zero solution of Maxwell's equations. Later, the proof was recognized to be valid in the real-analytic category [17] or, more generally, when the underlying CR space is embeddable ('realizable') [16]. To appreciate these subtleties, it is desirable to consider three different definitions of CR manifolds; this is presented here in Section 3.

The notation in this paper is standard; for example, the symbols $\otimes$, $\wedge$ and $\lrcorner$ denote the tensor, exterior and interior products, respectively. If V is a vector space with a scalar product and $K \subset V$, then $K^{\perp}$ is the vector subspace of V consisting of all vectors that are orthogonal to all elements of K. Manifolds and maps among them are smooth (of class C^{∞}). If $N \to M$ is a vector bundle, then $\mathsf{Sec}\, N$ is the module over $C^{\infty}(M)$ of its sections. If $f : M \to M$ is a diffeomorphism and Y is a tensor field on M, then $f^{*}Y$ is its pull-back by f, e.g., if $Y \in C^{\infty}(M)$, then $f^{*}Y = Y \circ f$. If (ϕ_t) is the flow generated by the vector field X, then $L_X Y = (\mathrm{d}/\mathrm{d}t)\phi_t^{*} Y|_{t=0}$ is the Lie derivative of Y with respect to X. All considerations are local.

2. Robinson and Hermite manifolds

2.1. Definition of Robinson manifolds

Recall first Robinson's definition of a shear-free congruence of null geodesics and its natural generalization. Let (M, g) be an oriented space-time. Using the Hodge duality operator $\star$ one associates with a real 2-form f (the electromagnetic field) the complex 2-form $F = f - \mathrm{i} \star f$ so that $\star F = \mathrm{i}F$ (F is 'self-dual') and Maxwell's equations without charges and currents are

$$\mathrm{d}F = 0.$$

The field f, assumed to be $\neq 0$, is null if, and only if, there is a real 1-form $\kappa = g(k) \neq 0$ such that

$$\kappa \wedge F = 0.$$

The trajectories of the null vector field k form an SNG. To generalize, note that

$$N = \{w \in \mathbb{C} \otimes TM \mid w \lrcorner F = 0\}$$

is a complex vector bundle with fibres that are maximal totally null (MTN). Maxwell's equations imply

$$[\mathsf{Sec}\, N, \mathsf{Sec}\, N] \subset \mathsf{Sec}\, N. \tag{1}$$

It is now easy to generalize:

Definition. A *Robinson manifold* is a triple (M, g, N), where M is an orientable manifold with g of signature $(2n-1, 1)$ and $N \subset \mathbb{C} \otimes TM$ is a fibre bundle with MTN fibres, satisfying the integrability condition (1).

Let N^0 be the annihilator of N,

$$N^0 = \{\alpha \in \mathbb{C} \otimes T^*M \mid w \lrcorner \alpha = 0 \quad \text{for every } w \in N\}.$$

The integrability condition is equivalent to

$$\mathrm{d}\mathsf{Sec}\, N^0 \subset \mathsf{Sec}\, N^0 \wedge \mathsf{Sec}\,(\mathbb{C} \otimes T^*M).$$

The signature being Lorentzian, $N \cap \bar{N} = \mathbb{C} \otimes K$, where $K \subset TM$ is a bundle of null directions.

Remark 1. It might be more appropriate to say that the bundle $N \to M$ with MTN fibres, satisfying (1), defines on M a *Robinson structure*. Clearly, N and $\bar{N}$ define the same Robinson structure. A manifold (M, g) may have several distinct Robinson structures: such is the case of Einstein space-times of type D. Minkowski space-time has an infinity of distinct Robinson structures.

The trajectories of the line bundle $K \to M$ define a congruence on (foliation of) M; assume from now on this foliation to be *regular* in the sense that the set $\mathcal{M}$ of all its leaves has the structure of a manifold such that the canonical projection $M \to \mathcal{M}$ is a submersion. In Section 4 it is shown that $\mathcal{M}$ is a CR manifold.

2.2. Definition of Hermite manifolds

Robinson manifolds are the Lorentzian analogues of Hermite manifolds of proper Riemannian geometry [9]. To see this, consider a proper Riemannian manifold (M, g) of even dimension. Let $N \subset \mathbb{C} \otimes TM$ be a vector bundle with MTN fibres. Since now g is positive-definite, the intersection $N \cap \bar{N}$ contains only zero vectors so that $\mathbb{C} \otimes TM = N \oplus \bar{N}$. The bundle N defines on M an *almost complex structure* $J \in \mathsf{Sec}\,\mathsf{End}\, TM$,

$$J(w + \bar{w}) = \mathrm{i}(w - \bar{w}) \quad \text{for} \quad w \in N,$$

which is orthogonal, $g(J(u), J(v)) = g(u, v)$ for every $u, v \in TM$. Therefore, N defines on (M, g) the structure of an *almost Hermite manifold*; it becomes a *Hermite manifold* when the integrability condition (1) is satisfied. A Hermite manifold is *Kähler* if J is covariantly constant; this is equivalent to the invariance of N with respect to parallel transport.

3. Three definitions of CR structures

In the theory of relativity, a CR structure appeared for the first time in Penrose's theory of twistors: the set of projective null twistors is a 5-dimensional CR manifold. The approach of pure mathematicians to the subject is presented in [5, 18].

Consider a smooth manifold $\mathcal{M}$ of dimension $2n - \varepsilon$, where $\varepsilon \in \{0, 1\}$. (If the dimension is 3, then I say that $\mathcal{M}$ is a *space*.)

(a) A *complex chart* on $\mathcal{M}$ is a pair (U, z), where U is an open subset of $\mathcal{M}$ and

$$z : U \to \mathbb{C}^n \cong \mathbb{R}^{2n}$$

is a smooth immersion, so that, at every $p \in U$,

$$\mathsf{span}_p\{\mathrm{Re}\,\mathrm{d}z_1, \ldots, \mathrm{Re}\,\mathrm{d}z_n, \mathrm{Im}\,\mathrm{d}z_1, \ldots, \mathrm{Im}\,\mathrm{d}z_n\} = T^*_p\mathcal{M}. \tag{2}$$

Two charts (U, z) and (U', z') are said to be *compatible* if there is a function $f : \mathbb{C}^n \to \mathbb{C}^n$ which is biholomorphic (i.e. both f and f^{-1} are holomorphic) and such that $z' = f \circ z$ on $U \cap U'$. A *CR atlas* is an atlas of pairwise compatible complex charts.

For $\varepsilon = 0$, a CR atlas defines on $\mathcal{M}$ the structure of a complex manifold of complex dimension n.

For $\varepsilon = 1$, a CR atlas defines on $\mathcal{M}$ the structure of a locally embeddable CR manifold.

(b) Given a complex chart (U, z), define

$$\mathcal{F} = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$$

so that

$$\mathrm{d}\mathcal{F} = 0. \tag{3}$$

If $\mathcal{F}'$ corresponds to a chart (U', z') compatible with (U, z), then

$$\mathcal{F}' = \det\left(\frac{\partial f}{\partial z}\right)\mathcal{F} \quad \text{on } U \cap U'$$

and the immersion property (2) gives

$$\text{if } w \in \mathbb{C} \otimes T\mathcal{M} \text{ and } w \lrcorner \mathcal{F} = 0,\ w \lrcorner \bar{\mathcal{F}} = 0,\ \text{then } w = 0.$$

(c) This being so, define

$$\mathcal{N} = \{w \in \mathbb{C} \otimes T\mathcal{M} \mid w \lrcorner \mathcal{F} = 0\},$$

then $\dim_{\mathbb{C}} \mathcal{N}_p = n - \varepsilon$ and (2) gives

$$\mathcal{N} \cap \bar{\mathcal{N}} = 0$$

and (3) implies

$$[\mathsf{Sec}\,\mathcal{N}, \mathsf{Sec}\,\mathcal{N}] \subset \mathsf{Sec}\,\mathcal{N}.$$

If $\varepsilon = 0$, then $\mathbb{C} \otimes T\mathcal{M} = \mathcal{N} \oplus \bar{\mathcal{N}}$ and $J : T\mathcal{M} \to T\mathcal{M}$ given by $J|\mathcal{N} = \mathrm{i}\,\mathrm{id}_{\mathcal{N}}$ is the complex structure of $\mathcal{M}$.

To summarize, there are three related notions of CR structure on a manifold $\mathcal{M}$ of dimension $2n - \varepsilon$:

A. $\mathcal{M}$ has a CR atlas.

B. There is a complex line bundle $\Omega \subset \mathbb{C} \otimes \wedge^n T^*\mathcal{M}$ such that

1. the elements of Ω are decomposable,
2. if $0 \neq \omega \in \Omega_p$ and $w \in \mathbb{C} \otimes T_p\mathcal{M}$, then

$$w \lrcorner \omega = 0 \ \text{ and } \ \bar{w} \lrcorner \omega = 0 \ \text{ imply } \ w = 0,$$

3. every $p \in \mathcal{M}$ has a neighborhood admitting a local, non-vanishing section $\mathcal{F}$ of Ω such that $\mathrm{d}\mathcal{F} = 0$.

C. There is a vector bundle $\mathcal{N} \subset \mathbb{C} \otimes T\mathcal{M}$ such that

1. $\dim_{\mathbb{C}} \mathcal{N}_p = n - \varepsilon$,
2. $\mathcal{N} \cap \bar{\mathcal{N}} = 0$,
3. $[\mathsf{Sec}\,\mathcal{N}, \mathsf{Sec}\,\mathcal{N}] \subset \mathsf{Sec}\,\mathcal{N}$.

Clearly, always **A** $\Rightarrow$ **B** and **B** $\Rightarrow$ **C**.

In the *analytic* category, all three definitions are equivalent. In the *smooth* category (C^∞), things are much more difficult:

If $\varepsilon = 0$, then also **C** $\Rightarrow$ **A**, but this is a highly non-trivial result of Newlander and Nirenberg (1957).

For $\varepsilon = 1$ define the *Levi form* h: let (m_μ) be a local basis of $\mathsf{Sec}\,N$, choose $l \in \mathsf{Sec}\,T\mathcal{M}$ so that $(m_\mu, \overline{m_\nu}, l)$, $\mu, \nu = 1, \ldots, n-1$ is a local basis of $\mathsf{Sec}\,(\mathbb{C} \otimes T\mathcal{M})$, put

$$[m_\mu, \overline{m_\nu}] = \mathrm{i} h_{\mu\nu} l + \ldots, \quad \text{then} \quad h_{\mu\nu} = \overline{h_{\nu\mu}}.$$

If the Hermitian form h is positive-definite, then $\mathcal{M}$ is said to be *pseudo-convex*. Then, assuming $\mathcal{M}$ to be pseudo-convex, one has [5]

if $n \geqslant 4$, then **C** $\Rightarrow$ **A**,

the case $n = 3$ appears not to have been settled,

for $n = 2$ there is an example constructed by Nirenberg (1974) of $\mathcal{M}$ such that **C** and not **A**.

Without the assumption of pseudo-convexity, LeBrun [6] constructed $\mathcal{M}$ of dimension 7 such that **B** and not **A**.

Remark 2. If $\mathcal{M}$ is a CR space ($n = 2$), then the fibres of $\mathcal{N} \to \mathcal{M}$ are complex one-dimensional; therefore, the integrability condition **C**3 is no restriction whatsoever.

4. The structure of Robinson manifolds

Theorem. *Consider a Robinson manifold (M, g, N) of dimension $2n$. Let (ϕ_t) be the flow generated by a vector field $k : M \to K = \mathrm{Re}\, N \cap \bar{N}$, then*

(i) *the bundle $N \to M$ is invariant with respect to the action of the flow (ϕ_t) and the trajectories of (ϕ_t) are null geodesics,*

(ii) *the bundle $N \to M$ defines a Cauchy–Riemann structure in the sense of definition* **C** *on the quotient manifold $\mathcal{M}$,*

(iii) *the $(2n-2)$-dimensional fibres of the bundle $K^\perp/K \to M$ have a complex structure and a positive-definite quadratic form, induced by g.*

Proof. (i) The integrability condition implies $[\mathsf{Sec}\,K, \mathsf{Sec}\,N] \subset \mathsf{Sec}\,N$: this gives the invariance property, $\phi_t^* N = N$; let $\kappa = g(k)$, then $\ker \kappa = N + \bar{N}$, therefore $L_k \kappa \| \kappa$ so that $\nabla_k k \| k$ and the geodesic property is established.
(ii) The bundle $N/N \cap \bar{N}$ is invariant with respect to the action of (ϕ_t) and projects to a bundle $\mathcal{N} \to \mathcal{M}$, with fibres of complex dimension $n-1$, such that **C**2 holds; the integrability condition (1) implies **C**3.
(iii) Only the complex structure requires a construction: since $K^\perp = \mathrm{Re}\,(N + \bar{N})$, one can put $J(w + \bar{w} \bmod K) = \mathrm{i}(w - \bar{w}) \bmod K$ for $w \in N$. □

There is a converse: given a CR manifold $\mathcal{M}$ of dim $2n-1$, one constructs $M = \mathbb{R} \times \mathcal{M}$ and makes it into a Robinson manifold (the bundle $N \to M$ is uniquely determined by the CR structure of $\mathcal{M}$, but there is a great freedom in choosing g).

5. Robinson space-times

The case of dimension 4 is well known, but, since it is also the most important one, it is worth-while to review it briefly here. In a sense made precise below, in this case, unlike as in higher dimensions, all information about the Robinson structure is encoded in the properties of the bundle K.

Let (M, g, N) be a space- and time-oriented Robinson manifold of dimension 4. The fibres of the bundle $K^{\perp}/K \to M$ are two-dimensional 'screen spaces'. According to part (iii) of the Theorem, each screen space has a complex structure, which, *in this case*, is equivalent to a conformal structure and an orientation; this being preserved by the flow is the classical geodesic and shear-free condition,

$$L_k g = \rho g + \kappa \otimes \xi + \xi \otimes \kappa, \tag{4}$$

for some function ρ and 1-form ξ. Conversely, given a bundle K of null directions, the space and time orientations of M induce an orientation in the screen spaces; together with the induced Euclidean metric this determines a complex structure J in each screen space. This complex structure defines the bundle

$$N = \{w \in \mathbb{C} \otimes K^{\perp} \mid J(w \bmod \mathbb{C} \otimes K) = \mathrm{i}w \bmod \mathbb{C} \otimes K\}$$

with MTN fibres. Equation (4) implies $[\mathsf{Sec}\, K, \mathsf{Sec}\, N] \subset \mathsf{Sec}\, N$; in view of Remark 2 this is enough to establish the validity of (1).

Goldberg and Sachs showed that, for Einstein space-times, the existence of an SNG congruence is equivalent to the algebraic degeneracy of the Weyl tensor of conformal curvature [3].

Problem 1. Find a generalization of the Goldberg–Sachs theorem to manifolds of dimension > 4.

Problem 2. In dimension 4, are there not conformally flat Lorentz manifolds that have 4 distinct SNGs? From the Goldberg–Sachs theorem it follows that such manifolds cannot be conformal to an Einstein space-time.

Nurowski found a not conformally flat space-time with three distinct Robinson structures [9].

6. The Cartan description of CR spaces; their lifts to space-times

Let $\mathcal{M}$ be a CR space in the weak sense **C**: there is complex line bundle $\mathcal{N} \subset \mathbb{C} \otimes T\mathcal{M}$ and $\mathcal{N} \cap \bar{\mathcal{N}} = 0$. Let $(m, \bar{m}, l)$ be a (local) basis of $\mathsf{Sec}\,(\mathbb{C} \otimes T\mathcal{M})$ with $m \in \mathsf{Sec}\,\mathcal{N}$ and $(\bar{\mu}, \mu, \kappa)$ the dual basis of 1-forms, with κ real. The pair (κ, μ), defined up to

$$\kappa \mapsto \kappa' = a\kappa, \quad \mu \mapsto \mu' = b\mu + c\kappa,$$

where $a \neq 0$ is a real function and $b \neq 0, c$ are complex functions, provides a convenient (Cartan) description of the CR structure on $\mathcal{M}$. Namely, $\mathcal{N}$ is the annihilator of $\mathsf{span}\{\kappa, \mu\}$.

6.1. Lifts

Consider $M = \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, pull κ and μ back to M, let r be a coordinate on $\mathbb{R}$, $k = \partial/\partial r$, P a nowhere vanishing function on M and ν a 1-form on M such that $k \lrcorner \nu \neq 0$, then the metric

$$g = P^2(\mu \otimes \bar{\mu} + \bar{\mu} \otimes \mu) + \kappa \otimes \nu + \nu \otimes \kappa \tag{5}$$

defines on M the structure of a Robinson manifold and every Robinson 4-manifold can be locally so obtained, as a *lift* of a CR space $\mathcal{M}$ to the Lorentz manifold M.

Assume now that the CR structure on $\mathcal{M}$ is of the stronger type **B**: there exists (locally) a non-vanishing function $A : \mathcal{M} \to \mathbb{C}$ such that the 2-form $\mathcal{F} = A\kappa \wedge \mu$ is closed. Pulled back to M, this form is also closed; it is self-dual for every metric of the form (5); its real part is a null Maxwell field associated with the SNG congruence generated by k.

Problem 3. Characterize the CR spaces that admit lifts to Minkowski space-time.

NB. The Kerr–Penrose theorem yields a method of constructing all analytic SNGs in Minkowski space-time, but does not provide a characterization of the CR structure in terms of the pair (κ, μ).

If the null shear-free congruence generated by k is non-twisting,

$$\kappa \wedge \mathrm{d}\kappa = 0,$$

then the associated CR geometry is trivial, locally: $\mathcal{M} = \mathbb{R} \times \mathbb{C}$. Plane-fronted waves and Robinson–Trautman metrics are of this type.

6.2. Embeddable CR spaces

Consider a space $\mathcal{M}$ with a CR structure (of type **C**) characterized by the pair (κ, μ). If the equation

$$\mathrm{d}z \wedge \kappa \wedge \mu = 0 \tag{6}$$

has two solutions z_1 and z_2 such that the map

$$(z_1, z_2) : \mathcal{M} \to \mathbb{C}^2 \tag{7}$$

is an immersion, then the structure is embeddable (i.e. of type **A** in the sense of Section 3). Note that if $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a biholomorphic map, then the pair (w_1, w_2), where $w_k = f_k(z_1, z_2)$, $k = 1, 2$, is also such an immersion. The image of $\mathcal{M}$ by (7) is a hypersurface in $\mathbb{C}^2$; let its equation be

$$G(z_1, z_2, \overline{z_1}, \overline{z_2}) = 0, \quad \mathrm{d}G \neq 0 \quad \text{on} \quad \mathcal{M}, \tag{8}$$

where G is a real, smooth function, defined in a neighbourhood of $\mathcal{M}$ in $\mathbb{C}^2$. Let $G_1 = \partial G/\partial z_1$ and $G_2 = \partial G/\partial z_2$. Note that, since G is real, one has $\overline{\partial G/\partial z_1} = \partial G/\partial \overline{z_1}$. One constructs on $\mathcal{M}$ the forms

$$\kappa' = \mathrm{i}(G_1 \mathrm{d}z_1 + G_2 \mathrm{d}z_2), \quad \mu' = \overline{G_2}\mathrm{d}z_1 - \overline{G_1}\mathrm{d}z_2.$$

The pair (κ', μ') defines the same CR structure as the pair (κ, μ).

Penrose conjectured that non-embeddable CR structures may be of physical relevance in quantum gravity [11].

7. Robinson's twisting congruence and the Hans Lewy equation

Consider the Minkowski line-element

$$g = \mathrm{d}X^2 + \mathrm{d}Y^2 - 2\mathrm{d}W\mathrm{d}r$$

and introduce new coordinates

$$X + \mathrm{i}Y = (r + \mathrm{i})(x + \mathrm{i}y), \quad W = u + \tfrac{1}{2}r(x^2 + y^2)$$

to obtain

$$g = (r^2 + 1)(\mathrm{d}x^2 + \mathrm{d}y^2) - 2\kappa\mathrm{d}r, \quad \kappa = \mathrm{d}u + x\mathrm{d}y - y\mathrm{d}x.$$

The congruence generated by $k = \partial/\partial r$ is null, shear-free, and twisting,

$$\kappa \wedge \mathrm{d}\kappa = 2\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

The complex 2-form

$$F = A(x, y, u, r)\kappa \wedge (\mathrm{d}x + \mathrm{i}\mathrm{d}y)$$

is self-dual and Maxwell's equations $\mathrm{d}F = 0$ reduce to $\partial A/\partial r = 0$ and the equation $Z(A) = 0$, where

$$Z = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} - \mathrm{i}(x + \mathrm{i}y)\frac{\partial}{\partial u}$$

is an operator on $\mathbb{R}^3$ introduced by Hans Lewy in 1957. He constructed a smooth function h such that the equation $Z(A) = h$ has no solution, even locally.

The underlying CR geometry on $\mathcal{M} = \mathbb{R}^3$ with coordinates u, x, y is given by the pair

$$(\kappa, \mu = \mathrm{d}x + \mathrm{i}\mathrm{d}y).$$

Two solutions of (6) are $z_1 = x + \mathrm{i}y$ and $z_2 = u + \frac{1}{2}\mathrm{i}(x^2 + y^2)$ so that equation (8) is now that of the hyperquadric, $\mathrm{i}(\overline{z_2} - z_2) = |z_1|^2$. The biholomorphic map

$$w_1 = \sqrt{2}\frac{z_1}{z_2 + \mathrm{i}}, \quad w_2 = \frac{z_2 - \mathrm{i}}{z_2 + \mathrm{i}}$$

transforms the hyperquadric into the 3-sphere of equation

$$|w_1|^2 + |w_2|^2 = 1.$$

This is the most symmetric, non-trivial, 3-dimensional CR geometry: its group of automorphisms is $\mathsf{SU}_{2,1}$. The CR structure on $\mathbb{S}_3$ can be viewed as obtained from the complex structure of $\mathbb{S}_2 = \mathbb{C}\mathsf{P}_1$ via the Hopf map.

There are several interesting Robinson manifolds that are lifts of this CR structure; in particular, the Gödel universe, the Taub–NUT solution and Hauser's waves of type N.

Élie Cartan solved the problem of equivalence for CR spaces and found all such spaces that are homogeneous. The maximal group of CR symmetries of such a space is either 8-dimensional (for the hyperquadric) or has dimension 3 [1].

8. Analogies between Robinson and Hermite manifolds

8.1. Spinor calculus in four dimensions

Recall that the space S of Dirac spinors, associated with a real 4-dimensional quadratic space (V, g), carries the Dirac representation of the Clifford algebra $\gamma : \mathsf{Cl}(V, g) \to \mathsf{End}\, S$. If (e_μ) is an orthonormal frame in V, then $\gamma_\mu = \gamma(e_\mu)$, $\mu = 1, \ldots, 4$, are the corresponding Dirac matrices and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The space of Dirac spinors decomposes into the direct sum of two complex 2-dimensional spaces of Weyl (chiral, reduced) spinors, $S = S_+ \oplus S_-$. Using the symbols for carrier spaces to denote representations of the spin groups, one has the following equivalences:

$$S_\pm^* \sim S_\pm,$$

$$S \otimes S \sim \wedge V,$$

$$S_+ \otimes S_+ \sim \wedge_+^{\mathrm{even}} V,$$

$$S_+ \otimes S_- \sim \wedge_+^{\mathrm{odd}} V,$$

where the lower sign $+$ on the right refers to self-duality of multivectors,

$$S_+ \otimes_{\mathrm{sym}} S_+ \sim \wedge_+^2 V.$$

There are also equivalences of representations that depend on the signature of the quadratic form on V:

in Lorentz signature $\overline{S_\pm} \sim S_\mp$,

in Euclidean signature $\overline{S_\pm} \sim S_\pm$ and the Hermitian scalar product on S is positive definite when restricted to $S_\pm$.

These differences between the properties of spinor representations have consequences for the Cartan–Petrov–Penrose classification of Weyl tensors and for the Goldberg–Sachs theorem in Euclidean signature; they are described in [13]; see also [9].

8.2. The analogies

There are interesting analogies between the Hermite and Robinson manifolds; some of them are briefly presented in the Table. More information on this subject is in [8, 9]. In the Table, $\varphi \neq 0$ denotes a spinor field of (say) positive chirality. It defines the bundle

$$N = \{w \in \mathbb{C} \otimes TM \mid \gamma(w)\varphi = 0\}$$

Einstein–Robinson manifolds with a null k satisfying $\nabla k = k \otimes p$ have been described by Ehlers and Kundt in [2].

Table. Summary of comparison.

Lorentz		(M,g)		Riemann
		$F=\varphi\otimes\varphi$ defines $N\subset\mathbb{C}\otimes TM$		
$(I+\mathrm{i}\gamma_5)k_\mu\gamma^\mu$	$=$	$\varphi\otimes\overline{\varphi}$	$=$	$\frac{1}{4}(I+\gamma_5+2\,\mathrm{i}J_{\mu\nu}\gamma^{[\mu}\gamma^{\nu]})\Vert\varphi\Vert^2$
k is null		where		$J^2=-\mathrm{id}$
k null geod. shear-free	$\Leftrightarrow$	$[\mathsf{Sec}\,N,\mathsf{Sec}\,N]\subset\mathsf{Sec}\,N$	$\Leftrightarrow$	J integrable
Robinson				Hermite
$Dk\Vert k$	$\Leftrightarrow$	N is invariant by parallel transport	$\Leftrightarrow$	$\nabla J=0$
Ehlers–Kundt				Kähler

9. A difficult problem in elementary form

Consider the following *local* problem in $\mathbb{R}^3$. Recall that if the functions $z_1,z_2:\mathbb{R}^3\to\mathbb{C}$ are smooth, then

$$\mathrm{div}\,(\mathrm{grad}\,z_1\times\mathrm{grad}\,z_2)=0.$$

Let $\overrightarrow{F}$ be a *smooth*, complex vector field on $\mathbb{R}^3$ such that

$$\mathrm{div}\,\overrightarrow{F}=0.$$

Do there exist functions z_1 and z_2 such that

$$\overrightarrow{F}=\mathrm{grad}\,z_1\times\mathrm{grad}\,z_2\ ?$$

A positive answer to this question is equivalent to the following

Conjecture. If a Robinson manifold admits a nowhere vanishing null solution of Maxwell's equations, then the associated CR space is embeddable.

In other words, the conjecture states that, in dimension 3, **B** $\Rightarrow$ **A**. This is known to be true under the assumption of analyticity, but the smooth case is open. It is known that if a CR space lifts to an Einstein–Robinson space, then **C** $\Rightarrow$ **A** [7].

10. Concluding remarks

The discovery by Ivor Robinson of the role played by shear-free congruences of null geodesics has had a profound, stimulating influence on research in general relativity theory, especially on work on exact solutions and gravitational waves; it contributed to the emergence of Penrose's twistors.

Algebraically special gravitational fields are exceptional, but very important: the dominant part of the curvature tensor of an asymptotically flat space-time is of this type (peeling-off). Their role is somewhat analogous to that of the completely integrable systems.

The Kerr solution has been recognized as representing the final state of a rotating black hole, but it was discovered by looking for Einstein–Robinson manifolds with a twisting congruence. The charged Kerr–Newman solution—also a Robinson manifold—has the gyromagnetic ratio predicted by the Dirac theory of the electron. This provides a somewhat mysterious, striking link between classical general relativity and quantum physics.

Penrose's twistor ideas, the relation between Cauchy–Riemann spaces and Robinson manifolds and the analogy between Robinson and Hermite manifolds opened up interesting and subtle connections between general relativity and complex analysis.

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