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Relativistic gravitational fields with close Newtonian analogs

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Abstract

Given a Newtonian velocity field $\mathbf{v}(\mathbf{x}, t)$, one considers the manifold $\mathbf{R}^4$ with the Lorentz metric $g = (\mathrm{d}\mathbf{x} - \mathbf{v}\mathrm{d}t)^2 - \mathrm{d}t^2$. The Riemann tensor is computed and used to characterize flat space-times with g of this form. Among non-flat solutions of Einstein's equations for such a g there are some cosmological models, the Schwarzschild and Kasner metrics and their generalizations to include matter fields and the cosmological constant. If $|\mathbf{v}| = 1$, then the vector field $\partial/\partial t$ is null and has vanishing divergence; it is geodetic and shear-free if, and only if, $\partial\mathbf{v}/\partial t$ is parallel to $\mathbf{v}$.

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1 Introduction

The relation between the Einstein theory of general relativity and the Newtonian theory is usually discussed for slow motions and weak gravitational fields and described in terms of suitable approximation methods; see, e.g. [2, 5]. There is, however, a class of Newtonian fields and motions with close and exact relativistic analogs [1, 7]. For those special motions, one can construct, in a simple manner, Lorentzian metrics satisfying the Einstein field equations. Among the metrics, which can be so obtained, are the Schwarzschild, Kasner and some cosmological solutions. Recently, there has been a renewal of interest in this approach because of its relation to the dimensional reduction of a multidimensional gravitational field admitting a null Killing vector field; see [3] and the references given there.

In this paper, we take up the method outlined in [7] and prove a few new facts about Lorentz metrics constructed from a Newtonian velocity field $\mathbf{v}(\mathbf{x}, t)$. In particular, we characterize the motions that lead, in this manner, to flat space-times. We present known solutions, such as the Kasner metric, in a new form, and show how a field $\mathbf{v}$, such that[4] $|\mathbf{v}| = 1$, can be used to construct a Lorentz metric with a shear-free congruence of non-diverging null geodesics.

2 Notation

We use the standard notation of general relativity theory [5]. Our model of space-time is the manifold $\mathbf{R}^4$ with the Cartesian coordinates (x^μ). The Minkowski metric tensor has components $(\eta_{\mu\nu})$ such that $\eta_{ij} = \delta_{ij}$, $\eta_{i4} = 0$ and $\eta_{44} = -1$, where $i, j = 1, 2, 3$. This tensor, and its inverse $(\eta^{\mu\nu})$, are used to lower and raise the Greek indices μ, ν, etc. $= 1, \ldots, 4$. We put $x^4 = t$ and often use the notation of vector calculus in $\mathbf{R}^3$. Thus, e.g., if $\mathbf{v} = (v^1, v^2, v^3)$ and $\mathbf{w} = (w^1, w^2, w^3)$, then $\mathbf{v} \cdot \mathbf{w} = v_i w_i$ is their scalar product and the ith component of the vector product $\mathbf{v} \times \mathbf{w}$ is $\epsilon_{ijk} v^j w^k$, where $\epsilon_{ijk} = \epsilon_{[ijk]}$ and $\epsilon_{123} = 1$. There is no need to distinguish between the covariant and contravariant position of Latin indices, $v^i = v_i$. The radius-vector is $\mathbf{x} = (x_1, x_2, x_3)$ and its length is denoted by r.

[4] We assume that the system of physical units is chosen so that the velocity of light and the gravitational constant are both equal to 1.

All maps are assumed to be smooth. If

$$\mathbf{v} : \mathbf{R}^4 \to \mathbf{R}^3 \tag{1}$$

is a *velocity field*, then $\partial v_i/\partial x_j = v_{i,j} = v_{(i,j)} + v_{[i,j]}$, and

$$v_{[i,j]} = \tfrac{1}{2}\mathrm{curl}_k \mathbf{v}\, \epsilon_{kji}.$$

Note that $\mathrm{curl}_i \mathbf{curl}\, \mathbf{v} = 2v_{[j,i]j}$. We also write $\mathrm{div}\, \mathbf{v} = v_{i,i}$. If V is a vector space and $f : \mathbf{R}^4 \to V$, then

$$\dot{f} = \partial f/\partial t + v_i\, \partial f/\partial x^i \quad \text{and} \quad \Delta f = f_{,ii}.$$

3 The metric, the curvature, and the Ricci tensors

Given a velocity field (1), one constructs the Lorentzian metric on $\mathbf{R}^4$,

$$g = (\mathrm{d}\mathbf{x} - \mathbf{v}\mathrm{d}t)^2 - \mathrm{d}t^2. \tag{2}$$

Introducing the globally defined orthonormal coframe (e^μ), $\mu = 1, \ldots, 4$,

$$e^i = \mathrm{d}x^i - v^i \mathrm{d}t, \;\; i = 1, 2, 3, \;\; e^4 = \mathrm{d}t,$$

and the associated connection coefficients $\omega_{\mu\nu}$, using

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \quad \text{and} \quad \mathrm{d}e^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0$$

one obtains

$$\omega_{ij} = v_{[j,i]} e^4 \quad \text{and} \quad \omega_{i4} = v_{(i,j)} e^j.$$

The curvature two-form $\Omega_{\mu\nu} = \mathrm{d}\omega_{\mu\nu} + \omega_\mu{}^\rho \wedge \omega_{\rho\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} e^\rho \wedge e^\sigma$ has the following components

$$R_{ijkl} = v_{(j,l)} v_{(i,k)} - v_{(i,l)} v_{(j,k)}, \tag{3}$$

$$R_{ijk4} = v_{[j,i]k}, \tag{4}$$

$$R_{4ij4} = \dot{v}_{(i,j)} + v_{(i,k)} v_{k,j} + v_{(k,j)} v_{[k,i]}. \tag{5}$$

The components of the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ are

$$R_{ij} = \dot{v}_{(i,j)} + \mathrm{div}\, \mathbf{v}\, v_{(i,j)} + \tfrac{1}{2}(v_{k,i} v_{k,j} - v_{i,k} v_{j,k}), \tag{6}$$

$$R_{4i} = -\tfrac{1}{2}\mathrm{curl}_i \mathbf{curl}\, \mathbf{v} \tag{7}$$

$$R_{44} = -\mathrm{div}\, \dot{\mathbf{v}} - v_{(i,j)} v_{i,j}. \tag{8}$$

4 The co-moving coordinate system

Any metric of the form (2) can be transformed, at least locally, to co-moving coordinates. Consider the system of three ordinary differential equations for the functions $\mathbf{x}(\mathbf{y}, t)$,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(\mathbf{x}, t), \tag{9}$$

with the initial conditions $\mathbf{x}(\mathbf{y}, 0) = \mathbf{y}$. For every $\mathbf{y} \in \mathbf{R}^3$ there is a neighborhood of $(\mathbf{y}, 0)$ in $\mathbf{R}^4$ such that the map $(\mathbf{y}, t) \mapsto (\mathbf{x}(\mathbf{y}, t), t)$, is a diffeomorphism of the neighborhood on its image. In other words, solutions of (9) provide local coordinate transformations in $\mathbf{R}^4$. Since (9) gives

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial y^j}\,\mathrm{d}y^j + \frac{\partial x^i}{\partial t}\,\mathrm{d}t = \frac{\partial x^i}{\partial y^j}\,\mathrm{d}y^j + v^i \mathrm{d}t,$$

the metric in the *co-moving coordinates* $(\mathbf{y}, t)$ is

$$\frac{\partial \mathbf{x}}{\partial y^i} \cdot \frac{\partial \mathbf{x}}{\partial y^j}\,\mathrm{d}y^i\,\mathrm{d}y^j - \mathrm{d}t^2.$$

5 Flat space-times

If $R_{\mu\nu\rho\sigma} = 0$, then eq. (4) gives $\mathbf{curl}\,\mathbf{v} = \mathbf{a}$, where $\mathbf{a}$ is a vector-valued function of t only. Therefore, there exists a 'potential' $f : \mathbf{R}^4 \to \mathbf{R}$ such that $\mathbf{v} = \mathbf{grad}\, f + \frac{1}{2}\mathbf{a} \times \mathbf{x}$. From (3) it follows that the matrix $(f_{,ij})$ is of rank no larger than 1: there exists $\mathbf{b} : \mathbf{R}^4 \to \mathbf{R}^3$ such that $f_{,ij} = b_i b_j$ and (5) reduces to

$$b_i c_j + c_i b_j = 0, \quad \text{where} \quad c_i = \dot{b}_i + \tfrac{1}{2}(\mathbf{b}^2 b_i - (\mathbf{a} \times \mathbf{b})_i).$$

The vector $\mathbf{a}$ can be reduced to 0 by a rotation of the axes (x^1, x^2, x^3) with the angular velocity $\mathbf{a}$. After this has been achieved, the flatness condition $R_{\mu\nu\rho\sigma} = 0$ for (2) is equivalent to the existence of a function $f : \mathbf{R}^4 \to \mathbf{R}$ such that $\mathbf{v} = \mathbf{grad}\, f$, the rank of $(f_{,ij})$ is < 2 and

$$\dot{b}_i + \tfrac{1}{2}\mathbf{b}^2 b_i = 0, \quad \text{where} \quad b_i b_j = f_{,ij}.$$

An example of a locally flat solution of this form is

$$(\mathrm{d}x - \sqrt{x}\,\mathrm{d}t)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2.$$

6 Non-trivial solutions

6.1 Equations for a perfect fluid

In this section we assume that the metric (2) satisfies the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{10}$$

with a cosmological constant Λ and with the energy-momentum tensor

$$T_{\mu\nu} = (\mu + p) u_\mu u_\nu + p g_{\mu\nu} \tag{11}$$

of a perfect fluid. The fluid is characterized by the four-velocity vector (u^μ), with

$$u^i = 0 \quad \text{for } i = 1, 2, 3 \text{ and } \quad u^4 = 1, \tag{12}$$

the energy density μ and the presure p. Using (10)-(12) one rewrites the Einstein equations in the form

$$R_{ij} = (\Lambda + \tfrac{1}{2}(\mu - p))\delta_{ij}, \tag{13}$$

$$R_{i4} = 0, \tag{14}$$

$$R_{44} = \tfrac{1}{2}(\mu + 3p) - \Lambda. \tag{15}$$

6.2 Spherically symmetric spaces

If the velocity field $\mathbf{v}$ is of the form $\mathbf{v} = \mathbf{grad} f$, where f is a function of t and r only, then the metric (2) has spherical symmetry. The dependence of such a $\mathbf{v}$ on t and r can be easily determined if one assumes the Einstein equations (13)-(15) with μ and p also depending on t and r only. One obtains

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{x} \sqrt{\tfrac{1}{3}\Lambda + r^{-3}(2m(t) + \int^r r'^2 \mu(r', t) \mathrm{d}r')}.$$

If, in addition, one assumes that both μ and p are independent of r, then the general solution to the Einstein equations (13)-(15) is given by the following two classes of velocity fields. Either

$$\mathbf{v} = \mathbf{x} \sqrt{\tfrac{1}{3}(\Lambda + \mu)}, \tag{16}$$

and

$$\dot{\mu} + \sqrt{3(\Lambda + \mu)}(\mu + p) = 0, \qquad \mu = \mu(t), \qquad p = p(t), \tag{17}$$

or

$$\mathbf{v} = \mathbf{x}\sqrt{2mr^{-3} + \tfrac{1}{3}(\Lambda + \mu)}, \tag{18}$$

and

$$\mu = -p = \text{const.}, \qquad m = \text{const.} \tag{19}$$

The equations (16)-(17) should be supplemented by an equation of state $F(p, \mu) = 0$ and, in particular cases, may be solved explicitly for μ and p (e.g., for the polytropes, when $p = (\gamma - 1)\mu$, $\gamma = \text{const.}$). Substituting $\mathbf{v}$ into (2) one gets Einstein metrics with a perfect fluid; in particular, the following well-known metrics are obtained as special cases:

(DS) If $\mu = p = 0$, then $\Lambda > 0$ and $\mathbf{v} = \mathbf{x}\sqrt{\Lambda/3}$ corresponds to the DeSitter metric,

(F) If $\Lambda = 0$, then $\mathbf{v} = \mathbf{x}\sqrt{\mu/3}$ with $\dot{\mu} + \sqrt{3\mu}(\mu + p) = 0$ defines the Friedmann universe with $K = 0$. In the case of a polytrope the explict expressions for the energy density and the pressure read

$$\mu = \frac{4}{3\gamma^2 t^2}, \qquad p = \frac{4(\gamma - 1)}{3\gamma^2 t^2}, \qquad \gamma = \text{const.} \neq 0,$$

so that the velocity field can be written as

$$\mathbf{v} = \frac{2}{3|\gamma|t}\mathbf{x}.$$

The equations (19) imply that the velocity field (18) describes the Einstein metric with the energy-momentum tensor of the cosmological-constant type. Thus, without losing generality, we can restrict to the case $\mu = p = 0$. The ensuing velocity field

$$\mathbf{v} = \mathbf{x}\sqrt{2mr^{-3} + \tfrac{1}{3}\Lambda}, \quad m = \text{const.},$$

corresponds to metrics which, as special cases, include the Schwarzschild solution ($\Lambda = 0$) and the DeSitter space ($m = 0$).

6.3 The Kasner solution

Let us now return to the case of empty space-times. Consider a velocity field linear in the coordinates (x^i),

$$v_i(\mathbf{x}, t) = A_{ij}(t)x^j + B_i(t).$$

The vector field B_i can be eliminated by the coordinate transformation $x_i \mapsto x_i - \int B_i(t)\mathrm{d}t$. Similarly, the antisymmetric part of A_{ij} can be reduced to 0 by a suitable, time-dependent rotation in $\mathbf{R}^3$. Since the matrix $A = (A_{ij})$ depends on t only, one has $\dot{A} = \mathrm{d}A/\mathrm{d}t$. Denoting by $\mathrm{Tr}\, A$ the trace of the matrix A, assuming that $B_i = 0$ and that A is symmetric, $A^T = A$, we obtain from $R_{44} = 0$ and $R_{ij} = 0$ the equations

$$\mathrm{Tr}\, \dot{A} + \mathrm{Tr}\, A^2 = 0 \quad \text{and} \quad \dot{A} + (\mathrm{Tr}\, A)\, A = 0.$$

By integration one obtains $A(t) = \alpha/t$, where α is a constant, symmetric matrix subject to

$$\mathrm{Tr}\, \alpha = \mathrm{Tr}\, \alpha^2 = 1. \tag{20}$$

By a (time-independent) rotation the matrix α can be brought to the diagonal form, $\alpha = \mathrm{diag}\,(p_1, p_2, p_3)$, and conditions (20) are equivalent to

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Since $\mathrm{d}x_i - p_i x_i t^{-1}\mathrm{d}t = t^{p_i}\mathrm{d}(x_i t^{-p_i})$, the metric reduces to the classical Kasner form, see §11.3 in [4]. The solution is non-flat if, and only if, the matrix α is of maximal rank so that $p_1 p_2 p_3 \neq 0$.

6.4 Perfect fluid generalizations of the Kasner solution

Using the velocity field

$$v_i(\mathbf{x}, t) = A_{ij}(t)x_j, \qquad A_{ij}(t) = A_{(ij)}(t)$$

of the same form as in the Kasner case, and assumming that μ and p are functions of t only, we construct solutions to the Einstein equations for a perfect fluid (13)-(15). We consider two cases: (A) when $\mu = p = 0$ and (B) when $\Lambda = 0$.

Case (A).
All solutions in this case are given by the velocity field $\mathbf{v}$ of the form

$$\mathbf{v} = \sqrt{\frac{\Lambda}{3}} \cot(t\sqrt{3\Lambda})\mathbf{x} - \frac{\sqrt{3\Lambda}}{\sin(t\sqrt{3\Lambda})} B\mathbf{x}, \qquad (21)$$

where $B\mathbf{x}$ is a vector with components $B_{ij}x_j$ and the constant matrix $B = (B_{ij})$ satisfies

$$B = B^T, \qquad \mathrm{Tr}\, B = -\tfrac{2}{3} + \mathrm{Tr}\, B^2 = 0. \qquad (22)$$

The solution given by (21) and (22) is meaningful also for $\Lambda \leq 0$. If $\Lambda = 0$ then $\mathbf{v} = (\mathbf{x} - 3B\mathbf{x})/3t$ and the solution reduces to the Kasner solution of Section 6.3. In the general case, the solution provides a known generalization of the Kasner metric to the case of vacuum Einstein equations with a cosmological constant Λ of either sign; see §11.3.2 in [4].
Case (B).
In this case the general solution is given in terms of the matrix A of the form

$$A = \pi(t)I + \tau(t)B,$$

where I is the identity matrix and the matrix B is constant, symmetric and traceless. The following two cases are worth distinguishing.

(B1) $\tau = 0$. In this case, any function $\pi(t)$ generates a solution to the equations. This is a special case ($\Lambda = 0$) of solutions of section 6.2 and we do not comment on it any further.

(B2) $\tau \neq 0$. Then any nonvanishing function $\tau(t)$ generates a solution to the equations provided that

$$\pi = -\dot{\tau}/3\tau.$$

It follows that in both cases (B1) and (B2), the energy density and the pressure of the fluid can be written as

$$\mu = 3\pi^2 - \tfrac{1}{2}\tau^2 \mathrm{Tr}\, B^2, \quad p = -2\dot{\pi} - 3\pi^2 - \tfrac{1}{2}\tau^2 \mathrm{Tr}\, B^2.$$

These relations need to be supplemented by an equation of state. The simplest polytrope equation $p = (\gamma - 1)\mu$ applied to the nonspherically symmetric case (B2), leads to the following equation for the function τ.

$$\frac{2}{3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\log\tau) - \frac{\gamma}{3}(\frac{\mathrm{d}}{\mathrm{d}t}(\log\tau))^2 + \frac{\gamma - 2}{2}\tau^2 \mathrm{Tr}\, B^2 = 0. \qquad (23)$$

The general solution of this equation in the case of $\gamma = 2$ generates the Einstein space-time associated with the velocity field of the form

$$\mathbf{v} = \frac{\mathbf{x} - 3B\mathbf{x}}{3t}, \qquad B = B^T, \qquad \mathrm{Tr}\, B = 0, \qquad \mathrm{Tr}\, B^2 \leq \frac{2}{3},$$

and for which

$$\mu = p = \frac{2 - 3\mathrm{Tr}\, B^2}{6t^2}.$$

This again provides a generalization of the Kasner solution.
If $\gamma \neq 2$, then the substitution $\tau = w\sqrt{2/(2-\gamma)\mathrm{Tr}\, B^2}$ transforms (23) into the equation

$$\ddot{w} = \frac{\gamma + 2}{2}\frac{\dot{w}^2}{w} + \frac{3}{2}w^3$$

for w; it has a first integral of the form

$$\dot{w}^2 = \frac{3}{2-\gamma}w^4 + 2\sqrt{3}cw^{\gamma+2}, \qquad c = \mathrm{const.} \tag{24}$$

For some values of the parameter γ one can solve the above equation explicitly. In particular, if $\gamma = 1$, then the general solution of (24) gives the following pure dust solution of the Einstein equations.

$$\mathbf{v} = \frac{2}{3}\frac{c^2 t}{c^2t^2 - 1}\mathbf{x} + \sqrt{\frac{2}{3\mathrm{Tr}\, B^2}}\frac{2c}{c^2t^2 - 1}B\mathbf{x}, \quad \mu = \frac{4}{3}\frac{c^2}{c^2t^2 - 1}, \quad p = 0.$$

where B is symmetric and traceless; *cf.* §12.4 in [4]

7 Congruences of null geodesics

If $|\mathbf{v}| = 1$, then the metric (2) reduces to

$$g = \mathrm{d}\mathbf{x}^2 - 2\mathbf{v}\cdot\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{25}$$

and the four-dimensional vector field $k = \partial/\partial t$ is null. Its four-divergence vanishes and the Lie derivative of g with respect to k is

$$-2(\partial\mathbf{v}/\partial t)\cdot\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Therefore, the congruence of null curves, generated by k, is geodetic and shear-free if, and only if [6]

$$\mathbf{v} \times \partial \mathbf{v}/\partial t = 0.$$

The one-form associated with k by g is

$$\lambda = g(k) = -\mathbf{v} \cdot \mathrm{d}\mathbf{x}.$$

The form λ is integrable ('k is hypersurface-orthogonal') if, and only if, $\mathbf{curl}\,\mathbf{v} = 0$. If this is so, then the metric (25) is flat.

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