# PIN STRUCTURES AND THE DIRAC OPERATOR ON REAL PROJECTIVE SPACES AND QUADRICS

*Summary of an invited lecture*
*given by Andrzej Trautman*
*at the IV International Conference*
*on Clifford Algebras and their Applications*
*Aachen, 28–31 May 1996*

Michel Cahen and Simone Gutt
*Département de Mathématiques*
*Université Libre de Bruxelles*
*Campus Plaine, C.P. 218*
*B-1050 Bruxelles, Belgium*

Andrzej Trautman*
*Instytut Fizyki Teoretycznej*
*Uniwersytet Warszawski*
*Hoża 69, 00-681 Warszawa, Poland*

## 1. Introduction

In recent years, there has been considerable interest in properties of the Dirac operator on manifolds; much research was done on estimating its first eigenvalue on compact Riemannian spaces (Friedrich, 1980), on harmonic, parallel (Lichnerowicz, 1963; Hitchin, 1974) and Killing spinors (Cahen *et al.*, 1986; Baum *et al.*, 1991), and on spin structures on symmetric spaces (Cahen and Gutt, 1988). Most of that work was restricted to simply-connected manifolds. Some time ago, we started a study of pin structures on non-orientable manifolds such as the even-dimensional real projective spaces (Dąbrowski and Trautman, 1986) and odd-dimensional real projective quadrics (Cahen *et al.*, 1995), in order to prepare ground for the determination of the spectrum of the Dirac operator on these spaces.

In this short article, based on the talk given by one of us (A.T.) at the Conference, we summarize our recent work on this subject; a fuller account can be found in a series of papers published in the *Journal of Geometry and Physics.*

* Research of this author was supported in part by the Polish Committee on Scientific Research (KBN) under Grant No. 2 P302 112 7.

## 2. Notation

A *quadratic space* is a pair $(V, h)$, where $V$ is a finite-dimensional real vector space and $h$ is a quadratic form on $V$ which is non-degenerate, but not necessarily definite. The corresponding *Clifford algebra* $\mathsf{Cl}(h)$ is an associative algebra over $\mathbb{R}$, containing $\mathbb{R}$ and generated by $V \subset \mathsf{Cl}(h)$; for every $v \in V$ one has $v^2 = h(v)$. This algebra is $\mathbb{Z}_2$-graded by the main automorphism $\alpha$ characterized by $\alpha(v) = -v$ and $\alpha(1) = 1$. A *unit vector* $v$ satisfies $h(v) = 1$ or $-1$. The group $\mathsf{Pin}(h)$ is a subset of $\mathsf{Cl}(h)$ consisting of Clifford products of all finite sequences of unit vectors; $\mathsf{Spin}(h)$ is its subgroup consisting of all even elements of $\mathsf{Pin}(h)$. The *twisted adjoint* representation $\rho$ of $\mathsf{Pin}(h)$ in $V$ is defined by

$$\rho(a)v = \alpha(a)va^{-1} \tag{1}$$

for every $a \in \mathsf{Pin}(h)$ and $v \in V$. Let $G = \mathsf{O}(h)$ be the group of automorphisms of $(V, h)$ and $H$ be either $\mathsf{Pin}(h)$ or $\mathsf{Pin}(-h)$. The group $H$ is an extension of $G$ by $\mathbb{Z}_2$: there is the exact sequence

$$1 \to \mathbb{Z}_2 \to H \overset{\rho}{\to} G \to 1.$$

If $h$ is a quadratic form of signature $(k, l)$ in $\mathbb{R}^{k+l}$, then we write $\mathsf{Pin}_{k,l}$ instead of $\mathsf{Pin}(h)$; a similar notation is used for the spin and orthogonal groups. The groups $\mathsf{Spin}_{k,0}$ and $\mathsf{Spin}_{0,k}$, which are isomorphic, are denoted $\mathsf{Spin}_k$. The volume element associated with an oriented $V$, and an $h$ of signature $(k, l)$, is $\eta_{k,l} = e_1 \cdots e_{k+l}$, where $(e_1, \ldots, e_{k+l})$ is an orthonormal frame of the given orientation. If $\eta_{k,l}^2 = 1$, then $\{1, \eta_{k,l}\}$ is a subgroup of $\mathsf{Pin}_{k,l}$. We put $\mathsf{Pin}_m = \mathsf{Pin}_{m,0}$ for $m \equiv 0$ or $1 \bmod 4$ and $\mathsf{Pin}_m = \mathsf{Pin}_{0,m}$ for $m \equiv 2$ or $3 \bmod 4$, and use a similar notation for the volume elements. In this notation, one has $\eta_m^2 = 1$ for $m$ odd or divisible by 4, but $\eta_{4k+2,0}^2 = \eta_{0,4k+2}^2 = -1$ for every $k$.

## 3. Pin structures and the modified Dirac operator

Let $(M, g)$ be a Riemannian space having $(V, h)$ as a local model. A *pin structure* on $(M, g)$ is given by the sequence of maps

$$H \to Q \overset{\chi}{\to} P \overset{\pi}{\to} M, \tag{2}$$

where $P$ is the $G$-bundle of all orthonormal frames on $(M, g)$, and $Q$, the *pin bundle*, is a principal $H$-bundle over $M$ and a principal $\mathbb{Z}_2$-bundle over $P$. Denoting by $\delta$ the right action of $H$ on $Q$ so that $\delta(a)(q) = qa$ and $(qa)b = q(ab)$ for $q \in Q$ and $a, b \in H$, one has $\chi(qa) = \chi(q)\rho(a)$ for every $q \in Q$ and $a \in H$. Sometimes, to be more explicit, if $H = \mathsf{Pin}(h)$ (resp., $\mathsf{Pin}(-h)$), then one says that (2) defines a $\mathsf{Pin}(h)$- (resp., $\mathsf{Pin}(-h)$-) structure. If $P$ admits a restriction $P'$ of

$G$ to a subgroup $G'$, then $Q$ has a restriction $Q' = \chi^{-1}(P')$ to the subgroup $H' = \rho^{-1}(G')$; the pin structure on $M$ can be obtained, in an obvious way, from the sequence of maps $H' \to Q' \to P' \to M$; the latter sequence is referred to, somewhat imprecisely, as the *restricted pin structure* on $M$. In particular, if $M$ is orientable, then $P$ can be restricted to the special orthogonal group $G' = \mathsf{SO}(h)$; moreover, if $M$ is pin, then it has a pin structure restricted to $H' = \mathsf{Spin}(h)$, called a *spin structure* on $M$. The space $M$ is then said to be *spin* and $Q'$ is its *spin bundle*.

Let $\gamma : H \to \mathsf{GL}(S)$ be a spinor representation in a complex vector space $S$; a *spinor field* of type $\gamma$ is a map $\psi : Q \to S$, equivariant with respect to the action of $H$, i.e. such that $\psi(qa) = \gamma(a^{-1})\psi(q)$ for every $q \in Q$ and $a \in H$.

Given a pin structure on $(M, g)$, the Levi-Civita connection form on $P$ lifts to a connection form $\omega$ on $Q$, with values in the Lie algebra of $H$. Let $\nabla$ be the basic horizontal $V^*$-valued vector field on $Q$ defined by $\omega$. The value $\nabla(q)$ of $\nabla$ at $q \in Q$ is a linear map $\nabla(q) : V \to T_qQ$ and for every $q \in Q$ and $a \in H$ one has

$$\nabla(qa) = T_q\delta(a) \circ \nabla(q) \circ \rho(a). \tag{3}$$

If $\psi : Q \to S$, then $\nabla\psi : Q \to \mathsf{Hom}(V, S)$. The representation $\gamma$ defines a Clifford evaluation map $\tilde{\gamma} : \mathsf{Hom}(V, S) \to S$ given by $\tilde{\gamma}(v^* \otimes s) = \gamma(\bar{h}(v^*))s$, where $v^* \in V^*$, $s \in S$ and $\bar{h}$ is the isomorphism $V^* \to V$ associated with $h$. The *classical Dirac operator* $D^{\mathrm{cl}}$ acts on a spinor field $\psi$ according to $D^{\mathrm{cl}}\psi = \tilde{\gamma} \circ \nabla\psi$. It follows from (1) and (3) that *if $\psi$ is a spinor field of type $\gamma$, then $D^{\mathrm{cl}}\psi$ is a spinor field of type $\gamma \circ \alpha$.* If $M$ is orientable, then one restricts $Q$ to $H' = \mathsf{Spin}(h)$ and then $\gamma \circ \alpha | H' = \gamma | H'$ so that $D^{\mathrm{cl}}\psi = \lambda\psi$ is meaningful also for $\lambda \neq 0$. However, if $M$ is pin, but not orientable, then one has to modify the Dirac operator in order to consider an eigenvalue problem for that operator. If the dimension $m$ of $M$ is *even*, then the Dirac representations $\gamma$ and $\gamma \circ \alpha$ are equivalent: there is an intertwiner $\Gamma \in \mathsf{GL}(S)$ such that $\Gamma^2 = \mathrm{id}_S$ or $-\mathrm{id}_S$ and $\gamma \circ \alpha(a) = \Gamma \circ \gamma(a) \circ \Gamma^{-1}$ for every $a \in H$. The *modified Dirac operator* $D = \Gamma D^{\mathrm{cl}}$ preserves the type of Dirac spinor fields. If the dimension $m$ of $M$ is *odd*, then $e_1 \cdots e_m$ is in the centre of $H$ and, therefore, the two irreducible Pauli representations of $H$ are not complex-equivalent. Let $\psi_+$ and $\psi_-$ be Pauli spinor fields of type $\gamma$ and $\gamma \circ \alpha$, respectively. By taking the 'Cartan representation' to be the direct sum of $\gamma$ and $\gamma \circ \alpha$, one can define the modified Dirac ('Cartan') operator to act on the Cartan spinor field so that $(\psi_+, \psi_-) \mapsto (D^{\mathrm{cl}}\psi_-, D^{\mathrm{cl}}\psi_+)$.

## 4. Spin structures on products

Consider two spin manifolds $(M_i, g_i)$, $i = 1, 2$, and their corresponding local models $(V_i, h_i)$. Let

$$\mathsf{Spin}(h_i) \to SQ_i \to SP_i \to M_i, \quad (i = 1, 2)$$

be the sequences defining their spin structures.

PROPOSITION 1. *The product $(M_1 \times M_2, g_1 \oplus g_2)$ of two manifolds is spin if, and only if, both factors are spin. The spin structure on the product can be obtained from the restricted spin structure*

$$\mathsf{Spin}(h_1) \cdot \mathsf{Spin}(h_2) \to SQ_1 \cdot SQ_2 \to SP_1 \times SP_2 \to M_1 \times M_2,$$

*where $\mathsf{Spin}(h_1) \cdot \mathsf{Spin}(h_2) = (\mathsf{Spin}(h_1) \times \mathsf{Spin}(h_2))/\mathbb{Z}_2$ is a subgroup of $\mathsf{Spin}(h_1 \oplus h_2)$ doubly covering the subgroup $\mathsf{SO}(h_1) \times \mathsf{SO}(h_2)$ of $\mathsf{SO}(h_1 \oplus h_2)$ and $SQ_1 \cdot SQ_2 = (SQ_1 \times SQ_2)/\mathbb{Z}_2$.*

For $i = 1$ and 2, let $\gamma_i$ be a Dirac or a Pauli representation of $\mathsf{Spin}(h_i)$ in $S_i$, depending on whether the dimension $m_i$ of $M_i$ is even or odd. Put $S = S_1 \otimes S_2$ if at least one of these two dimensions is even and $S = \mathbb{C}^2 \otimes S_1 \otimes S_2$ if both $M_1$ and $M_2$ are odd-dimensional. The representations $\gamma_1$ and $\gamma_2$ extend to a representation $\gamma$ of $\mathsf{Spin}(h_1 \oplus h_2)$ in $S$. Let $v_i \in V_i$ for $i = 1$ or 2. If $m_1$ is even, then $\gamma(v_1, v_2) = v_1 \otimes \mathrm{id}_{S_2} + \Gamma_1 \otimes v_2$, where $\Gamma_1$ is an intertwiner of the representations $\gamma_1$ and $\gamma_1 \circ \alpha$ such that $\Gamma_1^2 = \mathrm{id}_{S_1}$. If both $m_1$ and $m_2$ are odd, then $\gamma(v_1, v_2) = \sigma_1 \otimes \gamma_1(v_1) \otimes \mathrm{id}_{S_2} + \sigma_2 \otimes \mathrm{id}_{S_1} \otimes \gamma_2(v_2)$, where $\sigma_1$ and $\sigma_2$ are the Pauli matrices. Tensor-multiplying spinor fields defined on the factors, one obtains spinor fields on the product. We have (Cahen *et al.*, 1995)

PROPOSITION 2. *Let $D_i^{\mathrm{cl}}$ be the classical Dirac operator on the spin manifold $M_i$, $i = 1$, 2. If $m_1$ is even, then the classical Dirac operator on $M_1 \times M_2$ is given by*

$$D^{\mathrm{cl}}(\psi_1 \otimes \psi_2) = D_1^{\mathrm{cl}}\psi_1 \otimes \psi_2 + \Gamma_1\psi_1 \otimes D_2^{\mathrm{cl}}\psi_2.$$

*If*

$$D_i^{\mathrm{cl}}\psi_i = \lambda_i\psi_i$$

*and*

$$\psi_\pm = (\lambda_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2} + \lambda_2\Gamma_1)\psi_1 \otimes \psi_2,$$

*then*

$$D^{\mathrm{cl}}\psi_\pm = \pm\sqrt{\lambda_1^2 + \lambda_2^2}\,\psi_\pm\,.$$

There is a similar result for the case when both $m_1$ and $m_2$ are odd.

## 5. The covering space of a pin manifold

There is a simple relation between pin structures on manifolds and their covers. Let $\Pi$ be the fundamental group of a connected manifold $M$. The universal covering manifold $\tilde{M}$ of $M$ is the total space of the principal $\Pi$-bundle $\xi : \tilde{M} \to M$. We write the left action of $\Pi$ on $\tilde{M}$ as $(c, x) \mapsto cx$, so that $\xi(cx) = \xi(x)$ for every $c \in \Pi$and $x \in \tilde{M}$. The principal $\mathsf{O}(h)$-bundle $\tilde{\pi} : \tilde{P} \to \tilde{M}$ of all orthonormal frames on $\tilde{M}$ can be identified with the bundle induced from $\pi : P \to M$ by $\xi$,

$$\tilde{P} = \{(x, p) \in \tilde{M} \times P : \xi(x) = \pi(p)\}.$$

The projection $\tilde{\pi} : \tilde{P} \to \tilde{M}$ is given by $\tilde{\pi}(x, p) = x$ and there is the map $\eta : \tilde{P} \to P$ such that $\eta(x, p) = p$. The group $\mathsf{O}(h)$ acts on $\tilde{P}$ so that $((x, p), A) \mapsto (x, pA)$, where $A \in \mathsf{O}(h)$; the map $\eta$ is equivariant: $\eta(x, pA) = \eta(x, p)A$. There is a natural lift of the action of $\Pi$ to $\tilde{P}$ given by $(c, (x, p)) \mapsto (cx, p)$. The lifted action commutes with that of $\mathsf{O}(h)$. We have

PROPOSITION 3. *A Riemannian space $M$, with a local model $(V, h)$, admits a pin structure if, and only if, there exists a pin structure*

$$\mathsf{Pin}(h) \to \tilde{Q} \xrightarrow{\tilde{\chi}} \tilde{P} \xrightarrow{\tilde{\pi}} \tilde{M} \tag{4}$$

*on its universal cover $\tilde{M}$ and an action of $\Pi = \pi_1(M)$ on $\tilde{Q}$, lifting the action of $\Pi$ on $\tilde{P}$ and commuting with the action of $\mathsf{Pin}(h)$. Moreover, a spinor field $\tilde{\psi} : \tilde{Q} \to S$ descends to a spinor field $\psi : Q \to S$ if, and only if, $\tilde{\psi}$ is invariant with respect to the action of $\Pi$ on $\tilde{Q}$ and every spinor field on $Q$ is so obtained.*

## 6. Pin structures on real projective spaces and quadrics

Proposition 3 is used to obtain, in an explicit manner, the pin (or spin) structures on real projective spaces and quadrics. For example, let $m$ be a positive integer; the sphere $\mathbf{S}_{2m}$ has a unique spin structure which can be extended to the pin structure

$$\mathsf{Pin}_{2m} \to \mathsf{Pin}_{2m+1} \to \mathsf{O}_{2m+1} \to \mathbf{S}_{2m}. \tag{5}$$

The fundamental group of the real projective space $\mathbf{P}_{2m}$ is generated by an element which acts on the covering space $\mathbf{S}_{2m}$ by sending $x$ to $-x$; since $\eta_{2m+1}^2 = 1$ and $\rho(\eta_{2m+1})x = -x$, this action lifts to $\mathsf{Pin}_{2m+1}$ in two inequivalent ways: $a \mapsto \pm a\eta_{2m+1}$, where $a \in \mathsf{Pin}_{2m+1}$. The centre of $\mathsf{Pin}_{2m+1}$ admits the two-element subgroups $\mathbb{Z}_2^{\pm} = \{1, \pm\eta_{2m+1}\}$ and the non-orientable space $\mathbf{P}_{2m}$ has two inequivalent $\mathsf{Pin}_{2m}$-structures

$$\mathsf{Pin}_{2m} \to \mathsf{Pin}_{2m+1}/\mathbb{Z}_2^{\pm} \to \mathsf{O}_{2m+1}/\mathbb{Z}_2 \to \mathbf{P}_{2m}. \tag{6}$$

These two structures will be referred to as the pin$^+$- and pin$^-$-structures on $\mathbf{P}_{2m}$; they are swapped by the change of orientation, which induces a change of sign of the volume element.

The real quadric $\mathbf{S}_{k,l}$ is the conformal compactification of $\mathbb{R}^{k+l}$ endowed with a quadratic form of signature $(k,l)$; as a manifold, it is diffeomorphic to $(\mathbf{S}_k \times \mathbf{S}_l)/\mathbb{Z}_2$. In particular, $\mathbf{S}_{k,0} = \mathbf{S}_k$; but we are concerned here only with proper quadrics for which $k$ and $l$ are both positive. To obtain the (s)pin structures on such a proper quadric one first finds, by referring to Proposition 1, the spin structure(s) on its cover $\mathbf{S}_k \times \mathbf{S}_l$ which is universal if both $k$ and $l$ are $> 1$. This product manifold admits two natural metric tensor fields: a proper Riemannian one and a pseudo-Riemannian of signature $(k,l)$. They both descend to the corresponding quadric. A quadric such that $kl \neq 0$ is orientable if, and only if, $k+l$ is even. In this case, one can consider on the quadric $\mathsf{Spin}_{k+l}$- and $\mathsf{Spin}_{k,l}$-structures. If $k+l$ is odd and $kl \neq 0$, then there can occur, depending on the particular values of $k$ and $l$, pin structures corresponding to the groups $\mathsf{Pin}_{k+l,0}$, $\mathsf{Pin}_{0,k+l,}$, $\mathsf{Pin}_{k,l}$ and $\mathsf{Pin}_{l,k}$. All these structures are easily obtained from Proposition 3. The cases when $k$ or $l = 1$ require special consideration because $\mathbf{S}_1$ has two inequivalent spin structures. We indicate, very briefly, how one can determine the pin structures on $\mathbf{S}_{k,l}$ when both $k$ and $l$ are larger than 1. Complete results are given in (Cahen *et al.*, 1993; Cahen *et al.*, 1995). Let $\eta_{k,0}$ (or $\eta_{0,k}$) and $\eta_{l,0}$ (or $\eta_{0,l}$) denote the volume elements in the mutually orthogonal subspaces $\mathbb{R}^k$ and $\mathbb{R}^l$ of $\mathbb{R}^{k+l}$. The quadric $\mathbf{S}_{k,l}$ admits two $\mathsf{Pin}_{k+l}$-structures whenever $(\eta_{k,0}\eta_{l,0})^2 = 1$; it has two $\mathsf{Pin}_{k,l}$-structures whenever $(\eta_{k,0}\eta_{0,l})^2 = 1$, etc. A simple evaluation leads to

PROPOSITION 4. *Let the integers $k$ and $l$ be larger than 1. If $k$ is even and $l$ is odd, then the quadric $\mathbf{S}_{k,l}$ has two $\mathsf{Pin}_{l,k}$- and two $\mathsf{Pin}_{0,k+l}$-structures for $k+l \equiv 1 \bmod 4$ and two $\mathsf{Pin}_{k+l,0}$- and two $\mathsf{Pin}_{k,l}$-structures for $k+l \equiv 3 \bmod 4$. If both $k$ and $l$ are even, then the quadric has two $\mathsf{Spin}_{k,l}$-structures for $k+l \equiv 0 \bmod 4$ and two $\mathsf{Spin}_{k+l}$-structures for $k+l \equiv 2 \bmod 4$. If both $k$ and $l$ are odd and $k+l \equiv 2 \bmod 4$, then there are on $\mathbf{S}_{k,l}$ two $\mathsf{Spin}_{k,l}$- and two $\mathsf{Spin}_{k+l}$-structures. If $k$ and $l$ are odd and $k+l$ is divisible by 4, then the quadric has no spin structure whatsoever.*

## 7. The spectrum of the Dirac operator on spheres

The spectrum of the Dirac operator on spheres is well-known; see (Trautman, 1995) and the references given there. In view of applications to $\mathbf{P}_{2m}$, we recall the relevant information for the even-dimensional spheres. The Dirac representation $\gamma : \mathsf{Pin}_{2m} \to \mathsf{GL}(S)$ admits an extension to an irreducible, but not faithful, Pauli representation $\gamma' : \mathsf{Pin}_{2m+1} \to \mathsf{GL}(S)$ such that $\gamma'(\eta_{2m+1}) = \mathrm{id}_S$. If $\psi : \mathsf{Pin}_{2m+1} \to S$ is a spinor field, then the map $a \mapsto \gamma'(a)\psi(a)$ is constant on the fibres of $\varpi : \mathsf{Pin}_{2m+1} \to \mathbf{S}_{2m}$, where $\varpi(a) = \alpha(a)e_{2m+1}a^{-1}$. Therefore, there is $\Psi_0 : \mathbf{S}_{2m} \to S$ such that

$$\Psi_0 \circ \varpi(a) = \gamma'(a)\psi(a) \tag{7}$$

for every $a \in \mathsf{Pin}_{2m+1}$. Since $\varpi(a\eta_{2m+1}) = -\varpi(a)$, if $\psi(a\eta_{2m+1}) = \pm\psi(a)$, then $\Psi_0(-x) = \pm\Psi_0(x)$ for every $x \in \mathbf{S}_{2m}$.

Let $\gamma_i = \gamma'(e_i)$ so that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2(-1)^m\delta_{ij}\,\mathrm{id}_S$$

for $i, j = 1, \ldots, 2m+1$. It is now convenient to take $\gamma_{2m+1}$ as the intertwiner of the representations $\gamma$ and $\gamma\circ\alpha$. The square of the modified Dirac operator $D = \gamma_{2m+1}D^{\mathrm{cl}}$ is a positive operator for every $m$; $D$ descends to an operator $\mathcal{D}$ acting on a spinor-valued function $\Psi_0$, given by (7), so that

$$(\mathcal{D}\Psi_0) \circ \varpi(a) = \gamma'(a)(D\psi)(a).$$

Explicitly, this operator is given by

$$\mathcal{D} = \sum_{i<j} \gamma_i\gamma_j(x_i\partial_j - x_j\partial_i) + m.$$

Put $\boldsymbol{\partial} = \sum_i \gamma_i\partial_i$ and denote by $\boldsymbol{x}$ another linear operator acting on $S$-valued functions on $\mathbb{R}^{2m+1}$ so that $(\boldsymbol{x}\Psi)(y) = \sum_i y_i\gamma_i\Psi(y)$, where $y \in \mathbb{R}^{2m+1}$. For every non-negative integer $n$, the vector space $H_{m,n}$ of harmonic, homogeneous of degree $n$, $S$-valued polynomials on $\mathbb{R}^{2m+1}$ admits a decomposition $H'_{m,n} \oplus H''_{m,n}$, where $H'_{m,n}$ is the kernel of $\boldsymbol{\partial} : H_{m,n} \to H_{m,n-1}$ and $H''_{m,n}$ is the image of $H'_{m,n-1}$ by $\boldsymbol{x}$. Since the sequence

$$\ldots \xrightarrow{\boldsymbol{\partial}} H_{m,n+1} \xrightarrow{\boldsymbol{\partial}} H_{m,n} \xrightarrow{\boldsymbol{\partial}} H_{m,n-1} \xrightarrow{\boldsymbol{\partial}} \ldots \xrightarrow{\boldsymbol{\partial}} H_{m,0} \xrightarrow{\boldsymbol{\partial}} \{0\}$$

is exact, $H'_{m,n}$ is also the image of $H_{m,n+1}$ by $\boldsymbol{\partial}$. The vector space $H'_{m,0}$ can be identified with $S$, whereas $H''_{m,0} = \{0\}$. For every $S$-valued polynomial $\Psi$ on $\mathbb{R}^{2m+1}$, let $\Psi_0$ denote its restriction to $\mathbf{S}_{2m}$. There holds

PROPOSITION 5. *If $\Psi \in H'_{m,n}$, where $n = 0, 1, \ldots$, then $\mathcal{D}\Psi_0 = (n+m)\Psi_0$. If $\Psi \in H''_{m,n}$, where $n = 1, 2, \ldots$, then $\mathcal{D}\Psi_0 = (1-n-m)\Psi_0$. Every eigenfunction of $\mathcal{D}$ on $\mathbf{S}_{2m}$ can be obtained in this manner.*

The spectrum of the Dirac operator on spheres is symmetric; if $\mathcal{D}\Psi_0 = \lambda\Psi_0$, then $\mathcal{D}\boldsymbol{x}\Psi_0 = -\lambda\boldsymbol{x}\Psi_0$. Note also that each eigenfunction of $\mathcal{D}$ has a definite parity. The dimension of the space of eigenfunctions of $\mathcal{D}$ with eigenvalue $\lambda$ such that $|\lambda| = m + n$ is

$$2^m\binom{n+2m-1}{n}.$$

There are similar statements for odd-dimensional spheres.

## 8. The spectrum of the Dirac operator on projective spaces and on quadrics

It is now clear how one can find the spectrum of the Dirac operator on real projective spaces and on quadrics by using Propositions 1-5. For a quadric, one first finds, by referring to Propositions 1 and 2, the spin structures and the spectrum of the Dirac operator $D$ on the product $\mathbf{S}_k \times \mathbf{S}_l$. Prop. 4—and its extension to the case when $k$ or $l = 1$—gives the (s)pin structures on $\mathbf{S}_{k,l}$. Since the quadric and its covering space are locally isometric, every eigenfunction of $D$ that descends to the quadric (Prop. 3) is also an eigenfunction of the Dirac operator with the same eigenvalue and all eigenfunctions on $\mathbf{S}_{k,l}$ can be so obtained.

The case of even-dimensional real projective spaces is particularly simple and instructive. According to Prop. 3 and Section 6, a spinor field on $\mathbf{S}_{2m}$, i.e. a $\mathsf{Pin}_{2m}$-equivariant map $\psi : \mathsf{Pin}_{2m+1} \to S$, descends to one or the other pin structure on $\mathbf{P}_{2m}$, given by (6), depending on whether $\psi(a\eta_{2m+1}) = \psi(a)$ or $-\psi(a)$. According to Section 7, such spinor fields define by (7), even and odd $S$-valued functions on the sphere, respectively. If $\Psi \in H_{m,n}$, then $\Psi_0(-x) = (-1)^n \Psi_0(x)$: therefore, the spectrum of the Dirac operator corresponding to the pin$^+$-structure on $\mathbf{P}_{2m}$ is the set

$$\Lambda^+_m = \{\lambda \in \mathbb{Z} | \lambda + \tfrac{1}{2}| = m + 2n + \tfrac{1}{2};\ n = 0, 1, 2, \dots\},$$

whereas the spectrum associated with the pin$^-$-structure is the 'opposite' set $\Lambda^-_m = \{-\lambda : \lambda \in \Lambda^+_m\}$. These spectra are asymmetric, $\Lambda^+_m \cap \Lambda^-_m = \emptyset$, and $\Lambda^+_m \cup \Lambda^-_m$ is the spectrum of the Dirac operator on $\mathbf{S}_{2m}$.

## References


Baum, H., Friedrich, Th., Grunewald, R. and Kath, I.: 1991, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Verlag, Leipzig-Stuttgart.

Cahen, M. and Gutt, S.: 1988, 'Spin structures on compact simply connected Riemannian symmetric spaces', *Simon Stevin Quart. J. Pure Appl. Math.* Vol. no. **62**, pp. 209–242.

Cahen, M., Gutt, S., Lemaire, L. and Spindel, T.: 1986, 'Killing spinors', *Bull. Soc. Math. Belg. Sér. A* Vol. no. **38**, pp. 75–102.

Cahen, M., Gutt, S. and Trautman, A.: 1993, 'Spin structures on real projective quadrics', *J. Geom. Phys.*, Vol. no. **10**, pp. 127–154.

Cahen, M., Gutt, S. and Trautman, A.: 1995, 'Pin structures and the modified Dirac operator', *J. Geom. Phys.*, Vol. no. **17**, pp. 283–297.

Dąbrowski, L. and Trautman, A.: 1986, 'Spin structures on spheres and projective spaces', *J. Math. Phys.* Vol. no. **27**, pp. 2022–2028.

Friedrich, Th.: 1980, 'Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung', *Math. Nachr.* Vol. no. **97**, pp. 117–146.

Hitchin, N.: 1974, 'Harmonic spinors', *Adv. Math.* Vol. no. **14**, pp. 1–55.

Lichnerowicz, A.: 1963, 'Spineurs harmoniques', *C. R. Acad. Sci. Paris Sér. A-B* Vol. no. **257**, pp. 7–9.

Trautman, A.: 1995, 'The Dirac operator on hypersurfaces', *Acta Phys. Polon. B* Vol. no. 7, pp. 1283–1310.