BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. V, No. 12, 1957

8.

THEORETICAL PHYSICS

Proof of the Non-Existence of Periodic Gravitational Fields Representing Radiation

by

A. TRAUTMAN

Presented by L. INFELD on October 9, 1957

One of the most interesting and, as yet, unsolved questions in general relativity is the problem of gravitational waves carrying energy. Even the concept of gravitational energy is not a simple one and causes some difficulties. Restricting ourselves to metric fields $g_{\alpha\beta}$ which are asymptotically Minkowskian, we can describe this energy by means of the so-called energy-momentum pseudotensor density $\mathfrak{t}_{\mu}^{\nu}$. The power radiated by the gravitational field is commonly [1] defined as the flux of the quantity $\mathfrak{t}_{0}^{k}$ through a two-dimensional closed surface "in spatial infinity".

The Hertz *dipole* is one of the simplest radiating systems in *electrodynamics*. It can produce a strictly periodic electromagnetic field carrying energy. Its *gravitational* analogue is the Einstein-Eddington *quadrupole* (physical model: a rod spinning around an axis perpendicular to it) *). Assuming uniform rotation of the rod, Eddington [2], [3] found that, in the first (linear) approximation, it creates a *periodic* gravitational field representing a non-vanishing mean value of radiated power. The question arises: are there exact, periodic solutions of the Einstein equations with matter, representing radiation understood as energy escape to infinity? We shall show that such fields do not really exist.

From Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi k}{c^4} T_{\mu\nu} \qquad (\mu, \nu = 0,1,2,3) \tag{1}$$

---

*) This is not a "pure" quadrupole; its energy-momentum tensor has the following components: $T^{00} = (m\delta + \tau^{kl}\delta_{,kl})c^2$, $T^{0k} = -\dot{\tau}^{kl}\delta_{,l}c$, $T^{ik} = \ddot{\tau}^{ik}\delta$, where $\delta = \delta(\vec{r})$ denotes three-dimensional Dirac's function, m — the total mass, τ^{ik} is a periodic function of $t = x^0/c$, $\dot{\tau}^{ik} = d\tau^{ik}/dt$, and $\delta_{,k} = \partial\delta/\partial x^k$.

there follows the existence of non-covariant conservation laws

$$(\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)_{,\nu} = 0 \text{ *)} . \tag{2}$$

Restricting ourselves to isolated distributions of matter, i. e. assuming that T_μ^ν vanishes outside a bounded 3-dimensional region (or that it diminishes to zero sufficiently quickly at spatial infinity), we can suppose asymptotically Minkowskian behaviour of the metric field. Roughly speaking, we assume the existence of a co-ordinate system, such that, for sufficiently large $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, we have

$$|g_{\mu\nu} - \eta_{\mu\nu}| < \frac{M}{r}, \quad |g_{\mu\nu,\varrho}| < \frac{M}{r}, \tag{3}$$

where $\eta_{\mu\nu}$ is the metric tensor of a flat space-time, and M — a constant.

The power radiated at $x^0 = a$ is defined as [1]

$$W(a) = \oint_{S(a)} (\mathfrak{T}_0^k + \mathfrak{t}_0^k) n_k dS, \qquad (k = 1, 2, 3), \tag{4}$$

where $S(a)$ denotes a two-dimensional surface "in infinity", lying on the hypersurface $x^0 = a$; n_k is a unit vector normal to S. The existence of the integral (4) is ensured by our assumptions concerning the metric field and the co-ordinate system as expressed by (3).

A gravitational field $g_{\mu\nu}$ can be called *periodic*, if there exists a co-ordinate system satisfying inequality (3) and a number τ such that

$$g_{\mu\nu}(x^0 + \tau, x^k) \equiv g_{\mu\nu}(x^0, x^k) . \tag{5}$$

We can now formulate the following theorem: *The mean value of power radiated by a periodic, asymptotically Minkowskian gravitational field is equal to zero.* The proof is immediate. From (1) it follows that $\mathfrak{T}_\mu^\nu$ is a periodic function of x^0; the same can be said of $\mathfrak{t}_\mu^\nu$, which is built of $g_{\alpha\beta}$. We can thus write

$$\int_{x^0=a} (\mathfrak{T}_0^0 + \mathfrak{t}_0^0) dV = \int_{x^0=a+\tau} (\mathfrak{T}_0^0 + \mathfrak{t}_0^0) dV, \tag{6}$$

where integrals are to be taken over the whole hypersurface $x^0 = \text{const.}$ Integrating both sides of Eq. (2) with $\mu = 0$ over a four-dimensional region between hypersurface $x^0 = a$ and $x^0 = a + \tau$, we obtain

$$\int_{x^0=a+\tau} (\mathfrak{T}_0^0 + \mathfrak{t}_0^0) dV - \int_{x^0=a} (\mathfrak{T}_0^0 + \mathfrak{t}_0^0) dV + \int_a^{a+\tau} W(x^0) dx^0 = 0 \tag{7}$$

*) Gothic letters denote densities: $\mathfrak{T}_\mu^\nu = \sqrt{-g}\, T_\mu^\nu$, $g = \det(g_{\mu\nu})$.

or

$$\overline{W} = \frac{1}{\tau} \int_{a}^{a+\tau} W(x^0)\, dx^0 = 0. \tag{8}$$

This can be expressed in the following way: periodic gravitational fields can describe standing-wave processes only (e. g., those obtained by the EIH method [6]). Traveling waves carrying energy (gravitational, electromagnetic or other) produce secular changes in the metric. A similar statement was formulated by Einstein and Rosen [4].

Formula (7) provides an answer to a question posed by Pirani [5]. Is gravitational radiation accompanied by changes in masses of radiating bodies? For the Schwarzschild field, the integrals appearing in Eq. (6) are known to be proportional to the masses producing the field. From Eq. (7) we see that the decrease in mass is equal (if $c=1$) to the energy outgoing from the system.

Our simple theorem reflects an essential difference between the gravitational field and other classical fields. In the electrodynamics of special relativity there are periodic fields representing radiation and behaving at large distances as $1/r$. This is possible because there exist forces, e. g. mechanical, which are *external* with respect to the electromagnetic field. The energy-momentum conservation law can be written here in the form $(\overset{\text{em}}{T}{}^{\nu}_{\mu} + \overset{\text{m}}{T}{}^{\nu}_{\mu})_{,\nu} = 0$, where $\overset{\text{em}}{T}{}^{\nu}_{\mu}$ is the energy tensor of the electromagnetic field, and $\overset{\text{m}}{T}{}^{\nu}_{\mu}$ represents the kinetic energy of charges and forces acting upon them. $\overset{\text{em}}{T}{}^{\nu}_{\mu}$ is periodic, if the field is such; $\overset{\text{m}}{T}{}^{\nu}_{\mu}$ is non-periodic, at least if radiation occurs. In general relativity theory the entire energy is coupled to the g-field; there is no kind of energy which could be considered external with respect to this field. The periodicity of $g_{\mu\nu}$ implies that the *total* energy-momentum tensor must also have this property.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] J. N. Goldberg, *Gravitational radiation*, Phys. Rev. **99** (1955), 1873.

[2] A. S. Eddington, *The propagation of gravitational waves*, Proc. Roy. Soc. (London) **A 102** (1922), 269.

[3] — *The mathematical theory of relativity*, 2nd ed. Cambridge University Press, New York, 1924.

[4] A. Einstein and N. Rosen, *On gravitational waves*, J. Franklin Inst. **223** (1937), 43.

[5] F. Pirani, *Invariant formulation of gravitational radiation theory*, Phys. Rev. **105** (1957), 1089.

[6] A. Einstein, L. Infeld, B. Hoffmann, *The gravitational equations and the problem of motion*, Ann. of Math. **39** (1938), 65.