Fock space description of simple spinors

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Cartan's *simple*—often called *pure*—spinors corresponding to even-dimensional complex vector spaces are defined in terms of the associated maximal totally null planes. Their geometrical properties are derived and described using notions familiar to physicists: Dirac and Weyl spinors, gamma matrices, tensors formed bilinearly from pairs of spinors, and creation and annihilation operators of Fermi states. A new theorem characterizes a simple spinor ϕ by the properties of the vector $t_\psi B\gamma^\mu\phi$, where ψ is an arbitrary spinor and B is the matrix connecting the gamma matrices with their transposes. The Cartan constraint equations on the components of simple spinors are given a new, geometrically transparent derivation based on the action on simple spinors of a maximal Abelian subgroup of the group Spin.

I. INTRODUCTION

During the winter semester of 1935–1936, Elie Cartan gave a course of lectures at the Sorbonne on spinors. The notes of these lectures, taken by Mercier, were published in 1938.[1] In 1966 an English translation by Streater appeared.[2] In the lectures, Cartan presented a new approach to spinors, associated with a vector space of n dimensions, based on their intimate relation to totally *null* (Cartan used the word *isotropic*) subspaces of maximal dimension. In fact, Cartan showed that for $n > 6$ not all spinors correspond to maximal, totally null subspaces; he called *simple* those that do and described their properties. Characteristically of Cartan, the lectures combine a depth and originality of ideas with only rough outlines of the proofs. The importance of the lectures was recognized by another outstanding mathematician, Chevalley, whose book[3] connects Cartan's ideas with the approach to spinors presented by Brauer and Weyl[4] and based on Clifford algebras. Chevalley's emphasis is on algebra: Spinors are identified with elements of a minimal left ideal of the Clifford algebra and most of the theorems are proved without restrictions on the basic field (it may be of characteristic 2, for example). The generality of Chevalley's exposition made his book difficult to use by physicists; there is a readable account of parts of it by Benn and Tucker.[5]

Following Chevalley, most of the authors of publications on spinors in English replace the adjective *simple* by *pure*. However, here we shall use the original Cartan expression: Otherwise, we would have to accept that the Dirac spinor is impure (cf. Proposition 3).

Weyl spinors and the related null geometrical elements are known to play an important role in general relativity,[6] twistor theory,[7,8] and optical geometry.[9] Recent work on fundamental interactions and their unification makes essential use of geometries of more than four dimensions. For this reason, nontrivial simple spinors—which occur for $n > 6$—now have more chance of becoming relevant to physics than they had at the time of the appearance of Cartan's lectures. Further remarks on this subject can be found in our recent publications.[10,11]

In this paper we present a straightforward and explicit description of simple spinors and their principal properties. A new theorem characterizes a simple spinor ϕ in terms of the properties of the vector bilinear in ϕ and another spinor which need not be simple. Our approach is based on the observation—which can be traced back to Brauer and Weyl—that to a complex vector space of dimension $2m$ or $2m+1$ there corresponds a spinor space S of complex dimension 2^m representable as the Fock space of m Fermi states. We show that every simple spinor can be used to define the "vacuum state" in S and then all eigenstates of the occupation number operators are also simple.

The study of simple spinors may be considered as a preliminary to the problem of "classification of spinors,"[12] which consists in finding the orbits of the Spin groups in spinor spaces, computing their stabilizers, and exhibiting the generators of the ring of invariants of the representation. Simple spinors correspond to the orbit of the lowest dimension. The classification problem is difficult and very little is known for $n > 14$. We hope our approach will also shed light on this problem.

We restrict ourselves here to complex vector spaces of even dimension $2m$. It is easy to extend our considerations to complex odd-dimensional spaces as well as to real spaces with a scalar product of signature (m,m) and $(m+1,m)$. Other signatures require a subtler study because in those cases, the dimension of the maximal totally null subspaces is less than m.

II. PRELIMINARIES: NOTATION, CLIFFORD ALGEBRAS, AND SPINORS[13]

Let V be a complex vector space of dimension $2m$ $(m = 1,2,\ldots)$ with a scalar product g. The Clifford algebra $\mathrm{Cl}(g)$ admits a faithful and irreducible representation in a complex spinor space S of dimension 2^m. To alleviate the notation we identify V with its image in $\mathrm{Cl}(g)$. Moreover, since the representation $\mathrm{Cl}(g) \to \mathrm{End}\, S$ is an isomorphism (of algebras), we can also identify $\mathrm{Cl}(g)$ with End S. Therefore, the same letter u denotes a vector, an element of $\mathrm{Cl}(g)$, and an endomorphism of S; if $u, v \in V$, then

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$$uv + vu = 2g(u,v).$$

The vector space V admits a (generalized) orthonormal basis $(\gamma_1,...,\gamma_{2m})$ such that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu},$$

where

$$g_{\mu\mu} = (-1)^{\mu+1},$$

$$g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \quad (\mu,\nu = 1,...,2m).$$

We shall write $\gamma^\mu = \Sigma_\nu g_{\mu\nu}\gamma_\nu$, so that $\gamma^\mu\gamma_\nu + \gamma_\nu\gamma^\mu = 2\delta^\mu_\nu$.

From the orthonormal basis one can construct a null basis $(n_1,...,n_m,p_1,...,p_m)$ by putting

$$n_\alpha = \tfrac{1}{2}(\gamma_{2\alpha-1} - \gamma_{2\alpha}), \quad p_\alpha = \tfrac{1}{2}(\gamma_{2\alpha-1} + \gamma_{2\alpha}),$$

so that

$$n_\alpha n_\beta + n_\beta n_\alpha = 0, \quad p_\alpha p_\beta + p_\beta p_\alpha = 0 \tag{1}$$

and

$$n_\alpha p_\beta + p_\beta n_\alpha = \delta_{\alpha\beta}, \tag{2}$$

where $\alpha,\beta = 1,...,m$. The null vectors (n_α) span a maximal totally null (MTN) subspace N of V:

$$N = \text{span}\{n_1,...,n_m\}.$$

Similarly,

$$P = \text{span}\{p_1,...,p_m\}.$$

is also a MTN subspace, $N \cap P = \{0\}$, and there is a decomposition of V into a direct sum

$$V = N \oplus P.$$

Conversely, given a pair (N,P) of MTN subspaces of V such that $N \cap P = \{0\}$, one can find a basis $(n_1,..,n_m)$ of N and a basis $(p_1,...,p_m)$ of P such that Eqs. (1) and (2) hold. Every vector u admits a unique decomposition

$$u = n + p,$$

where $n \in N$ and $p \in P$. Writing

$$n = \sum_{\alpha=1}^{m} x_\alpha n_\alpha, \quad p = \sum_{\alpha=1}^{m} y_\alpha p_\alpha$$

one can express the fundamental quadratic form of V as

$$g(u,u) = x_1 y_1 + \cdots + x_m y_m.$$

The orthogonal group $\text{O}(g)$ acts transitively on the set of all MTN subspaces of V; this set has a natural structure of complex manifold of dimension $m(m-1)/2$.

Assume now that V has a preferred orientation and the basis (γ_μ) agrees with the orientation. The volume element

$$\Gamma = \gamma_1\gamma_2\cdots\gamma_{2m}$$

can be expressed in terms of the null basis as

$$\Gamma = [n_1,p_1][n_2,p_2]\cdots[n_m,p_m], \tag{3}$$

where square brackets denote commutators. Note, also, that Γ changes sign when the orientation of V is reversed. Since $\Gamma^2 = I$ (the unit automorphism of S), the eigenvalues of Γ are 1 and -1; the corresponding eigenvectors are *Weyl spinors* of positive and negative *helicity*, respectively. There is the decomposition

$$S = S_+ \oplus S_-,$$

where

$$S_\pm = \{\phi \in S \mid \Gamma\phi = \pm\phi\}.$$

The transposed endomorphisms (matrices) ${}^t\gamma_\mu$ define a representation of $\text{Cl}(g)$ in the space S^* dual to S. Since $\text{Cl}(g)$ is simple, there is an isomorphism $B: S \to S^*$ such that

$${}^t\gamma_\mu = B\gamma_\mu B^{-1}. \tag{4}$$

One shows that

$${}^tB = (-1)^{m(m-1)/2}B \tag{5}$$

and

$${}^t\Gamma B = (-1)^m B\Gamma. \tag{6}$$

If $\phi \in S$ is a "contravariant" and $\psi^* \in S^*$ a "covariant" spinor, then $\langle\psi^*,\phi\rangle$ is the evaluation ("contraction") of ψ^* on ϕ. The isomorphism of End S with $S \otimes S^*$ makes it possible to consider $\phi \otimes \psi^*$ as the endomorphism of S such that $(\phi \otimes \psi^*)(\psi) = \langle\psi^*,\psi\rangle\phi$ for every $\psi \in S$. Clearly,

$$\text{Tr}(\phi \otimes \psi^*) = \langle\psi^*,\phi\rangle, \quad \text{Tr}\, I = 2^m.$$

If $A \in \text{End}\, S$, then

$$A \circ (\phi \otimes \psi^*) = (A\phi) \otimes \psi^*. \tag{7}$$

The isomorphism B defines a bilinear map

$$S \times S \ni (\psi,\phi) \to \langle B\psi,\phi\rangle \in \mathbb{C},$$

which is invariant with respect to the action of the group[14] $\text{Pin}(g)$: If u is a unit vector, then

$$\langle Bu\psi,u\phi\rangle = \langle B\psi,\phi\rangle. \tag{8}$$

With every pair of spinors ψ, ϕ one can associate a sequence $B_k(\psi,\gamma)$, $k = 0,1,2,...,2m$ of multivectors over V: Their components with respect to an orthonormal basis (γ_μ) are given by

$$B_k^{\mu_1\cdots\mu_k}(\psi,\phi) = \langle B\psi,\gamma^{\mu_k}\cdots\gamma^{\mu_1}\phi\rangle, \tag{9}$$

with

$$1 \leqslant \mu_1 < \cdots < \mu_k \leqslant 2m. \tag{10}$$

With the understanding that the product of an empty sequence of the gammas is the unit automorphism I, Eq. (9) makes sense for $k = 0$ and $B_0(\psi,\phi) = \langle B\psi,\phi\rangle$. The set of all products $\gamma_{\mu_1}\cdots\gamma_{\mu_k}$, where $k = 0,1,...,2m$ and the indices satisfy (10), is a basis of End S. Therefore, there is a decomposition[15]

$$\phi \otimes B\psi = 2^{-m}\sum_k B_k(\psi,\phi), \tag{11}$$

where

$$B_k(\psi,\phi) = \sum_{(10)} B_k^{\mu_1\cdots\mu_k}\gamma_{\mu_1}\cdots\gamma_{\mu_k},$$

which is proved by noting that the trace of the product

$$\gamma^{\nu_l}\cdots\gamma^{\nu_1}\gamma_{\mu_1}\cdots\gamma_{\mu_k}, \quad \text{where} \quad \nu_1 < \cdots < \nu_l,$$

is 2^m for $k = \ell$, $\mu_1 = \nu_1,...,\mu_k = \nu_k$ and zero otherwise.

The symmetry properties (4) and (5) imply

$$B_k(\phi,\psi) = (-1)^{(k(k-1)+m(m-1))/2}B_k(\psi,\phi), \tag{12}$$

so that $B_k(\phi,\phi) = 0$ for $m \equiv 0,1$ and $k = 2,3$ or $m \equiv 2,3$ and $k = 0,1 \pmod 4$. Equation (6) implies

$$B_k(\Gamma\psi,\phi) = (-1)^{m-k} B_k(\psi,\Gamma\phi). \quad (13)$$

Therefore, if ψ and ϕ are Weyl spinors, then

$$B_k(\psi,\phi) = 0 \quad \text{if the helicities of } \psi \text{ and } \phi \text{ are } \begin{cases} \text{equal and } m-k \text{ is odd} \\ \text{opposite and } m-k \text{ is even} \end{cases} \quad (14)$$

In particular, if ϕ is a Weyl spinor, then

$$B_k(\phi,\phi) = 0 \quad \text{for} \quad m-k \equiv 1,2,3 \pmod 4. \quad (15)$$

Since

$$\Gamma\gamma^1\cdots\gamma^k = (-1)^k \gamma^{k+1}\cdots\gamma^{2m},$$

there is a convenient way of defining *Hodge duality* of multivectors by means of the gammas. For every k-vector F with the components $F^{\mu_1\cdots\mu_k}$ with respect to the orthonormal basis (γ_μ), one defines its *dual* to be the $(2m-k)$-vector $^*F$ with the components given by

$$\sum {}^*F^{\nu_{k+1}\cdots\nu_{2m}}\gamma_{\nu_{k+1}}\cdots\gamma_{\nu_{2m}} = \sum F^{\mu_1\cdots\mu_k}\Gamma\gamma_{\mu_1}\cdots\gamma_{\mu_k}, \quad (16)$$

where the sums are taken over all strictly increasing sequences of the indices. Since $\Gamma^2 = I$ one has $^{**}F = F$.

Replacing ϕ by $\Gamma\phi$ in Eq. (11) and using (7) and (16) one obtains

$$^*B_k(\psi,\phi) = B_{2m-k}(\psi,\Gamma\phi). \quad (17)$$

Therefore, if ϕ is a Weyl spinor $\Gamma\phi = \pm\phi$, then

$$^*B_k(\psi,\phi) = \pm B_{2m-k}(\psi,\phi) \quad (18)$$

and, in particular,

$$^*B_m(\psi,\phi) = \pm B_m(\psi,\phi). \quad (19)$$

The only essential component of $B_{2m}(\psi,\phi)$ is the pseudoscalar $\langle B\psi,\Gamma\phi\rangle$.

Let F be a k-vector and $u = u^\mu\gamma_\mu \in V$; then the *contraction* $u \lrcorner F$ of u with F is the $(k-1)$-vector with the components

$$(u \lrcorner F)^{\mu_2\cdots\mu_k} = \sum_{\mu\nu} g_{\mu\nu} u^\nu F^{\mu\mu_2\cdots\mu_k}$$

and the *exterior product* $u \wedge F$ is the $(k+1)$ vector obtained from the tensor product $u \otimes F$ by the alternating map ("antisymmetrization over all indices"). The isomorphism (of vector spaces) $\mathrm{Cl}(g) = \Lambda V$ leads to the following useful formula[16]:

$$uF = u \lrcorner F + u \wedge F, \quad (20)$$

where uF is the Clifford product of u and F.

Computing $u\phi \otimes B\psi$, where $u \in V$ and using Eqs. (7), (11), and (20) one obtains

$$B_k(\psi,u\phi) = u \lrcorner B_{k+1}(\psi,\phi) + u \wedge B_{k-1}(\psi,\phi) \quad (21)$$

for $k = 0, 1,\ldots,2m$; it is understood that B_{-1} and B_{2m+1} are zero.

III. DEFINITION OF SIMPLE SPINORS AND AN EXAMPLE

The vector space *associated* with a spinor $\phi \in S$,

$$M(\phi) = \{u \in V \mid u\phi = 0\}, \quad (22)$$

depends only on the direction of ϕ. For $\phi \neq 0$ this vector space is totally null: If $u, v \in M(\phi)$, then $g(u,v) = 0$.

Definition: A nonzero spinor is said to be simple if its associated totally null vector space is maximal.

In other words, if V is $2m$-dimensional, then simplicity of ϕ is equivalent to $\dim M(\phi) = m$. To see that simple spinors exist in every dimension, consider a MTN subspace N of V and a basis $(n_1,\ldots,n_m)$ of N. Since the representation of $\mathrm{Cl}(g)$ in S is faithful, there exists a spinor χ such that

$$\omega = n_1 n_2 \cdots n_m \chi \quad (23)$$

is nonzero; then $M(\omega) = N$ and ω is simple. On the other hand, not all spinors are simple, as may be seen from the following example, which is familiar to physicists.

Example: If V is four-dimensional, $m = 2$, then S is also:

$${}^tB = -B, \quad B\Gamma = {}^t\Gamma B.$$

Let $\phi = \phi_+ + \phi_-$ be the decomposition of a Dirac spinor ϕ into its Weyl components, $\phi_\pm = \frac{1}{2}(I \pm \Gamma)\phi$.

For every $u \in V$, the condition $u\phi = 0$ is equivalent to $u\phi_+ = 0$ and $u\phi_- = 0$; this shows that

$$M(\phi) = M(\phi_+) \cap M(\phi_-).$$

To determine the spaces $M(\phi_\pm)$, consider the endomorphisms $\phi_\pm \otimes B\phi_\pm$. From (15) it follows that only the term with $k = 2$ is present in the decomposition (11):

$$\phi_\pm \otimes B\phi_\pm = F_\pm, \quad \text{where} \quad F_\pm = \tfrac{1}{4}B_2(\phi_\pm,\phi_\pm) \quad (24)$$

and (15) and (19) imply

$$\Gamma F_\pm = {}^*F_\pm = \pm F_\pm. \quad (25)$$

Similarly,

$$\phi_\pm \otimes B\phi_\mp = \tfrac{1}{2}(1 \pm \Gamma)k, \quad (26)$$

where k is a vector. Using (21) for $\psi = \phi = \phi_\pm$ one obtains that

$$u\phi_\pm = 0 \quad \text{is equivalent to} \quad u \lrcorner F_\pm = 0 \quad \text{and} \quad u \wedge F_\pm = 0. \quad (27)$$

In particular, computing $k\phi_\pm$ and using (26) and $\langle B\phi_\pm,\phi_\pm\rangle = 0$ one obtains

$$k\phi_\pm = 0. \quad (28)$$

Equation (28) implies that the two-forms F_+ and F_- are decomposable and have the null vector k as their common eigenvector. The spaces $M(\phi_+)$ and $M(\phi_-)$ coincide if and only if both ϕ_+ and ϕ_- are zero. If $\phi_+ \neq 0$ and $\phi_- \neq 0$, then the intersection

$$M(\phi_+) \cap M(\phi_-) = \mathbb{C}k \quad (29)$$

is one-dimensional and the Dirac spinor $\phi_+ + \phi_-$ is not simple. If $\phi_+ \neq 0$ and $\phi_- = 0$, the $M(\phi) = M(\phi_+)$ is two-dimensional and the spinor $\phi = \phi_+$ is simple: similarly, the same holds when the roles of ϕ_+ and ϕ_- are interchanged.

Parenthetically, we should mention that in Minkowski space, which can be defined by embedding $\mathbb{R}^4$ in $\mathbb{C}^4$, so that $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ are replaced by $(\gamma_1,i\gamma_2,\gamma_3,\gamma_4)$, the null vector k can be made real by taking the charge conjugate of ϕ_+ for ϕ_-. With the direction of such a vector there is an associated two-dimensional space of "null electromagnetic fields"[8,9,17]. Remembering that in signature (3,1), the square of the dual is $-id$, one can write

$$F_{\pm} = f \mp i{}^*f,$$

where the null two-form f, representing the electromagnetic field, is now real. The totally null subspaces $M(\phi_+)$ and $M(\phi_-)$ are complex conjugate to each other and $\mathbb{R}k$ is their only real direction. The spinorial representation of null electromagnetic waves, contained in Eq. (24), gives rise to various extensions and generalizations to gravitation and other fields. They play a major role in the Penrose program and the Newman–Penrose formalism for the treatment of algebraically special metrics,[8] as well as in the treatment of null strings.[18] There is also a related spinorial form of the Enneper–Weierstrass formula for solutions of the equation for minimal surfaces and its extension to strings.[19]

IV. A FOCK REPRESENTATION BASIS IN THE SPACE OF SPINORS

The vectors (endomorphisms of S) n_α and p_α ($\alpha = 1,...,m$) defined in Sec. II fulfill the anticommutation relations (1) and (2), which are identical to those satisfied by the annihilation and creation operators of states subject to Fermi statistics. This observation—made previously by Brauer and Weyl[4]— can be used to construct a convenient basis in S.

Given the decomposition of V into a direct sum of MTN subspaces N and P and the corresponding null basis $(n_1,...,n_m,p_1,...,p_m)$ one can find a nonzero spinor ω of the form (23) and interpret it as the "vacuum state": It is annihilated by all the "operators" $n_1,...,n_m$. By acting on ω with products of the "creation operators" p_α corresponding to all sequences (μ_i) subject to (10) one obtains a collection of 2^m spinors:

$$\begin{aligned} &\omega;\quad p_1\omega,...,p_m\omega; \\ &p_1p_2\omega,...,p_{m-1}p_m\omega; \\ &\cdots;\quad p_1\cdots p_m\omega. \end{aligned} \tag{30}$$

The collection (30) is linearly independent, as may be easily shown using (1) and (2) only. All the spinors occurring in the sequence (30) are simple, e.g.,

$$\begin{aligned} &M(p_{k+1}p_{k+2}\cdots p_m\omega) \\ &\quad = \mathrm{span}\{n_1,...,n_k,p_{k+1},p_{k+2},...,p_m\} \end{aligned}$$

is totally null and of dimension m. The basis (30) can be used to show that the direction of a simple spinor is determined by the associated MTN subspace. Indeed, let $\omega' \in S$ be such that $M(\omega') = M(\omega) = N$ and let

$$\begin{aligned} \omega' = {} & \xi_0\omega + \xi_1 p_1\omega + \cdots + \xi_{2\cdots m}p_2\cdots p_m\omega \\ & + \xi_{12\cdots m}p_1p_2\cdots p_m\omega. \end{aligned}$$

Multiplying both sides of the above equation by $n_1n_2\cdots n_m$ yields $\xi_{12\cdots m} = 0$; multiplying it next by $n_2\cdots n_m$ leads to $\xi_{2\cdots m} = 0$, etc. This proves the following proposition.

Proposition 1: There is a natural, one-to-one correspondence between the set of all MTN subspaces of V and the set of directions of all simple spinors.

Every simple spinor ω can be taken to be the first, "vacuum" or "standard spinor"[2] of the sequence (30). The spinor ω determines only the subspace $N = M(\omega)$: There is still considerable freedom in choosing the complementary MTN subspace P and the null basis adapted to the decomposition $V = N \oplus P$.

Proposition 2: Let ω and ϕ be linearly independent simple spinors. There is a basis in S of the form (30) such that ϕ is one of the basis vectors, other than ω.

Indeed, it follows from Proposition 1 that $N = M(\omega) \neq M(\phi) = N'$. Let

$$k = \dim N \cap N';$$

then $0 \leqslant k < m$ and there exists a basis $(n_1,...,n_m)$ of N adapted to the subspace $N\cap N'$ in the sense that

$$N \cap N' = \mathrm{span}\{n_1,...,n_k\}.$$

Similarly, there is basis $(n'_1,...,n'_m)$ of N' such that $n'_1 = n_1,...,n'_k = n_k$. The matrix of solar products

$$g(n_a,n'_b), \quad \text{where} \quad a,b = k+1,...,m$$

is nonsingular: Otherwise there would be a vector $\lambda_{k+1}n'_{k+1} + \cdots + \lambda_m n'_m$ different from zero, null and orthogonal to N, but not in N. This would contradict the maximality of N among totally null subspaces. Therefore, by a linear transformation of the vectors n'_a ($a = k+1,...,m$) one can achieve $g(n_a,n'_b) = \frac{1}{2}\delta_{ab}$. One can now put

$$p_{k+1} = n'_{k+1},...,p_m = n'_m$$

and complete the sequence $(p_{k+1},...,p_m)$ to a basis $(p_1,...,p_m)$ of a MTN subspace P complementary to N and such that relations (1) and (2) hold. Since now

$$M(\phi) = \mathrm{span}\{n_1,...,n_k,p_{k+1},...,p_m\},$$

it is clear that after rescaling,

$$\phi = p_{k+1}\cdots p_m\omega. \tag{31}$$

Proposition 3: Simple spinors are Weyl.

The proof is straightforward: Since $[n_1,p_1]n_1 = n_1$ and $[n_1,p_1]p_1 = -p_1$, etc., it is clear that with Γ given by (3) and ϕ by (31), one has

$$\Gamma\phi = (-1)^{m-k}\phi.$$

Choosing the orientation in V so that Γ—rather than $-\Gamma$—is of the form (3) is equivalent to assigning a positive helicity to ω. Since parallel spinors have equal helicity, this notion is transferred, via Proposition 1, to MTN subspaces of V. We should also mention that if $n > 6$ there are Weyl spinors that are not simple. For example, for $n = 8$, the spinor $\phi = (1 + p_1p_2p_3p_4)\,\omega$ is not simple because $u\phi = 0$ implies $u = 0$.

If u is a unit vector and ϕ is a simple spinor, then $u\phi$ is also simple and of opposite helicity to ϕ. Indeed,

$$M(u\phi) = uM(\phi)u, \; \Gamma u\phi = -u\Gamma\phi.$$

Remembering that the Pin(g) group consists of products of unit vectors and Spin(g) is its subgroup, consisting of products of even sequences of such vectors, one arrives at once at the following proposition.

Proposition 4: The group Pin(g) acts transitively on the set of directions of all simple spinors. The group Spin(g) acts transitively on each of the two sets of directions of simple spinors of equal helicity.

Indeed, Eq. (31) can be written as

$$\phi = (n_{k+1} + p_{k+1}) \cdots (n_m + p_m)\omega,$$

where each factor $n_\alpha + p_\alpha$ is a unit vector; their product belongs to Spin(g) whenever $m - k$ is even, i.e., whenever ω and ϕ are of equal helicity. Otherwise, the product belongs to Pin(g), but not to Spin(g), and the spinors ω and ϕ are of opposite helicity.

The helicities of the spinors ω and ϕ related by (31) are equal or opposite depending on whether $m - k$ is even or odd. Therefore, the dimension k of $M(\omega) \cap M(\phi)$ is even if and only if m is even and the helicities are equal or m is odd and the helicities are opposite. Thus, for example, the complementary MTN subspaces N and P are of equal helicity if and only if m is even. One also proves the following proposition.

Proposition 5 (Ref. 3, Proposition III. 1.12): If ω and ϕ are linearly independent simple spinors, then $\omega + \phi$ is simple if and only if

$$\dim M(\omega) \cap M(\phi) = m - 2;$$

then

$$M(\omega) \cap M(\phi + \omega) = M(\omega) \cap M(\phi).$$

The "if" part of the proposition is immediate: Taking ϕ to be of the form (31) with $k = m - 2$ one has

$$M(\phi + \omega) = \mathrm{span}\{n_1,\dots,n_{m-2}, n_{m-1} + p_m, n_m - p_{m-1}\}.$$

The *Lie algebra* spin(g) of the group Spin(g) can be identified[20] with the subspace $[V,V]$ of Cl(g) spanned by all the commutators $[u,v]$ where $u, v \in V$. Every MTN subspace P defines the subalgebra

$$[V,P] = \{A \in \mathrm{spin}(g) \mid A\psi = \lambda\psi, \lambda \in \mathbb{C}\}, \tag{32}$$

where ψ is a simple spinor such that $P = M(\psi)$. If (p_α) is a basis in P, then

$$A \in [V,P] \leftrightarrow A = \sum_{\alpha=1}^{m} [v_\alpha, p_\alpha],$$

where $v_\alpha \in V$. The commutator subalgebra of $[V,P]$ is the Abelian Lie algebra $[P,P]$ consisting of all those elements of spin(g) that annihilate ψ. If

$$A = \sum_{\alpha<\beta} A_{\alpha\beta} p_\alpha p_\beta \in [P,P],$$

where $A_{\alpha\beta} \in \mathbb{C}$, then the element

$$a = \exp A = \prod_{\alpha<\beta} (1 + A_{\alpha\beta} p_\alpha p_\beta) \in \mathrm{Spin}(g) \tag{33}$$

leaves invariant all elements of P: If $p \in P$, then $apa^{-1} = p$. If N is a MTN subspace complementary to P, then the Abelian subalgebra $[P,P]$ is complementary, as a vector subspace, to $[V,N]$ in spin(g). The subgroup of Spin(g) corresponding to the subalgebra $[V,N]$ leaves the direction of ω invariant; the subgroup corresponding to $[P,P]$ "moves" ω, but its action is not transitive on the set of directions of simple spinors of equal helicity. Indeed, if ϕ and ϕ' are two such spinors and there is an element (33) such that $\phi' = \mathrm{const}\, a\phi$, then $M(\phi') = aM(\phi)a^{-1}$. Since $p \in P$ implies $p = apa^{-1}$ one obtains

$$M(\phi') \cap P = M(\phi) \cap P$$

as a necessary condition for the existence of a. This condition is also sufficient, as asserted in the following proposition.

Proposition 6: Let N and P be two complementary MTN subspaces of V and ϕ a simple spinor such that $M(\phi) \cap P$ is k-dimensional. There then exists an element a of the form (33) such that

$$\phi = \lambda a p_1 \cdots p_k \omega, \tag{34}$$

where $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $M(\omega) = N$, and the vectors $(p_1,\dots,p_k)$ constitute a basis of $M(\phi) \cap P$.

Proof: Let $(p_1,\dots,p_m)$ be a basis of P adapted to $M(\phi) \cap P$, i.e., such that

$$M(\phi) \cap P = \mathrm{span}\{p_1,\dots,p_k\}.$$

Let $(n_1,\dots,n_m)$ be a basis of N such that Eqs. (2) hold and consider the MTN subspaces

$$N_k = \mathrm{span}\{p_1,\dots,p_k, n_{k+1},\dots,n_m\},$$

$$P_k = \mathrm{span}\{n_1,\dots,n_k, p_{k+1},\dots,p_m\}.$$

Clearly, $N_k \cap P_k = \{0\}$, but, also, $M(\phi) \cap P_k = \{0\}$ because if $u \in M(\phi)$, then $g(u,p_\alpha) = 0$ for $\alpha = 1,\dots,k$. Therefore, if $u \in M(\phi) \cap P_k$, then $u \in \mathrm{span}\{p_{k+1},\dots,p_m\} \subset P$, so that $u \in M(\phi) \cap P$, but $M(\phi) \cap P \cap P_k = \{0\}$. Complete now the basis $(p_1,\dots,p_k)$ of $M(\phi) \cap P$ to a basis $(p_1,\dots,p_k, n'_{k+1},\dots,n'_m)$ of M(ϕ), so that

$$n_\alpha n'_\kappa + n'_\kappa n_\alpha = 0, \quad \text{for} \quad \alpha = 1,\dots,k, \quad \kappa = k+1,\dots,m; \tag{35}$$

$$n'_\kappa p_\lambda + p_\lambda n'_\kappa = \delta_{\kappa\lambda}, \quad \text{for } \kappa,\lambda = k+1,\dots,m. \tag{36}$$

Writing $n'_\kappa = n_\kappa - v_\kappa$; computing the scalar products $g(v_\kappa,p)$, $p \in P$ and $g(v_\kappa,n_\alpha)$, $\alpha = 1,\dots,k$; and using Eqs. (2), (35), and (36) one finds that $v_\kappa \in \mathrm{span}\{p_{k+1},\dots,p_m\}$, i.e., there is a matrix $(A_{\kappa\lambda})$ such that

$$v_\kappa = \sum_\lambda A_{\kappa\lambda} p_\lambda. \tag{37}$$

The vectors n'_κ belong to an MTN subspace, $g(n'_\kappa, n'_\lambda) = 0$. Therefore,

$$g(n_\kappa, v_\lambda) + g(v_\kappa, n_\lambda) = 0$$

and the matrix $(A_{\kappa\lambda})$ is antisymmetric. Let

$$A = \sum_{k+1 \leqslant \kappa < \lambda \leqslant m} A_{\kappa\lambda} p_\kappa p_\lambda; \tag{38}$$

then

$$n_\kappa A - A n_\kappa = \sum_\lambda A_{\kappa\lambda} p_\lambda.$$

Therefore, if a is given by (3) with A determined from (37) and (38), then

$$a p_\alpha a^{-1} = p_\alpha \, (\alpha = 1,\dots,k),$$

$$a n_\kappa a^{-1} = n'_\kappa \, (\kappa = k+1,\dots,m).$$

In other words, the element a of the group Spin(g) transforms N_k into $M(\phi)$ preserving $N_k \cap M(\phi) = P \cap M(\phi)$. Since

$$N_k = M(p_1 \cdots p_k \omega),$$

the simple spinor ϕ is proportional to $a p_1 \cdots p_k \omega$, as claimed.

Remark 1: The element (38) of Cl(g) is nilpotent: There exists an integer l such that $2l \leqslant m-k+2$ and $A^l = 0$. In particular, if $k = m-2$, then

$$A^2 = 0.$$

In this case A is proportional to $p_{m-1}p_m$, $\exp tA = 1 + tA$ and the straight line in S,

$$t \to (1+tA)p_1 \cdots p_{m-2}\omega, \quad 0 \leqslant t \leqslant 1,$$

connects the simple spinor $\phi = p_1 \cdots p_{m-2}\,\omega$ with the simple spinor $\phi = \psi$, where $\psi \sim p_1 \cdots p_m \omega$. This sheds some light on the property of simple spinors described in Proposition 5.

V. MULTIVECTORS ASSOCIATED WITH SIMPLE SPINORS

The decomposition formula (11) associates with a pair of spinors a sequence of multivectors; they provide useful information, often with a clear geometrical interpretation. This is especially so when one of the spinors—or both—are simple.

For the sequel we need the following useful result.

Lemma (Ref. 3, Proposition III. 2.4): If ω and ψ are simple spinors, then

$$M(\omega) \cap M(\psi) \neq \{0\} \leftrightarrow B_0(\psi,\omega) = 0. \tag{39}$$

Indeed, if $u \in M(\omega) \cap M(\psi)$ and $u \neq 0$, then there is a vector $v \in V$ such that $uv + vu = 1$ and since $u\omega = u\psi = 0$, one obtains

$$B_0(\psi,\omega) = \langle B\psi, uv\omega\rangle = \langle {}^t uB\psi, v\omega\rangle = \langle Bu\psi, v\omega\rangle = 0.$$

Conversely, if $M(\omega) \cap M(\psi) = \{0\}$, then one can take $N = M(\omega)$, $P = M(\psi)$ and construct a null basis $(n_1,\ldots,n_m, p_1,\ldots,p_m)$ of V and the "Fock basis" (30) of S. If ϕ is any spinor from the sequence (30) other than $\psi = p_1 \cdots p_m \omega$, then $M(\omega) \cap M(\phi) \neq \{0\}$ and thus $B_0(\phi,\omega) = 0$. Since B is an isomorphism, the form B_0 is nondegenerate and therefore, $B_0(\psi,\omega) \neq 0$; this completes the proof of the lemma.

We may now prove the following proposition.

Proposition 7: A necessary and sufficient condition for a spinor $\omega \neq 0$ to be a simple and have N as its associated MTN subspace is that the vector $B_1(\phi,\omega)$ belongs to N for every spinor ϕ.

Indeed, let ω be a simple spinor $N = M(\omega)$ and let ϕ be any spinor. For $k = 0$ the recurrence relation (21) gives

$$u \,\lrcorner\, B_1(\phi,\omega) = B_0(\phi,u\omega). \tag{40}$$

If $u \in N$, then $u\omega = 0$ and (40) yields $u \,\lrcorner\, B_1(\phi,\omega) = 0$. Therefore, the vector $B_1(\phi,\omega)$ is orthogonal to N and as such, contained in N. Conversely, let $B_1(\phi,\omega)$ belong to N for every $\phi \in S$. If $u \in N$, then $B_0(\phi,u\omega) = 0$ for every ϕ; therefore, $u\omega = 0$, i.e., $N = M(\omega)$ and ω is simple.

Can every element of N be represented as $B_1(\phi,\omega)$ with a suitable choice of ϕ? To show that this is so, we first observe that the lemma implies

$$\text{if } \psi = p_1 \cdots p_m \omega, \text{ then } \langle B\psi,\omega\rangle \neq 0. \tag{41}$$

To prove that the map $S \to N$ given by $\phi \to B_1(\phi,\omega)$ is surjective we consider the spinors $\phi_\alpha = n_\alpha \psi$ and notice that by virtue of (40),

$$p_\alpha \,\lrcorner\, B_1(\phi_\beta,\omega) = \langle B\psi,\omega\rangle \delta_{\alpha\beta}, \quad (\alpha,\beta = 1,\ldots,m),$$

and the collection of vectors $B_1(\phi_\alpha,\omega)$, $\alpha = 1,\ldots,m$ constitutes a basis of N.

The characterization of simple spinors contained in Proposition 7 seems to be new.

The following proposition is derived from the work by Cartan.[2]

Proposition 8: If ω is a simple spinor, then $B_k(\omega,\omega) = 0$ for $k \neq m$ and the m-vector $B_m(\omega,\omega)$ is proportional to the product $n_1 \cdots n_m$ of the vectors constituting a basis of $M(\omega)$.

Indeed, if ω is simple, then it is a Weyl spinor and (15) gives $B_k(\omega,\omega) = 0$ unless $k \equiv m \pmod 4$. If $n \in M(\omega)$, then $n\omega = 0$ and (21) yields

$$n \,\lrcorner\, B_{k+2}(\omega,\omega) + n \wedge B_k(\omega,\omega) = 0. \tag{42}$$

If $k \equiv m \pmod 4$, then $B_{k+2}(\omega,\omega) = 0$ and (42) implies

$$n \wedge B_k(\omega,\omega) = 0, \quad \text{for every } n \in N. \tag{43}$$

A similar argument also gives

$$n \,\lrcorner\, B_k(\omega,\omega) = 0, \quad \text{for every } n \in N.$$

The only nonzero solution of (43) is for $k = m$. Therefore,

$$\omega \otimes B\omega = n_1 \cdots n_m \langle B\omega, p_m \cdots p_1 \omega\rangle, \tag{44}$$

where the numerical coefficient is determined by putting $\phi = \psi = \omega$ in (11), multiplying on the lhs by $p_m \cdots p_1$, taking the trace of both sides, and noting that $\operatorname{Tr}(p_m \cdots p_1 n_1 \cdots n_m) = 1$.

Chevalley proved also the converse[21] of Proposition 8: If $\omega \neq 0$ is a Weyl spinor and $B_k(\omega,\omega) = 0$ for $k \neq m$, then ω is simple. Therefore, Eq. (44), with the provision that $\omega \neq 0$ is a Weyl spinor, provides a definition of simple spinors which is equivalent to the one based on the maximality of the associated totally null plane $M(\omega)$ given by (22).

It is now easy to see that all Weyl spinors in spaces of dimension $n \leqslant 6$ are simple. Indeed, it follows from (15) that for every Weyl spinor ω corresponding to a space of dimension $2m \leqslant 6$ one has

$$B_k(\omega,\omega) = 0, \quad \text{for } k = 0,1,\ldots,m-1.$$

In dimension (7 and) 8 one encounters the first quadratic constraint on simple spinors, namely $B_0(\omega,\omega) = 0$. In higher dimensional spaces there is a sequence of such constraints, namely

$$B_k(\omega,\omega) = 0,$$

where $k \equiv m \pmod 4$ and $k < m$. Since, for Weyl spinors, ${}^*B_k(\omega,\omega) = \pm B_{2m-k}(\omega,\omega)$, it is enough to consider the constraints for $k < m$.

Consider now a linearly independent pair of simple spinors ω and ϕ. According to Proposition 2, one can find a null basis $(n_1,\ldots,n_m,p_1,\ldots,p_m)$ of V such that

$$M(\omega) = \operatorname{span}\{n_1,\ldots,n_m\},$$
$$M(\phi) = \operatorname{span}\{n_1,\ldots,n_k,p_{k+1},\ldots,p_m\},$$

and

$$\phi = p_{k+1} \cdots p_m \omega,$$

where $k = \dim M(\omega) \cap M(\phi)$. Since

$$p_m n_1 \cdots n_m = \tfrac{1}{2}(-1)^{m-1} n_1 \cdots n_{m-1}(1 + [p_m,n_m])$$

one easily proves by induction that $p_{k+1}\cdots p_m n_1\cdots n_m$ is proportional to $n_1\cdots n_k$ + multivectors of degrees $k+2$, $k+4,...,2m-k$. By multiplying $\omega\otimes B\omega$ with $p_{k+1}\cdots p_m$ on the left and using the last observation one arrives at the following proposition.

Proposition 9: If ω and ϕ are simple spinors, then the dimension of the intersection $M(\omega)\cap M(\phi)$ is the least integer k such that $B_k(\omega,\phi)\neq 0$. The multivector $B_k(\omega,\phi)$ is then proportional to the product of the vectors of a basis of the intersection. This proposition generalizes Proposition 8 and the lemma.

Remark 2: The Abelian subgroup $G(P)$ of $\mathrm{Spin}(g)$ corresponding to the subalgebra $[P,P]$ of $\mathrm{spin}(g)$ is of dimension $m(m-1)/2$, equal to that of the manifold Σ of directions of simple spinors of one helicity coinciding, say, with that of ω: Its action on Σ is not transitive. For example, if P is a complementary subspace to the MTN subspace $M(\omega)$, then the direction dir ψ of the spinor $\psi=p_1...p_m\omega$ is left invariant by $G(P)$. However, this action is "almost transitive"[22] in the sense that the orbit of $G(P)$ containing dir ω is an open submanifold of Σ and its complement is a submanifold Σ_1 of lower dimension. Indeed, according to Proposition 6 and the lemma, the direction of a simple spinor ϕ does not belong to the orbit containing dir ω if and only if it satisfies the homogeneous equation $B_0(\psi,\phi)=0$ defining the submanifold Σ_1 of Σ. Therefore, a simple spinor ϕ, of the same helicity as ω, can be said to be in a generic position with respect to P if $B_0(\psi,\phi)\neq 0$; there then exists an antisymmetric matrix $(A_{\alpha\beta})$, $\alpha,\beta=1,..,m$ and a number $\xi_0\neq 0$ such that

$$\phi=\xi_0(\exp A)\omega, \tag{45}$$

where

$$A=\sum_{\alpha<\beta} A_{\alpha\beta}p_\alpha p_\beta.$$

(Note that in the expansion of $\exp A$ only a finite number of terms are different from 0.) On the other hand, every spinor can be expressed in terms of the basis (30), as was already done in Sec. IV:

$$\phi=\sum_{k=0}^{m}\sum \xi_{\alpha_1\cdots\alpha_k}p_{\alpha_1}\cdots p_{\alpha_k}\omega, \tag{46}$$

where the second sum is over all the sequences (α_i) such that

$$1\leqslant\alpha_1<\cdots<\alpha_k\leqslant m \tag{47}$$

and the term corresponding to $k=0$ is $\xi_0\omega$. Comparing (45) and (46) one obtains

$$\xi_{\alpha_1\cdots\alpha_k}=0,\quad \text{for } k \text{ odd},$$

a condition resulting from the fact that ω and ϕ have equal helicities, and

$$\xi_{\alpha_1\alpha_2}=\xi_0 A_{\alpha_1\alpha_2},\quad \text{for } k=2. \tag{48}$$

Taking (33) into account gives

$$\phi=\xi_0\prod_{\alpha<\beta}\left(1+\frac{\xi_{\alpha\beta}}{\xi_0}p_\alpha p_\beta\right)\omega.$$

By comparing the other terms with even k one obtains

$$l!\sum \xi_{\alpha_1\cdots\alpha_{2l}}p_{\alpha_1}\cdots p_{\alpha_{2l}}=\xi_0 A^l, \tag{49}$$

where the sum is over the sequences (α_i) satisfying (47) with $k=2l$. Since $2l\leqslant m$, there is only a finite set of such relations. Using (48) to eliminate A one obtains a sequence of constraints on the components ξ of a generic simple spinor ϕ of the same helicity as ω; these constraints coincide with the set of equations (a) in Sec. 92 of Cartan's lectures.[1,2] For example, for $l=2$ one obtains

$$\xi_0=\xi_{\alpha_1\alpha_2\alpha_3\alpha_4}=\xi_{\alpha_1\alpha_2}\xi_{\alpha_3\alpha_4}+\xi_{\alpha_1\alpha_3}\xi_{\alpha_4\alpha_2}+\xi_{\alpha_1\alpha_4}\xi_{\alpha_2\alpha_3}.$$

If m is even, then for $l=m/2$ the constraint (49) relates $\xi_{1...m}$ to the Pfaffian of the antisymmetric matrix $(\xi_{\alpha\beta})$; with a suitable generalization of the notion of the Pfaffian one can extend this observation to other values of l; see Ref. 23.

We hope to have convinced the reader that simple spinors are simple indeed.

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