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7.

THEORETICAL PHYSICS

On the Conservation Theorems and Co-ordinate Systems in General Relativity

by

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Introduction

It is well known that in the framework of special relativity every field theory contains 10 conservation laws. They correspond to the 10 first integrals of the Newtonian mechanics. It is easy to express a Lorentz-covariant theory in a general covariant manner, but the same cannot be said about the corresponding conserved quantities (energy-impulse tensor and angular momentum tensor).

It seems natural to require that the conserved quantities and the conservation laws should be defined in such a way that:

(a) these quantities and the corresponding conservation theorems have an invariant meaning;

(b) the conservation laws can be formulated in an *integral form*;

(c) for a flat space-time the conservation laws become identical with those defined in special relativity.

In general relativity one introduces a symmetric tensor of matter $T^{\alpha\beta}$, divergenceless by virtue of the Einstein equations

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{1}$$

But the 4 equations (1) do not fulfil the conditions (b) and (c).

The aim of the present paper is to give a general definition of conserved quantities for fields interacting with "pole-particles". The conservation laws appear to be closely connected with the symmetry properties of the Riemannian space-time V_4 [2], [7]. In this connection we shall formulate some remarks on the role of "preferred" co-ordinate systems in general relativity.

Notations, equations of motion

We deal [1] with a tensor field $\psi_A(x^\nu)$ (Greek indices run from 0 to 3, capital Roman — from 1 to N; summation convention is used throughout;

Gothic letters denote tensor densities), whose equations follow from a variational principle $\delta W = 0$, where

$$(2)\qquad W = W_f + W_p = \int_R d_4x \Big(\mathfrak{L} + \int_{-\infty}^{\infty} \Lambda \delta_4(x-\xi)\, ds\Big), \quad W_f = \int_R d_4 x \mathfrak{L},$$

and

$$\mathfrak{L} = \sqrt{-g}\, L, \qquad L = L(g_{\mu\nu}, g_{\mu\nu,\sigma}, \psi_A, \psi_{A,\alpha}),$$

$$g = \det(g_{\alpha\beta}), \qquad \Lambda = \Lambda(\psi_A, \dot{\xi}^\alpha), \qquad \dot{\xi}^\alpha = d\xi^\alpha/ds,$$

$\xi^\alpha = \xi^\alpha(s)$ are the co-ordinates of a particle, $\delta_4(x)$ is the four-dimensional Dirac function *). L and Λ are assumed to be scalars depending on their arguments in the same way for all co-ordinate systems.

Varying W with respect to ψ_A we obtain the field equations

$$(3)\qquad \mathfrak{L}^A = \mathfrak{j}^A,$$

where

$$\mathfrak{L}^A = \sqrt{-g}\, L^A \underset{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \psi_A} - \left(\frac{\partial \mathfrak{L}}{\partial \psi_{A,\nu}}\right)_{,\nu}; \qquad \mathfrak{j}^A = \sqrt{-g}\, j^A \underset{\text{def}}{=} -\int_{-\infty}^{\infty} ds \frac{\partial \Lambda}{\partial \psi_A} \delta_4(x-\xi).$$

We define the symmetric tensor of matter as

$$\frac{1}{2}\sqrt{-g}\, T^{\alpha\beta} = \frac{1}{2}\mathfrak{T}^{\alpha\beta} \underset{\text{def}}{=} \sqrt{-g}\frac{\delta W}{\delta g_{\alpha\beta}} = \frac{\partial \mathfrak{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial \mathfrak{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu} + \frac{1}{2}\int_{-\infty}^{\infty}\Big(\Lambda - \frac{\partial \Lambda}{\partial \dot{\xi}^\nu}\dot{\xi}^\nu\Big)\dot{\xi}^\alpha \dot{\xi}^\beta \delta_4 ds,$$

$$\overset{f}{T}{}^{\alpha\beta} \underset{\text{def}}{=} 2\frac{\delta W_f}{\delta g_{\alpha\beta}}; \quad \overset{p}{T}{}^{\alpha\beta} = 2\frac{\delta W_p}{\delta g_{\alpha\beta}}, \quad T^{\alpha\beta} = \overset{f}{T}{}^{\alpha\beta} + \overset{p}{T}{}^{\alpha\beta}.$$

The equations of motion of particles can be obtained in several ways:

1⁰ by varying

$$W_p = \int_{\lambda_1}^{\lambda_2} \Lambda(\psi_A(\xi), \dot{\xi}^\alpha)\sqrt{g_{\mu\nu}\frac{d\xi^\mu}{d\lambda}\frac{d\xi^\nu}{d\lambda}}\, d\lambda$$

with respect to ξ;

2⁰ by a variational process with a subsidiary condition $\dot{\xi}^\alpha \dot{\xi}_\alpha = 1$ [3];

3⁰ from the integrability conditions of the gravitational field equations $T^{\alpha\beta}{}_{;\beta} = 0$.

All these methods lead to the same result, viz.:

$$(4)\qquad \Omega_\alpha \underset{\text{def}}{=} \frac{D}{ds}\Big[\Big(\Lambda - \frac{\partial \Lambda}{\partial \dot{\xi}^\nu}\dot{\xi}^\nu\Big)\dot{\xi}_\alpha + \frac{\partial \Lambda}{\partial \dot{\xi}^\alpha}\Big] - \frac{\partial \Lambda}{\partial \psi_A}\psi_{A;\alpha} = 0.$$

*) The generalisation for Lagrangians depending on higher derivatives of the potentials and for a system of particles presents no difficulties.

Let us take an infinitesimal co-ordinate transformation $x^\nu \to x^{\nu'} = x^\nu + \delta\zeta^\nu$, where $\delta\zeta^\nu(x^\alpha)$ is a vector field.

We define the substantial variation of ψ_A:

$$\delta^*\psi_A \underset{\text{def}}{=} \psi'_A(P') - \psi_A(P),$$

where P' is a point whose new co-ordinates are numerically equal to the old ones of P. We assume [4]

$$\delta^*\psi_A = -\psi_{A,\nu}\delta\zeta^\nu + F_{A\mu}{}^{B\nu}\psi_B\delta\zeta^\mu{}_{,\nu}. \tag{5}$$

Conservation laws for the system: field and particle

The point transformations generated by the solutions of Killing's equations

$$\delta^* g^{\alpha\beta} \equiv \delta\zeta^{\alpha;\beta} + \delta\zeta^{\beta;\alpha} = 0 \tag{6}$$

are called motions (isometries) of our V_4 [5].

THEOREM 1. *To each isometry in V_4 generated by $\delta\zeta^\alpha$ corresponds a conservation law*

$$\left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\delta^*\psi_A + \mathfrak{L}\,\delta\zeta^\nu\right)_{,\nu} + \int_{-\infty}^{\infty} ds\,\delta_4(x-\xi)\frac{d}{ds}\left\{\left[\left(\Lambda - \frac{\partial\Lambda}{\partial\dot\xi^\nu}\dot\xi^\nu\right)\dot\xi_\mu + \frac{\partial\Lambda}{\partial\dot\xi^\mu}\right]\delta\zeta^\mu\right\} = 0, \tag{7}$$

or in the "integral" form:

$$\int_\sigma \left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\delta^*\psi_A + \mathfrak{L}\,\delta\zeta^\nu\right)d\sigma_\nu + \left[\left(\Lambda - \frac{\partial\Lambda}{\partial\dot\xi^\nu}\dot\xi^\nu\right)\dot\xi_\mu + \frac{\partial\Lambda}{\partial\dot\xi^\mu}\right]\delta\zeta^\mu\bigg|_\sigma = \text{const.}, \tag{7'}$$

where σ is any space-like hypersurface.

Proof. $\mathfrak{L}$ being a scalar density, we have

$$\delta^*\mathfrak{L} + (\mathfrak{L}\,\delta\zeta^\nu)_{,\nu} \equiv 0. \tag{8}$$

Evaluating explicitly the left-hand side we obtain

$$\frac{1}{2}\overset{f}{\mathfrak{T}}{}^{\alpha\beta}\delta^* g_{\alpha\beta} + \mathfrak{L}^A\delta^*\psi_A + \left(\frac{\partial\mathfrak{L}}{\partial g_{\alpha\beta,\nu}}\delta^* g_{\alpha\beta} + \frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\delta^*\psi_A + \mathfrak{L}\,\delta\zeta^\nu\right)_{,\nu} \equiv 0. \tag{9}$$

Contracting the expression in square brackets in (4) with $\delta\zeta^\alpha$ and differentiating with respect to s we obtain

$$\frac{d}{ds}\left\{\left[\left(\Lambda - \frac{\partial\Lambda}{\partial\dot\xi^\nu}\dot\xi^\nu\right)\dot\xi_\mu + \frac{\partial\Lambda}{\partial\dot\xi^\mu}\right]\delta\zeta^\mu\right\} - \Omega_\nu\delta\zeta^\nu + \frac{\partial\Lambda}{\partial\psi_A}\delta^*\psi_A + \frac{1}{2}\left(\Lambda - \frac{\partial\Lambda}{\partial\dot\xi^\nu}\dot\xi^\nu\right)\dot\xi^\alpha\dot\xi^\beta\delta^* g_{\alpha\beta} \equiv 0. \tag{10}$$

Multiplying (10) by $\delta_4(x-\xi)$, integrating with respect to s and adding to (9), we obtain

$$\frac{1}{2}\mathfrak{T}^{\alpha\beta}\delta^* g_{\alpha\beta} + (\mathfrak{L}^A - \mathfrak{j}^A)\,\delta^*\psi_A + \left(\frac{\partial\mathfrak{L}}{\partial g_{\alpha\beta,\nu}}\delta^* g_{\alpha\beta} + \frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\delta^*\psi_A + \mathfrak{L}\delta\zeta^\nu\right)_{,\nu} + \int_{-\infty}^{\infty} ds\,\delta_4(x-\xi)\left[-\Omega_\nu\delta\zeta^\nu + \frac{d}{ds}\left\{\left[\left(\Lambda - \frac{\partial\Lambda}{\partial\dot\xi^\nu}\dot\xi^\nu\right)\dot\xi_\mu + \frac{\partial\Lambda}{\partial\dot\xi^\mu}\right]\delta\zeta^\mu\right\}\right] = 0. \tag{11}$$

The Eq. (7) follows from (11) if we put $\delta^* g_{\alpha\beta} = 0$, $\Omega_\alpha = 0$ and $L^A = j^A$. Integrating (7) between two space-like hypersurfaces we obtain (7′).

Our conservation laws (7) or (7′) fulfil all the requirements (a), (b), (c), but their number is limited by the symmetry properties of V_4. Indeed, *the number of such laws is equal to the number of independent isometries in the space-time continuum*. We have then 10 conservation laws only in spaces of constant curvature and no conserved quantities in wholly asymmetric V_4's.

The *identity* (11) is based on the only assumption that L and Λ are scalar functions; $\delta\zeta^\nu$ denotes in (11) an arbitrary vector field. Let us take a region R, and a field $\delta\zeta^\nu$ vanishing with first derivatives on the boundary of R. Then, integrating (11) over R, we obtain the following *identity* *):

$$\left(T_\alpha{}^\beta - (L^A - j^A)F_{A\alpha}{}^{B\beta}\psi_B\right)_{;\beta} - (L^A - j^A)\psi_{A;\alpha} - \int_{-\infty}^{\infty}\Omega_\alpha\delta_4(x-\xi)\,ds = 0. \tag{12}$$

This identity can facilitate the understanding of the logical structure of the EIH method of deriving the equations of motion. If we assume $L^A = j^A$ and take into account that the equations of gravity imply $T^{\alpha\beta}{}_{;\beta} = 0$, then (12) yields without further assumptions the equations of motion $\Omega_\alpha = 0$.

Other identities may be obtained from (9) and (10) following a method presented in [8].

Conservation laws for particles

The free-field case (no particles) leads to conservation laws in the form of an ordinary divergence [2]. Obviously, when particles interact with a field, there are in general no conserved quantities which could be associated with the sole particles. But the "energy" $mc^2/\sqrt{1-v^2/c^2} + eA_0$ is conserved for a (test) charge e of rest-mass m moving with the velocity v in an electrostatic field of potential A_0. We shall now give a ge-

*) It is evident that this identity is of the type investigated by E. Noether [6]. This paper is nothing but an application of her ideas.

neral prescription for constructing such quantities (first integrals of the equations of motion).

THEOREM 2. *If $\delta\zeta^\alpha$ is an infinitesimal motion and if the field ψ_A is invariant with respect to the field $\delta\zeta^\alpha$:*

$$\delta^*\psi_A = 0\,, \tag{13}$$

then

$$\left[\left(\Lambda - \frac{\partial \Lambda}{\partial \dot{\xi}^\nu}\dot{\xi}^\nu\right)\dot{\xi}_\mu + \frac{\partial \Lambda}{\partial \dot{\xi}^\mu}\right]\delta\zeta^\mu = \text{const.} \tag{14}$$

along the world line of the particle.

The proof follows directly from (10).

If we put $\Lambda = -mc - e/c\,A_\nu\dot{\xi}^\nu$ (electromagnetic interaction) and assume that the V_4 is flat and the A's are invariant with respect to $\delta\zeta^\nu \overset{*}{=} \delta_0^\nu$*) (i. e., with respect to translations along the time axis), we obtain the example mentioned above.

Let us write (14) for the case of a free gravitating particle ($\Lambda = -mc = \text{const.}$; the condition (13) holds automatically):

$$m\frac{d\xi^\nu}{ds}\delta\zeta_\nu = \text{const.} \tag{15}$$

The corresponding equations of motion are those of a geodesic line, $D\dot{\xi}^\alpha/ds = 0$. It is well known [5] in mathematics that, when $\delta\zeta^\alpha$ generates a motion, then (and only then) the expression $\dot{\xi}^\alpha\delta\zeta_\alpha$ is a first integral of the geodesic equation. Our Theorem 2 gives a generalisation and a physical interpretation of this statement. If we take a flat V_4 and the 10 corresponding motions $\delta\zeta^\alpha$, then (15) yields the 10 classical conserved quantities.

Conservation of energy

A space-time V_4 is called *stationary* if there exists an isometry with time-like trajectories. One can choose then the co-ordinate system ($\varkappa$) so that

$$\delta\zeta^\nu \overset{*}{=} -\delta_0^\nu. \tag{16}$$

Eq. (6) then implies $g_{\alpha\beta,0} \overset{*}{=} 0$. The corresponding conservation law for the free-field case resulting from (7) is

$$(\sqrt{-g}\,I^\nu)_{,\nu} = 0\,, \tag{17}$$

where

$$I^\nu = \frac{\partial L}{\partial \psi_{A,\nu}}\delta^*\psi_A + L\delta\zeta^\nu \tag{18}$$

*) The sign $\overset{*}{=}$ denotes an equality valid in a specified (used in the formula) co-ordinate system.

and can be interpreted as the conservation principle of energy. I^a is the *energy-impulse vector*. It is interesting that the time-like character of $\delta\zeta^\nu$ assures the same for I^a in the case of the electromagnetic and the meson fields; this can be verified by a straight forward computation.

For a free particle moving in a stationary V_4 we have the following constant of motion

$$m\dot{\xi}^\nu \delta\zeta_\nu \stackrel{*}{=} -m\, d\xi_0/ds = \text{const}.$$

For weak fields and slow motions this becomes

$$mc^2 + \tfrac{1}{2} mv^2 + m\varphi = \text{const.},$$

where

$$g_{00} = 1 + 2\varphi/c^2$$

and φ is the Newton potential.

We see thus that the conception of energy is closely connected with the stationary character of space-time. We do not see any reasonable way of defining the (conserved) energy in a world which does not possess this property.

On preferred co-ordinate systems in general relativity

We want to show that the "privileged" or "preferred" co-ordinate systems are also related to the symmetry properties of V_4.

It seems that one can accept the following definition: preferred are such co-ordinate systems in which the (basic) laws of nature are expressed in the simplest form.

Let us take an example. In a stationary V_4 the energy-impulse vector I^a given by (18) and the equation of continuity (17) have a meaning invariant with respect to general co-ordinate transformations. But if we restrict the allowable co-ordinates to those in which (16) holds, we can write I^a in a simplified form

$$I^\nu \stackrel{*}{=} \frac{\partial L}{\partial \psi_{A,\nu}} \psi_{A,0} - \delta_0^\nu L, \tag{19}$$

not involving the field $\delta\zeta^a$ *explicitly*. Further, the right-hand side of (19) is a vector with respect to transformations leaving $\delta\zeta^a$ *invariant in form*, i. e.,

$$\begin{aligned} x^{0'} &= x^0 + \varphi(x^k), \qquad k = 1,2,3. \\ x^{k'} &= x^{k'}(x^l), \qquad \det(x^{k'}{}_{,l}) \neq 0. \end{aligned} \tag{20}$$

These transformations form a subgroup of the group of general co-ordinate transformations. All the co-ordinate systems in which (16) holds can be considered as preferred in a stationary V_4. The transformation laws from a preferred co-ordinate system to another are given by (20).

In a flat space-time we define the Lorentz group as the group generated by 10 infinitesimal motions. The Lorentz transformations leave the metric invariant in form and have an invariant meaning.

According to this point of view one cannot consider as preferred the so-called "harmonic" (de Donder) co-ordinate systems [7]. In fact, they are not connected with any symmetry property of the space-time continuum. The "Lorentz transformations" considered by Fock (linear orthogonal transformations in curvilinear co-ordinates) cannot be defined on a physical (geometrical) ground.

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