Spinor structures on spheres and projective spaces

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An explicit construction of spinor structures on real, complex, and quaternionic projective spaces is given for all cases when they exist. The construction is based on a theorem describing the bundle of orthonormal frames of a homogeneous Riemannian manifold. This research is motivated by a remarkable coincidence of spinor connections on low-dimensional spheres with simple, topologically nontrivial gauge configurations.

I. INTRODUCTION

Spinors—and structures associated with them—are indispensable in physics and important in geometry. They have become an essential tool in theoretical physics of particles and nuclei; they are also useful in the study of gravitation.[1] A proper treatment of spinors on manifolds, with an account of their topology, is relatively recent.[2] It has led to the deep idea of spin cobordism,[3] to a study of harmonic spinors[4] and of the index theorem for the Dirac operator.[5]

In physics, spinors have recently acquired a new significance through the twistor program,[6] work on supersymmetry and unified theories based on higher-dimensional geometries of the Kaluza–Klein type. There are interesting ideas on the possible physical relevance of inequivalent spinor structures.[7] The Feynman method of quantization based on sums over classical histories applied to gravity coupled to fermions requires an analysis of nontrivial spinor configurations.

A somewhat unexpected link between spinors and another part of physics consists in the recognition of coincidences between spinor connections on low-dimensional spheres and simple, topologically nontrivial gauge configurations.[8] Indeed, any sphere S_n of dimension $n \geqslant 2$ has a unique spinor structure. The Levi-Civita connection corresponding to the standard Riemannian metric on S_n lifts to a spinor connection, which may be interpreted as a "gauge configuration" for the group $\mathrm{Spin}(n)$. This configuration is invariant under the action of $\mathrm{Spin}(n+1)$ and satisfies the Yang–Mills equations on S_n. For example, the cases $n = 2,3$, and 4 correspond to the Dirac magnetic pole of lowest strength, the meron solution, and the instanton-cum-anti-instanton system, respectively. Landi[9] has shown that the spinor connection on S_8 concides with a gauge configuration described recently by Grossman, Kephart, and Stasheff (GKS).[10] Rawnsley[11] generalized the duality properties of the instanton and of the GKS solution to the gauge field obtained from the spinor connection on any $4k$-dimensional sphere. The local, differential-geometric properties of the spinor gauge fields and Riemannian curvature tensors of spheres are the same, but their global properties are different; in particular, they have different values of "topological charge." For example, the Levi-Civita connection on S_2 corresponds to a magnetic pole of strength twice the lowest, Dirac value. The meron charge is related to the Chern–Simons[12] conformal invariants.

These considerations have led us to study spinor structures on projective spaces, which are, after spheres, the simplest homogeneous manifolds. The natural spinor connections on these spaces also may be interpreted as simple gauge configurations, but we postpone their description to another work. In this paper, we restrict ourselves to the construction of the spinor structures themselves.

In order to find the spinor structures on a Riemannian manifold it is convenient to know its bundle of orthonormal frames. For a "generic" manifold without isometries there is not much one can say about this bundle: it is, for example, a parallelizable manifold, but this does not help much in constructing spinor structures. If, however, the manifold is homogeneous, i.e., admits a transitive Lie group G of isometries, then its bundle of frames can be explicitly described in terms of G and its subgroups. Moreover, the bundle of orthonormal frames can be restricted to a subgroup of the full orthogonal group. Such a restriction is convenient because it allows one to work with a bundle of lower dimension than that of the bundle of all orthonormal frames.

If a Riemannian n-manifold M is orientable, then its bundle of frames can be restricted to $\mathrm{SO}(n)$. This group admits a unique, nontrivial (for $n > 1$) double covering by the spin group, $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. A spin structure on M is a "prolongation" of the bundle of frames that "agrees" with this covering. (The precise definition is recalled in Sec. II.) It exists if, and only if, the second Stiefel–Whitney class of M is zero. In the nonorientable case the situation is somewhat more complicated (Whiston[13]). The full orthogonal group $\mathrm{O}(n)$ has, in general, several inequivalent double coverings. For example, for $n = 1$, $\mathrm{Spin}(1) = \mathbb{Z}_2$ and $\mathrm{SO}(1) = 1$, but the orthogonal group $\mathrm{O}(1) = \mathbb{Z}_2$ has two different coverings:

$$\rho_+ : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \quad \text{and} \quad \rho_- : \mathbb{Z}_4 \to \mathbb{Z}_2 .$$

In any dimension n, two such coverings ρ_+ and ρ_- can be obtained from Clifford algebras of $\mathbb{R}^n$ equipped, respectively, with quadratic forms ϕ and $-\phi$, where

$$\phi(x) = x_1^2 + \cdots + x_n^2 .$$

The topological obstructions to the existence of prolongations of the bundle of frames associated with ρ_+ and ρ_- are different from each other. We use the term "pin$^\pm$ structure"

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for such a prolongation corresponding to $\rho^{\pm}$. In Sec. III we show that a real projective space of dimension $2k$ admits two inequivalent pin$^+$ or pin$^-$ structures depending on whether k is even or odd. We also give an explicit description of the spin structures on odd-dimensional complex projective spaces in terms of metaunitary groups.

II. PRELIMINARIES: SPINOR GROUPS AND STRUCTURES

This chapter contains a brief summary of the basic definitions and results related to spinor groups and structures that are needed in the sequel. We follow rather closely the articles by Atiyah, Bott, and Shapiro,[14] Atiyah and Bott,[15] and Karoubi,[16] but we adapt the notation and terminology to our needs.[17]

Let (e_i), $i = 1,\dots,n$, be the standard orthonormal frame in $\mathbb{R}^n$. We denote by $C^+(n)$ and $C^-(n)$ the two related Clifford algebras generated by the e's subject to the relations

$$e_i e_j + e_j e_i = +2\delta_{ij} \quad \text{and} \quad -2\delta_{ij} \quad (i,j = 1,\dots,n),$$

respectively. In any of the Clifford algebras we have the main involution α and the main anti-involution β. The groups $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ consist of all invertible elements of $C^+(n)$ and $C^-(n)$, respectively, which preserve the underlying vector space $\mathbb{R}^n$ under the twisted adjoint representation $\rho_{\pm}$,

$$\rho_{\pm}(s)v = \alpha(s)vs^{-1}, \quad \text{where} \quad v \in \mathbb{R}^n, \quad s \in C^{\pm}(n),$$

and are normalized by $|\beta(s)s| = 1$. [The last condition is meaningful because the previous ones imply $\beta(s)s \in \mathbb{R}$.] In general, the groups $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ are nonisomorphic. The connected components of the identity of these groups consist of even elements and are isomorphic to each other; they are both denoted by $\mathrm{Spin}(n)$. The sequences

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}^{\pm}(n) \xrightarrow{\rho_{\pm}} \mathrm{O}(n) \to 1$$

and

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \to 1, \quad \text{where} \quad \rho = \rho_{\pm} \,|\, \mathrm{Spin}(n),$$

are exact. We use the generic term "spinor group" to denote one of the groups $\mathrm{Spin}(n)$, $\mathrm{Pin}^+(n)$, or $\mathrm{Pin}^-(n)$, $n = 1,2,\dots$. The centers of these groups are as shown in Table I. Here $\mathbb{Z}_2 = \{1, -1\}$, $\mathbb{Z}_2^+ = \{1, \epsilon\}$, $\mathbb{Z}_2^- = \{1, -\epsilon\}$, $\mathbb{Z}_4 = \{1, \epsilon, -1, -\epsilon\}$, and $\epsilon = e_1 e_2 \cdots e_n$ is the "volume element." The products occurring in Table I are direct. Note also that if $\epsilon \in \mathrm{Pin}^{\pm}(n)$, then

$$\epsilon^2 = (\pm 1)^n (-1)^{n(n-1)/2}.$$

TABLE I. Centers of spinor groups.

n	$\mathrm{Spin}(n)$	$\mathrm{Pin}^+(n)$	$\mathrm{Pin}^-(n)$
$4k$	$\mathbb{Z}_2^+ \times \mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$4k+1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^+ \times \mathbb{Z}_2^-$	$\mathbb{Z}_4$
$4k+2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$4k+3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2^+ \times \mathbb{Z}_2^-$

The existence of spinor structures on projective spaces depends crucially on the structure of the center of an appropriate spinor group. It is convenient to define

$$\mathrm{Pin}(n) = \begin{cases} \mathrm{Pin}^+(n), & \text{for } n \equiv 0,1 \bmod 4, \\ \mathrm{Pin}^-(n), & \text{for } n \equiv 2,3 \bmod 4, \end{cases}$$

and make a corresponding notational convention for the covering map ρ. If Σ is one of the groups $\mathrm{Spin}(4k)$ or $\mathrm{Pin}(2k+1)$, $k = 1,2,\dots$, then $[s]_{\pm}$ denotes the coset $s\mathbb{Z}_2^{\pm} \in \Sigma/\mathbb{Z}_2^{\pm}$, i.e., if $s,t \in \Sigma$, then

$$[s]_{\pm} = [t]_{\pm} \quad \text{iff } s = t \text{ or } s = \pm t\epsilon.$$

We inject $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by sending $(x_1,\dots,x_n)$ into $(x_1,\dots,x_n,0)$ and extend this to injections of the corresponding Clifford algebras and spinor groups.

In order to adapt to our purposes the classical definition of a spinor structure on the tangent bundle of Riemannian manifold M (see Haefliger and Milnor in Ref. 2), consider the following. Let M be n dimensional with a positive-definite metric tensor g. Let Ω be a closed subgroup of $\mathrm{O}(n)$ and $\Sigma = \rho_{\pm}^{-1}(\Omega) \subset \mathrm{Pin}^{\pm}(n)$. Assume that $F \subset F_g$ is a restriction to Ω of the bundle F_g of all linear frames on M that are orthonormal with respect to g. A spinor structure on M is defined by giving a prolongation of $\pi: F \to M$ to the group Σ, i.e., principal Σ-bundle σ: $S \to M$ and a morphism of bundles $\eta: S \to F$ such that there is a commutative diagram

$$\begin{array}{ccccc} S \times \Sigma & \longrightarrow & S & \overset{\sigma}{\searrow} & \\ \eta \times \rho_{\pm} \big\downarrow & & \big\downarrow \eta & & M, \\ F \times \Omega & \longrightarrow & F & \underset{\pi}{\nearrow} & \end{array}$$

where the horizontal arrows denote the action maps. If $\Omega \subset \mathrm{SO}(n)$, then M is orientable and one has a spin structure. If M is nonorientable, then F_g is connected, but Ω is not, and one has a pin$^{\pm}$ structure. The two structures pin$^+$ and pin$^-$ corresponding to the two covering maps ρ^+ and ρ^- are, in general, inequivalent. In some cases one exists whereas the other does not as may be seen from the topological conditions for their existence[16]: $w_2 = 0$ for pin$^+$, $w_1^2 + w_2 = 0$ for pin$^-$, and $w_1 = 0$, $w_2 = 0$ for spin. (Here w_1 and w_2 denote, respectively, the first and second Stiefel–Whitney classes of the tangent bundle of M.) We sometimes say "pin structure" when we mean one of the two and we use the generic term "spinor structure" to denote a pin or spin structure.

It is clear that, given the bundles described above, one can always extend their structure groups Ω and Σ to $\mathrm{O}(n)$ and $\mathrm{Pin}^{\pm}(n)$, respectively. The extended bundles provide a classical description of pin structure. Conversely, given such a classical pin structure on M, say

$$\mathrm{Pin}^{\pm}(n) \to P \xrightarrow{f} F_g \to M,$$

and a restriction F of F_g to $\Omega \subset \mathrm{O}(n)$, one can restrict the structure group $\mathrm{Pin}^{\pm}(n)$ of P to Σ by taking the induced bundle $S = f^{-1}(F)$. Similar remarks apply to spin structures.

The classical definition of equivalence of spinor structures can be easily adapted to our considerations. Let, for simplicity, η_a: $S_a \to F$ $(a = 1,2)$ be two spin structures, where each S_a is the total space of a principal Σ-bundle over M. They are equivalent if there is a based isomorphism

i: $S_1 \to S_2$ of principal Σ-bundles such that $\eta_2 \circ i = \eta_1$. The bundles $\pi \circ \eta_a$: $S_a \to M$ may be isomorphic, as principal Σ-bundles, without defining equivalent spin structures. The equivalence classes of isomorphic spinor structures are in a bijective correspondence with the elements of $H^1(M,\mathbb{Z}_2)$, the first cohomology group of M with coefficients in $\mathbb{Z}_2$ (see Milnor[2] and Whiston[13]).

III. FRAME BUNDLES OF HOMOGENEOUS SPACES

We restrict our study to very regular situations: all manifolds and maps are *smooth*, Lie groups and other spaces are *compact*, and subgroups are *closed*. All Riemannian manifolds are *proper*, i.e., their metric tensors are positive definite. The italicized adjectives will be omitted from now on.

Let M be a manifold admitting a transitive left action γ: $G \times M \to M$ of a Lie group G. Denoting $\gamma_a(x) = \gamma(a,x)$ one has $\gamma_a \circ \gamma_b = \gamma_{ab}$, for any $a,b \in G$, and $\gamma_1 = \text{id}$, where 1 is the unit of G. Let $H = \{a \in G: \gamma_a(x) = x\}$ be the *stability* ("little") group at $x \in M$. The homogeneous space M is canonically diffeomorphic to the quotient space G/H. The diffeomorphism h: $G/H \to M$, mapping the coset bH, $b \in G$, into $\gamma_b(x)$, intertwines the actions of G in G/H and M, $h \circ \gamma_a = \gamma_a \circ h$ for all $a \in G$ (cf., for example, Bredon[18]). We shall often identify G/H with M and, by doing so, make h disappear.

Let γ'_a denote the tangent map to γ_a at x. For any $a \in H$, this map is a linear automorphism of the tangent space $T_x M$ to M at x, and

$$\gamma': H \to \text{GL}(T_x M)$$

is a homomorphism of groups. Its kernel N is a normal subgroup of H and, therefore, also a subgroup—but not normal, in general—of G. According to the general theory of fiber bundles (Steenrod[19]) these data define a principal H/N bundle

$$F = G/N \to G/H = M, \tag{1}$$

where the action of H/N in F is given by $(aN)(bN) = abN$, $a \in G$ and $b \in H$.

Let now M be an n-dimensional Riemannian manifold with a metric tensor g admitting a group G of isometries acting transitively on M. The preceding construction leads to the following theorem.

Theorem: The bundle π: $F \to M$, defined by (1), is a restriction to the group H/N of the bundle π_g: $F_g \to M$ of all linear frames on M, orthonormal with respect to the metric tensor g.

To prove the theorem, it suffices to give an injective immersion i: $F \to F_g$ and a monomorphism of Lie groups j: $H/N \to \text{O}(n)$ such that

$$i((aN)(bN)) = i(aN)\, j(bN) \quad \text{and} \quad \pi_g \circ i = \pi. \tag{2}$$

Recall that an orthonormal frame in an n-dimensional vector space V may be identified with an isometry from $\mathbb{R}^n$ to V. Let f be a frame at x, orthonormal with respect to g. This frame is unchanged by γ'_a, $a \in H$, if and only if $a \in N$. For any $a \in G$, the composition $\gamma'_a \circ f$ is a frame at $\gamma_a(x)$, also orthonormal with respect to g. The maps i and j are now defined by

$$i(aN) = \gamma'_a \circ f, \quad a \in G,$$

and

$$j(bN) = f^{-1} \circ \gamma'_b \circ f, \quad b \in H,$$

where

$$f^{-1}: T_x M \to \mathbb{R}^n$$

is the inverse, or "dual," frame with respect to f. The morphism properties (2) are now easy to verify.

Example 1: The $(n-1)$-dimensional sphere S_{n-1} with its standard Riemannian metric admits $\text{SO}(n)$ as a group of isometries. The action γ of $\text{SO}(n)$ on S_{n-1} is transitive and the stability group of any point is isomorphic to $\text{SO}(n-1)$, whereas $N = \ker \gamma'$ reduces to the identity. The $\text{SO}(n-1)$-bundle,

$$\text{SO}(n) \to \text{SO}(n)/\text{SO}(n-1) = S_{n-1},$$

is simply the bundle of orthonormal frames of S_{n-1} with coherent orientation. For n even, $n = 2k$, the group $\text{SO}(2k)$ contains a subgroup $\text{U}(k)$, which also acts transitively on S_{2k-1}. The stability group is $\text{U}(k-1)$ and

$$\text{U}(k) \to \text{U}(k)/\text{U}(k-1) = S_{2k-1}$$

is the bundle of "unitary frames." Similarly, for $n = 4l$, there is the bundle of "symplectic frames"

$$\text{SP}(l) \to \text{Sp}(l)/\text{Sp}(l-1) = S_{4l-1}.$$

Example 2: Let K denote one of the three number fields: $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The set K^{n+1} is a right module (a vector space if $K = \mathbb{R}$ or $\mathbb{C}$) over K: if $q = (q_\alpha) \in K^{n+1}$, $\alpha = 1,\dots,n+1$, and $\lambda \in K$, then

$$q\lambda = (q_\alpha \lambda) \in K^{n+1},$$

so that

$$(q\lambda)\mu = q(\lambda\mu), \quad q(\lambda + \mu) = q\lambda + q\mu, \text{ etc.},$$

for any $\lambda, \mu \in K$. If $q \in K^{n+1}$ and $q \neq 0$, then the direction of q is the set

$$\text{dir}\, q = \{q\lambda: 0 \neq \lambda \in K\}$$

and the set of all such directions is the projective n-dimensional space over K,

$$KP_n = \{\text{dir}\, q: 0 \neq q \in K^{n+1}\}.$$

The module K^n has a natural, positive-definite quadratic form ϕ given by

$$\phi(q) = \sum_{\alpha=1}^{n} \bar{q}_\alpha q_\alpha,$$

where $\bar{\lambda} = \lambda$ for $K = \mathbb{R}$ and $\bar{\lambda}$ is the conjugate of λ otherwise. Let $\text{U}(n,K)$ be the set of all maps a: $K^n \to K^n$ such that $\phi \circ a = \phi$, $a(q\lambda) = a(q)\lambda$, and $a(q+q') = a(q) + a(q')$ for any $\lambda \in K$ and $q,q' \in K^n$. With respect to composition of maps this set is a group, namely

$$\text{U}(n,K) = \begin{cases} \text{O}(n), \\ \text{U}(n), \\ \text{Sp}(n), \end{cases} \quad \text{for } K = \begin{cases} \mathbb{R}, \\ \mathbb{C}, \\ \mathbb{H}. \end{cases}$$

The action of $\text{U}(n+1,K)$ in KP_n given by

$$\gamma_a(\text{dir}\, q) = \text{dir}\, a(q)$$

is transitive. Let e_α denote the element of K^{n+1} consisting of 1 at the αth place and zeros elsewhere so that

$$q=\sum_{\alpha=1}^{n+1} e_\alpha q_\alpha .$$

The stability group H of $x=\operatorname{dir} e_{n+1}\in KP_n$ may be computed as follows; let

$$a(e_\beta)=\sum_\alpha e_\alpha a_{\alpha\beta} ,$$

where $a_{\alpha\beta}\in K$ and $\alpha,\beta=1,...,n+1$. The condition $\operatorname{dir} a(e_{n+1})=\operatorname{dir} e_{n+1}$ implies $a_{\alpha,n+1}=0$ for $\alpha=1,...,n$. Since $\phi\circ a=\phi$ is equivalent to

$$\sum_\gamma \bar{a}_{\gamma\alpha} a_{\gamma\beta}=\delta_{\alpha\beta} ,$$

we obtain also $a_{n+1,\alpha}=0$, for $\alpha=1,...,n$, so that H is isomorphic to $U(1,K)\times U(n,K)$. The isomorphism is realized as follows: if $\lambda\in U(1,K)$ and $b\in U(n,K)$, then the corresponding element of H is represented by the matrix

$$a=\begin{pmatrix} b & 0\\ 0 & \lambda\end{pmatrix}. \tag{3}$$

Let y: $\mathbb{R}\to KP_n$ be a curve through x, $y(0)=x$. For sufficiently small $|t|$ one can write

$$y(t)=\operatorname{dir}(e_{n+1}+q(t)),$$

where

$$q(t)=\sum_{\alpha=1}^{n} e_\alpha q_\alpha(t) , \quad q(0)=0 ,$$

is a curve in K^n. The tangent vector to y at $t=0$ is represented by

$$\dot{q}(0)=v=\sum_{\alpha=1}^{n} e_\alpha v_\alpha .$$

If $a\in H$ is as in (3), then the tangent vector to the curve

$$\begin{aligned} t&\to\gamma_a(y(t))\\ &=\operatorname{dir} a(e_{n+1}+q(t))=\operatorname{dir}(e_{n+1}\lambda+bq(t))\\ &=\operatorname{dir}(e_{n+1}+bq(t)\lambda^{-1})\end{aligned}$$

is represented by

$$\gamma'_a(v)=bv\lambda^{-1}\in K^n .$$

Therefore, the kernel N of γ' consists of all matrices of the form (3) such that $bv=v\lambda$ for any $v\in K^n$. It follows that b is λ times the unit automorphism of K^n and λ belongs to the center of K. The group N is thus isomorphic with the center of $U(n+1,K)$,

$$N=\begin{cases}\mathbb{Z}_2 ,\\ U(1) ,\\ \mathbb{Z}_2 ,\end{cases} \text{for } K=\begin{cases}\mathbb{R} ,\\ \mathbb{C} ,\\ \mathbb{H} .\end{cases}$$

Taking into account

$$U(1,K)=\begin{cases}\mathbb{Z}_2 ,\\ U(1) ,\\ Sp(1) ,\end{cases} \text{for } K=\begin{cases}\mathbb{R} ,\\ \mathbb{C} ,\\ \mathbb{H} ,\end{cases}$$

we can compute the structure groups

$$H/N=\begin{cases}O(n) ,\\ U(n) ,\\ (Sp(n)\times Sp(1))/\mathbb{Z}_2 ,\end{cases} \text{for } \begin{cases}\mathbb{R}P_n ,\\ \mathbb{C}P_n ,\\ \mathbb{H}P_n ,\end{cases}$$

and the reduced bundles of orthonormal frames

$$G/N=\begin{cases}O(n+1)/\mathbb{Z}_2 ,\\ U(n+1)/U(1) ,\\ Sp(n+1)/\mathbb{Z}_2\end{cases} \text{for } \begin{cases}\mathbb{R}P_n ,\\ \mathbb{C}P_n ,\\ \mathbb{H}P_n .\end{cases}$$

For n even, $n=2k$, the quotient $O(2k+1)/\mathbb{Z}_2$ can be identified with $SO(2k+1)$: the bundle of all orthonormal frames of $\mathbb{R}P_{2k}$ is connected, i.e., $\mathbb{R}P_{2k}$ is not orientable. Note that the quotient $O(2k+1)/\mathbb{Z}_2$ may also be represented as $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^+\times\mathbb{Z}_2^-$. For n odd, $n=2k-1$, the quotient $O(2k)/\mathbb{Z}_2$ is the disjoint sum of two copies of $SO(2k)/\mathbb{Z}_2$. Therefore, $\mathbb{R}P_{2k-1}$ is orientable and its bundle of orthonormal frames of coherent orientation is diffeomorphic to $SO(2k)/\mathbb{Z}_2$. The bundle $U(n+1)/U(1)$ is diffeomorphic to the quotient $SU(n+1)/Z_{n+1}$ of the group $SU(n+1)$ by its center.

Example 3: Consider a Lie group G with a bi-invariant metric; e.g., if G is semisimple, then such a metric is obtained from the Killing form. In this case, the manifold of G is a homogeneous Riemannian space with respect to the action of $G\times G$ given by

$$\gamma_{(a,b)}(c)=acb^{-1} ,$$

for any $a,b,c\in G$. The stability group at the unit element of G is isomorphic to G embedded diagonally in $G\times G$. For any $a\in G$, the map $\gamma'_{(a,a)}$ coincides with Ad_a, where

$$\mathrm{Ad}: G\to GL(\mathfrak{g})$$

is the adjoint representation of G in its Lie algebra $\mathfrak{g}=T_1G$. The kernel of Ad is the center $Z(G)$ of G and the reduced bundle of orthonormal frames is a $G/Z(G)$-bundle

$$(G\times G)/Z(G)\to G . \tag{4}$$

Note that, unless G is Abelian, the total space of the bundle (4) is "larger" than that obtained by considering G as a homogeneous space with respect to left translations.

IV. SPINOR STRUCTURES ON SPHERES AND PROJECTIVE SPACES

In this section we use our description of the restricted bundle of orthonormal frames to construct spinor structures on spheres and on the projective spaces: $\mathbb{R}P_n$ ($n=1$ or $n>1$ and $\neq 1 \bmod 4$), $\mathbb{C}P_{2k-1}$ ($k=1,2,...$), and $\mathbb{H}P_n$ ($n=1,2,...$). The case of spheres is easy and well known. For an orientable projective space over K, the crucial information is contained in the structure of the center $Z(n,K)$ of the group $\Sigma(n,K)=\rho^{-1}$ $(U(n,K)\cap SO(m))$, where ρ: $\mathrm{Spin}(m)\to SO(m)$ is the covering map and $m=n \dim_{\mathbb{R}} K$. If the center is a direct product of the form $\mathbb{Z}_2\times\Lambda(n,K)$, then there is a spin structure on KP_{n-1} given by the sequence

$$\Sigma(n,K)/\Lambda(n,K)\to\Sigma(n,K)/Z(n,K)\to KP_{n-1} . \tag{5}$$

A similar statement applies to the nonorientable space $\mathbb{R}P_{2k}$; here the relevant groups are $\mathrm{Pin}(2k+1)$ and its center. Each real projective space other than $\mathbb{R}P_{4l+1}$ ($l=1,2,...$) has two pin or spin structures. We construct them both and show that they are inequivalent.

A. Spheres

The circle S_1 has two inequivalent spin structures (Milnor[2]). Since both S_1 and $SO(2)$ can be identified with $U(1)$, and $\mathrm{Spin}(1) = \mathbb{Z}_2$, these structures are given by the maps

$$\mathrm{U}(1)\times\mathbb{Z}_2 \underset{\mathrm{pr}_1}{\to} \mathrm{U}(1) \underset{\mathrm{id}}{\to} \mathrm{U}(1)$$

and

$$\mathrm{U}(1) \underset{\mathrm{square}}{\to} \mathrm{U}(1) \underset{\mathrm{id}}{\to} \mathrm{U}(1)\,.$$

For any $n\geqslant 2$, there is a unique spinor structure given by

$$\mathrm{Spin}(n+1)\to \mathrm{SO}(n+1)\to S_n\,.$$

For $n = 4l-1$ ($l = 1,2,...$) one can restrict the bundle of frames to $\Omega = \mathrm{Sp}(l-1)$ and the spinor bundle to $\Sigma = \mathrm{Sp}(l-1)\times\mathbb{Z}_2$. The (restricted) spinor structure is

$$\mathrm{Sp}(l)\times\mathbb{Z}_2\to\mathrm{Sp}(l)\to S_{4l-1}\,.$$

There is an analogous restriction of the spinor bundle of S_{2k-1} to the metaunitary group $\mathrm{MU}(k)\subset\mathrm{Spin}(2k)$, cf. Sec. IV C.

B. Real projective spaces

(i) Consider first the case of odd dimension. The one-dimensional real projective space is diffeomorphic to the circle S_1; its spin structures have already been given. Let now the dimension $n = 2k-1$ be greater than 1. The space $\mathbb{R}P_{2k-1}$ is orientable and the fundamental group Π_1 of its bundle of frames $\mathrm{SO}(2k)/\mathbb{Z}_2$ may be computed by considering three curves in $\mathrm{Spin}(2k)$ joining 1 to ϵ, $-\epsilon$, and -1, respectively. Each of these curves projects to a loop in $\mathrm{SO}(2k)/\mathbb{Z}_2$ and defines a nontrival element of Π_1. No two of these elements coincide and, since $\epsilon^2 = (-1)^k$, one has $\Pi_1 = \mathbb{Z}_4$ for k odd and $\Pi_1 = \mathbb{Z}_2\times\mathbb{Z}_2$ for k even. The group $\mathrm{SO}(2k-1)$ is the fiber of

$$\mathrm{SO}(2k)/\mathbb{Z}_2\to\mathbb{R}P_{2k-1}$$

and its fundamental group $\mathbb{Z}_2$ is embedded in Π_1 as follows:

if k is odd, $k\geqslant 3$,

then $\mathbb{Z}_2\to\mathbb{Z}_4$ is given by 1 mod 2$\mapsto$2 mod 4;

if k is even,

then $\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$ is the diagonal map.

To check this for odd k one can consider the projection to $\mathrm{SO}(2k)/\mathbb{Z}_2$ of the curve in $\mathrm{Spin}(2k)$ joining 1 and -1. This projection is the square of the loop obtained by projecting the curve joining 1 and ϵ. The square is represented by 2 mod 4 in $\mathbb{Z}_4$ and, since it is noncontractible, it is homotopic to a nontrivial loop in $\mathrm{SO}(2k-1)$. It is now clear that $\mathbb{R}P_{4l+1}$ ($l = 1,2,...$) has no spinor structure: a noncontractible loop in a fiber of its bundle of frames ("rotation by 360°") can be continuously deformed into the square of a loop in the bundle (Clarke[20]). This result is, of course, well known: $\mathbb{R}P_{4l+1}$ has $w_1 = 0$ and $w_2\neq 0$ for $l = 1,2,...$.

The space $\mathbb{R}P_{4l-1}$ has $w_1 = 0$, $w_2 = 0$, and $\pi_1 = \mathbb{Z}_2$. There are, therefore, two inequivalent spin structures on $\mathbb{R}P_{4l-1}$ ($l = 1,2,...$). They are

$$\pi^\pm:\ \mathrm{Spin}(4l)/\mathbb{Z}_2^\pm\to\mathrm{SO}(4l)/\mathbb{Z}_2\,,$$

where the $\pi^\pm$ are obvious projections and the action of $\mathrm{Spin}(4l-1)$ in $\mathrm{Spin}(4l)/\mathbb{Z}_2^\pm$ is obtained from the natural action of $\mathrm{Spin}(4l-1)$ in $\mathrm{Spin}(4l)$ by passing to the quotient. To see that π^+ and π^- define inequivalent spin structures consider a curve in $\mathrm{Spin}(4l)$ connecting 1 with ϵ. Its projection to $\mathrm{SO}(4l)/\mathbb{Z}_2$ is a loop. There are exactly two lifts of this loop to $\mathrm{Spin}(4l)/\mathbb{Z}_2^+$ and they are both closed curves (loops). There are also exactly two lifts of this loop to $\mathrm{Spin}(4l)/\mathbb{Z}_2^-$ and neither of them is closed. This contradicts the existence of a bundle isomorphism h: $\mathrm{Spin}(4l)/\mathbb{Z}_2^+\to\mathrm{Spin}(4l)/\mathbb{Z}_2^-$ such that $\pi^-\circ h = \pi^+$.

There is, however, an orientation–reversing isometry

$$j:\ \mathbb{R}P_{4l-1}\to\mathbb{R}P_{4l-1}\,,$$

$$\mathrm{dir}(\rho(a)e_{4l})\mapsto\mathrm{dir}(\rho(e_{4l}a)e_{4l}),$$

which lifts to an isomorphism of one spin structure onto the other, given by $[a]_+\mapsto[e_{4l}ae_{4l}]_-$.

(ii) The even-dimensional real projective spaces are nonorientable; they will be shown to admit pin structures. For any $k = 1,2,...$, the space $\mathbb{R}P_{2k}$ admits two inequivalent pin structures. Depending on whether k is even or odd, one has to consider the covering map $\mathrm{Pin}(2k)\to\mathrm{O}(2k)$ corresponding to a pin group associated with an Euclidean space $\mathbb{R}^{2k}$ with a quadratic form that is positive or negative, respectively (cf. Sec. II).

The pin structures on $\mathbb{R}P_{2k}$ are

$$\pi^\pm:\ \mathrm{Pin}(2k+1)/\mathbb{Z}_2^\pm \to\mathrm{Pin}(2k+1)/\mathbb{Z}_2^+\times\mathbb{Z}_2^- = \mathrm{SO}(2k+1),$$

where the projections $\pi^\pm$ are obvious and the action δ of $\mathrm{Pin}(2k)$ in $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^\pm$ comes from the natural embedding $\mathrm{Pin}(2k)\to\mathrm{Pin}(2k+1)$ by passing to the quotient, i.e.,

$$\delta_b([a]_\pm) = [ab]_\pm\,,$$

for any $a\in\mathrm{Pin}(2k+1)$ and $b\in\mathrm{Pin}(2k)$. The inequivalence of π^+ and π^- may be seen as follows. Consider a curve in $\mathrm{Pin}(2k+1)$ beginning at 1 and ending at $e_1\epsilon$. Its projection to $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^+\times\mathbb{Z}_2^-$ has the property that its end is obtained by applying $\rho(e_1)\in\mathrm{O}(2k)$ to its beginning. There are again exactly two lifts of this curve to each $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^+$ and $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^-$. The starting and end points of the curves in $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^\pm$ are related to each other by the action of $\pm e_1$, respectively. This difference in sign implies that there is no isomorphism of bundles h such that $\pi^-\circ h = \pi^+$.

The total spaces $\mathrm{Pin}(2k+1)/\mathbb{Z}_2^\pm$ are both diffeomorphic to $\mathrm{Spin}(2k+1)$. More precisely, let

$$\sigma:\ \mathrm{Spin}(2k+1)\to\mathbb{R}P_{2k} \qquad (6a)$$

be the projection $a\mapsto\mathrm{dir}\,\rho(a)e_{2k+1}$ and

$$\delta^\pm:\ \mathrm{Spin}(2k+1)\times\mathrm{Pin}(2k)\to\mathrm{Spin}(2k+1) \qquad (6b)$$

be right actions defined by

$$\delta_b^\pm(a) = \delta^\pm(a,b) = \begin{cases} ab\,, & \text{if } b \text{ is even}\,,\\ \pm\epsilon ab\,, & \text{if } b \text{ is odd}\,.\end{cases}$$

The two maps $h^\pm:\mathrm{Pin}(2k+1)/\mathbb{Z}_2^\pm\to\mathrm{Spin}(2k+1)$ given by

$$[a]_\pm\mapsto\begin{cases} a\,, & \text{if } a \text{ is even}\,,\\ \pm\epsilon a\,, & \text{if } a \text{ is odd}\,,\end{cases}$$

define, respectively, isomorphisms of the two principal bundles (6) with the bundles

$$\sigma^{\pm}: \mathrm{Pin}(2k+1)/\mathbf{Z}_2^{\pm} \to \mathbf{R}P_{2k}, \tag{7}$$

where $\sigma^{\pm} = \pi \circ \pi^{\pm}$ and

$$\pi: \mathrm{SO}(2k+1) \to \mathbf{R}P_{2k}.$$

We have indeed

$$\sigma \circ h^{\pm} = \sigma^{\pm} \quad \text{and} \quad h^{\pm} \circ \delta_b = \delta_b^{\pm} \circ h^{\pm},$$

for any $b \in \mathrm{Pin}(2k)$.

Even though the two spin structures on $\mathbf{R}P_{2k}$ are inequivalent, the two bundles (6) are *isomorphic* to each other when considered as principal bundles over $\mathbf{R}P_{2k}$. Indeed, a based isomorphism

$$i: \mathrm{Pin}(2k+1)/\mathbf{Z}_2^{+} \to \mathrm{Pin}(2k+1)/\mathbf{Z}_2^{-}$$

is given by

$$[a]_{+} \mapsto [\alpha(a) e_{2k+1}]_{-}.$$

C. Complex projective spaces

It is well known that even-dimensional complex projective spaces have no spinor structure. In order to understand the difference between even and odd dimensions and to construct the spin structure in the latter case, it is convenient to consider the metaunitary group $\mathrm{MU}(n)$ (see Rf. 21) and find its center. This group may be defined as that subgroup of $\mathrm{Spin}(2n)$ that (doubly) covers the unitary group $\mathrm{U}(n)$ considered as a subgroup of $\mathrm{SO}(2n)$:

$$\begin{array}{ccc} \mathrm{MU}(n) & \xrightarrow{\text{inj}} & \mathrm{Spin}\ (2n) \\ \rho \downarrow & & \downarrow \rho\,. \\ \mathrm{U}(n) & \xrightarrow{\text{inj}} & \mathrm{SO}\ (2n) \end{array}$$

Let $(e_1,\ldots,e_{2n})$ be an orthonormal frame in $\mathbf{R}^{2n}$ embedded in the Clifford algebra $C^{+}(2n)$. Let $J \in \mathrm{SO}(2n)$, given by

$$J(e_\alpha) = \begin{cases} -e_{n+\alpha}, & \text{for } \alpha = 1,\ldots,n, \\ e_{\alpha-n}, & \text{for } \alpha = n+1,\ldots,2n, \end{cases}$$

define a complex structure in $\mathbf{R}^{2n}$ so that $\mathrm{U}(n) = \{a \in \mathrm{SO}(2n): J \circ a = a \circ J\}$. The center of $\mathrm{U}(n)$ is isomorphic to $\mathrm{U}(1)$ and consists of all elements of $\mathrm{SO}(2n)$ of the form $\cos 2t + J \sin 2t = \exp 2tJ$, $0 \leqslant t < \pi$. Let

$$\iota = e_1 e_{n+1} + \cdots + e_n e_{2n} \in \mathrm{spin}(2n),$$

then

$$\rho(\pm \exp t\iota) = \exp 2tJ.$$

Any element of $\mathrm{Spin}(2n)$ commuting with ι projects by ρ to an element of $\mathrm{SO}(2n)$ commuting with J. One can, therefore, define the metaunitary group as follows:

$$\mathrm{MU}(n) = \{s \in \mathrm{Spin}(2n): s\iota = \iota s\}.$$

Its Lie algebra is spanned by the set of n^2 elements

$$e_\alpha e_{n+\beta} + e_\beta e_{n+\alpha}, \quad 1 \leqslant \alpha, \beta \leqslant n.$$

Any element of the center of $\mathrm{MU}(n)$ is of the form $\exp t\iota$ or $-\exp t\iota$ for some $t \in \mathbf{R}$. Since

$$\exp t\iota = (\cos t + e_1 e_{n+1} \sin t) \cdots (\cos t + e_n e_{2n} \sin t),$$

one sees that $\exp \frac{1}{4}\pi\iota$ covers J and $\exp \frac{1}{2}\pi\iota = \epsilon$. Moreover,

$$\exp \pi\iota = \epsilon^2 = (-1)^n,$$

and the center of $\mathrm{MU}(n)$ is the set

$$\{\exp t\iota: 0 \leqslant t < 2\pi\} = \mathrm{U}(1), \quad \text{for } n \text{ odd},$$

and

$$\{\pm \exp t\iota: 0 \leqslant t < \pi\} = \mathbf{Z}_2 \times \mathrm{U}(1), \quad \text{for } n \text{ even}.$$

If n is odd, then the spinor structure on $\mathbf{C}P_n$ can be described as follows. Let $\mathrm{U}(1)$ be embedded in $\mathrm{MU}(n+1)$ so as to coincide with the connected component of the identity of its center,

$$\exp 2t\sqrt{-1} \mapsto \exp t\iota, \quad 0 \leqslant t < \pi,$$

and put

$$S = \mathrm{MU}(n+1)/\mathrm{U}(1).$$

A right action of $\mathrm{MU}(n)$ in S is obtained by passing to the quotient with the action defined by the natural embedding $\mathrm{MU}(n) \to \mathrm{MU}(n+1)$. On quotienting, the double cover $\mathrm{MU}(n+1) \to \mathrm{U}(n+1)$ passes to a double cover of the unitary frame bundle E_n,

$$S \to \mathrm{U}(n+1)/\mathrm{U}(1) = \mathrm{SU}(n+1)/\mathbf{Z}_{n+1} = E_n,$$

and the action of $\mathrm{MU}(n)$ in S projects to the action of $\mathrm{U}(n)$ in $\mathrm{U}(n+1)/\mathrm{U}(1)$, as defined in Sec. III.

The nonexistence of a spinor structure in $\mathbf{C}P_{2k}$ results from $w_1 = 0$ and $w_2 \neq 0$ for such a space. It also may be deduced directly from a comparison of the fundamental groups of the total space of the fibration $E_{2k} \to \mathbf{C}P_{2k}$ and of its fiber $\mathrm{U}(2k)$. We have indeed

$$\pi_1(\mathrm{U}(2k)) = \mathbf{Z}$$

and

$$\pi_1(E_{2k}) = \mathbf{Z}_{2k+1}.$$

The injection $\mathrm{U}(2k) \to E_{2k}$ defines a homomorphism $\mathbf{Z} \to \mathbf{Z}_{2k+1}$ such that $2k+1 \mapsto 0 \bmod (2k+1)$. This contradicts the existence of a spinor structure.[22] It is known, however, that all complex projective spaces admit a natural spin^c-structure.[14,23] Recently, Robinson and Rawnsley[24] have shown that any symplectic manifold admits a complex metaplectic structure. The metaplectic structure on $\mathbf{C}P_{2k+1}$ gives rise to symplectic spinors.[25]

D. Quaternionic projective spaces

This is the simplest and easiest case: since $w_1 = 0$ and $w_2 = 0$ for $\mathbf{H}P_n$, $n = 1,2,\ldots$, any such space admits a unique spinor structure given by the sequence

$$S = \mathrm{Sp}(n+1) \to \mathrm{Sp}(n+1)/\mathbf{Z}_2 = F \to \mathbf{H}P_n.$$

The right action of $\Sigma = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ in S is obtained from the natural embedding. Incidentally, our considerations prove the existence of a natural monomorphism of groups

$$\mathrm{Sp}(n) \times \mathrm{Sp}(1) \to \mathrm{Spin}(4n), \tag{8}$$

which covers the injection $(\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbf{Z}_2 \to \mathrm{SO}(4n)$.

V. CONCLUDING REMARKS

Most of the work on spinor structures is based on methods of algebraic topology and concentrates on problems of existence. Our approach is differential geometric and Lie-group theoretic. It yields an explicit construction of all spaces and maps occurring in the description of spinor structures on projective spaces. It can be extended to other homogeneous spaces, such as the Grassmannians, as well as to pseudo-Riemannian manifolds.

Besides the two coverings of the orthogonal group, which we have used in the case of real projective spaces, there are coverings not coming from the Clifford scheme. The analogous coverings—Clifford and not—can be defined also for the pseudo-orthogonal groups and related to the transformation properties of fermions under space-time reflections considered by physicists.[26] Our method can also be used to construct "extended spinor structures" such as the spin^c and complex metaplectic structures. It is clear from this work that, in the nonorientable case, the topological condition for the existence of a pin structure depends on which particular double cover of the orthogonal group is being considered. It would also be of some interest to study the spinor connections on projective and other homogeneous spaces. Stiefel bundles over Grassmannians, together with their canonical connections, are universal. Can one give a meaning to the idea of "universal spinor structures and connections"?

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