Conformal geometry of flows in n dimensions

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Flows generated by smooth vector fields are considered from the point of view of conformal geometry. A flow is defined to be conformally geodesic if it preserves the distribution of vector spaces orthogonal to the lines of the flow. It is shear-free if, moreover, it preserves the conformal structure on these vector spaces. Differential equations characterizing such flows are derived for the general case of an n-dimensional conformal space of arbitrary signature. In the special case of null flows in spacetime, one obtains a refined version of the theorem connecting null solutions of Maxwell's equations with null flows that are geodesic and shear-free.

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I. INTRODUCTION

It has been known since the work of Hermann Weyl[1] that there are two essential geometric ingredients in relativistic theories of spacetime: a family of null cones, defined by the rays of light, and the projective structure, related to freely falling particles. The latter structure is usually described by means of an infinitesimal connection, while the former leads to conformal geometry. When suitable compatibility relations between these structures are assumed, they can be shown to arise from Riemannian geometry.[2] There is a part of physics which depends only on the conformal structure: not only Maxwell's equations,[3] but essentially all equations of massless particles,[4,5] including the Yang–Mills equations,[6] are conformally invariant. Moreover, in the domain of phenomena where particles are produced and scattered with energies much larger than their masses, one can use an approximate description of their behavior, based on neglecting their masses and assuming conformal invariance of the fundamental laws. Conformal ideas also underlie the twistor program of Penrose.

Since the propagation of light is closely related to conformal geometry, it is not surprising that the geodesic, or nongeodesic, character of a null curve is invariant under conformal changes of the metric. This is not true of timelike and spacelike curves. Moreover, congruences of null geodesics are intimately linked to null solutions of Maxwell's equations. It is easy to see that a null electromagnetic field—a field which at any point has the same algebraic form as a plane wave—has a stress–energy tensor of the form $k^a k^b$. Since that tensor is always traceless, the vector field k is always null; and since it is also conserved, the field k is tangent to a congruence of geodesics. There is another, subtler property of k which, together with the null geodesic property, characterizes completely the vector fields associated with null electromagnetic fields.[7] The additional property is related to the absence of shear or distortion: One can visualize it by thinking of the congruence generated by k as representing a beam of light. The study of shear-free congruences of null geodesics (rays) played an important role in the investigation of exact solutions to Einstein's equations. Goldberg and Sachs[8] showed that an empty spacetime admits such a congruence of rays if, and only if, its curvature tensor is algebraically degenerate. This theorem was generalized in a conformally invariant manner by Robinson and Schild.[9] Moreover, a systematic search for solutions of Einstein's equations admitting congruences of rays led to the discovery of simple spherical gravitational waves[10] and of the Kerr black hole[11] as well as of other metrics.[12]

This paper contains a generalization of the notion of a congruence of rays to n-dimensional spaces of arbitrary signature. It turns out that, although in conformal geometry there is no way of assigning the geodesic property to a single nonnull curve, there is a well-defined notion of a conformally geodesic flow. All relevant notions—geodesity, expansion, vorticity, and shear—are defined in terms which are manifestly invariant under conformal transformations.

II. NOTATION

To a large extent, this paper follows the terminology and notation prevalent in differential geometry and mathematical physics.[13–15] All manifolds and maps are of class C^∞. If $f: M \to N$ is a map of a manifold M into another manifold N, then $Tf: TM \to TN$ is the *tangent*, or derived, map of the corresponding tangent bundles. If g is a covariant tensor field on N, then f^*g denotes its *pullback* to M. If k is a vector field on M and α is a p-form on M, then $k \lrcorner \alpha$ is the $(p-1)$-form on M obtained by *contracting* α with k: that is, if $x \in M$ and $u_2,...,u_p \in T_x M$, then

$$(k \lrcorner \alpha)(u_2,...,u_p) = \alpha(k(x),u_2,...,u_p). \tag{2.1}$$

A vector field k on M *generates a flow* (φ_t) on M, i.e., a local one-parameter group of local transformations of M. For any (sufficiently small) $t \in R$, φ_t is a diffeomorphism of an open submanifold of M onto another such submanifold. If both t and s are sufficiently small,

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \tag{2.2}$$

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provided that the domains of the maps occurring in this equation are restricted to make both its sides meaningful. Since φ_t is a (local) diffeomorphism, it can be used to pull back any tensor field A on M, and the last equation implies that

$$\varphi_t^*(\varphi_s^* A) = \varphi_{t+s}^* A \,. \tag{2.3}$$

Differentiating both sides with respect to s at $s = 0$, one obtains

$$\frac{d}{dt}\varphi_t^* A = \varphi_t^* \mathscr{L}_k A \,, \tag{2.4}$$

where

$$\mathscr{L}_k A = \frac{d}{dt}\varphi_t^* A \bigg|_{t=0} \tag{2.5}$$

is the *Lie derivative* of A with respect to k. It is known that the Lie derivative of a p-form α can be evaluated from the formula

$$\mathscr{L}_k \alpha = k \lrcorner d\alpha + d(k \lrcorner \alpha) \,, \tag{2.6}$$

where d denotes the *exterior derivative*. The *exterior product* of forms is denoted by the wedge symbol. Both d and the interior product by k, $k\lrcorner$, are antiderivations of the Cartan algebra of differential forms on M; e.g.,

$$k \lrcorner (\alpha \wedge \beta) = (k \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (k \lrcorner \beta) \,, \tag{2.7}$$

for any p-form α and form β.

By a *Riemannian space* we understand a manifold M with a metric tensor g which is nonsingular, but not necessarily positive-definite. In terms of local coordinates (x^a), $a = 1,...,n$, the metric tensor may be written as $g_{ab} dx^a \otimes dx^b$; and a similar notation is used whenever it is convenient to represent tensors by their components. For example, if k and l are vector fields on M, then $\kappa = g(k)$ is a 1-form, $l \lrcorner\, g(k) = g(k,l)$, and

$$k = k^a \partial/\partial x^a, \quad \kappa = k_a dx^a, \quad \text{where } k_a = g_{ab} k^b \,. \tag{2.8}$$

The *Levi-Civita connection* ∇ associated with g is metric and has no torsion, so that one can write

$$d\kappa = k_{a;b} dx^b \wedge dx^a = g_{ac} k^c{}_{;b} dx^b \wedge dx^a \,, \tag{2.9}$$

if it is accepted that $(k^a{}_{;b})$ denotes the components of k with respect to the local coordinates (x^a). One writes ∇_ℓ for covariant differentiation along a vector field ℓ.

A conformal structure $[g]$ on M is an equivalence class of Riemannian metrics on M. Two metrics, g and g', belong to the same class if and only if there is a positive function h on M such that $g' = hg$. One also says that the metrics g and g' are *conformally related* to each other. In *conformal geometry*, one is interested in the relations which derive from the conformal structure: i.e., those which do not change under a replacement of g by $g' = hg$. Such relations, properties, and structures are said to be *conformally invariant*. For example, assume M to be n-dimensional, and put $\gamma = \det(g_{ab})$. Then

$$\hat{g}_{ab} = |\gamma|^{-1/n} g_{ab} \tag{2.10}$$

defines on M a conformally-invariant tensor density g of weight $2/n$. The *signature* of g is also a conformal invariant. A *conformal spacetime* is a four-dimensional manifold with a conformal structure and the same signature as that of Minkowski space.

On an oriented Riemannian space, there is a standard definition of the *Hodge duals* of differential forms. In this paper, however, we use $\hat{g}$ to define duals relative to the conformal structure of M. Thus, if ω is a p-form of weight q, then $*\omega$ is the $(n-p)$-form, of weight $q - 1 + 2(n-p)/n$, defined by

$$\omega \wedge \hat{g}(u_{p+1}) \wedge \cdots \wedge \hat{g}(u_n) = *\omega(u_{p+1},...,u_n)\mathrm{vol} \tag{2.11}$$

when $u_{p+1},...,u_n$ are arbitrary vectors, and vol denotes the volume element on M. If M is even-dimensional, then the $(n/2)$-forms α and $*\alpha$ have the same weight. If k is a vector field and $\hat{\kappa} = \hat{g}(k)$ the 1-form corresponding to it in a conformally invariant way, then

$$k \lrcorner * \omega = *(\omega \wedge \hat{\kappa}) \tag{2.12}$$

for any p-form ω.

III. CONFORMALLY GEODESIC FLOWS

Let k be a nowhere-vanishing vector field on M. It defines a distribution K of one-dimensional subspaces of the tangent spaces to M: For any $x \in M$, the subspace $K(x)$ consists of all vectors parallel to $k(x)$. The distribution K is invariant under the action of the flow (φ_t) generated by k. A conformal structure $[g]$ on M, together with K, defines a distribution $K^\perp$ of $(n-1)$-dimensional subspaces: $K^\perp(x)$ is the set of all vectors tangent to M at x and orthogonal to $k(x)$ relative to $[g]$. At any point $x \in M$, the sign $\epsilon(x)$ of $g(k(x), k(x))$ is a conformal invariant. We assume k to be such that ϵ is constant on M. The distributions K and $K^\perp$ define vector bundles over M. If $\epsilon \neq 0$, then $K(x) \cap K^\perp(x) = \{0\}$, and

$$TM = K \oplus K^\perp \,, \tag{3.1}$$

whereas in the null case, $\epsilon = 0$, $K \subset K^\perp$, and one can form the exact sequence of vector bundles over M,

$$0 \to K \to K^\perp \to K^\perp/K \to 0 \,. \tag{3.2}$$

The fibers of the quotient bundle $K^\perp/K$ are $(n-2)$-dimensional, and correspond to the "screen spaces" considered in early analyses of the geometry of null geodesic congruences.[16,17]

Definition 1: The flow (φ_t) generated by k is said to be *conformally geodesic* if it preserves the distribution $K^\perp$:

$$T\varphi_t \circ K^\perp = K^\perp \circ \varphi_t \,. \tag{3.3}$$

Sometimes, when there is no danger of confusion, we describe a flow as *geodesic*, although it is actually conformally geodesic. For any $g \in [g]$, one defines the 1-form $\kappa = g(k) = g_{ab} k^b dx^a = k_a dx^a$, and obtains $K^\perp(x) = \ker \kappa(x)$. If $g' = hg$, then $\kappa' = g'(k) = h\kappa$ has the same kernels as κ.

Theorem 1: The following conditions are equivalent:

(i) $K^\perp$ is invariant under (φ_t);

(ii) $\kappa \wedge \mathscr{L}_k \kappa = 0$; (3.4)

(iii) there exists a metric $g \in [g]$ such that $g(k, k)$ is constant on M and the congruence defined by (φ_t) consists of affinely parameterized Riemannian geodesics of g.

Remark: Condition (ii) is invariant under a conformal change: that is, under a replacement of κ by $\kappa' = h\kappa$. In general, it is not invariant under changes in the parameterization of the congruence: that is, under the replacement of k by fk, where f is a function on M. It is invariant under the latter change, however, if the congruence is null.

Proof of Theorem 1: Since $K^{\perp}(x) = \ker \kappa(x)$, invariance of K under φ_t is equivalent to $\kappa \wedge \varphi_t^* \kappa = 0$, and, consequently, from (2.4), to (ii). Next choose $g \in [g]$ so that $k \rfloor \kappa = g(k, k) = \text{const}$ on M. From (2.6),

$$\begin{aligned} \mathscr{L}_k \kappa &= k \rfloor d\kappa \\ &= k \rfloor (k_{a;b} dx^b \wedge dx^a) = k_{a;b} k^b dx^a , \end{aligned} \tag{3.5}$$

so that (ii) is equivalent to

$$k^{[a} k^{b]}{}_{;c} k^c = 0 . \tag{3.6}$$

If the vector field k is nonnull, then (3.6) implies

$$k^b{}_{;c} k^c = 0 , \tag{3.7}$$

and the parameter t of the flow (φ_t) is affine. If k is null, then (3.7) can be imposed by a conformal change of the metric tensor. This completes the proof of Theorem 1.

IV. THE PROPERTY OF BEING SHEAR-FREE

The vector bundle $K^{\perp} \to M$ has a conformal structure induced by $[g]$. In other words, for any $x \in M$, the vector space $K^{\perp}(x)$ is endowed with an equivalence class of scalar products, two scalar products being equivalent if and only if one is a positive multiple of the other. The scalar products on $K^{\perp}(x)$ need not be nonsingular, even though g has been assumed to be such.

If the flow (φ_t) preserves $K^{\perp}$, then the vector spaces $T\varphi_t(K^{\perp}(x))$ and $K^{\perp}(\varphi_t(x))$ coincide; and one can compare the conformal structure on $K^{\perp}(\varphi_t(x))$ with that on $K^{\perp}(x)$ by "dragging" (Lie transporting) tangent vectors by means of $T\varphi_t$. These considerations suggest the following:

Definition 2: A conformally geodesic flow (φ_t) is said to be *shear-free* if it preserves the conformal structure of $K^{\perp}$. In other words, (φ_t) is shear-free if, for any $t \in \mathbb{R}$ and $x \in M$, there exists $h \in \mathbb{R}$, $h > 0$, such that

$$u, v \in K^{\perp}(x) \quad \text{implies} \quad (\varphi_t^* g)(u, v) = h g(u, v) . \tag{4.1}$$

This is equivalent to the existence of a function h_t and a 1-form ξ_t such that

$$\varphi_t^* g = h_t g + \xi_t \otimes \kappa + \kappa \otimes \xi_t . \tag{4.2}$$

By differentiation, this implies

$$\mathscr{L}_k g = 2a g + \xi \otimes \kappa + \kappa \otimes \xi , \tag{4.3}$$

where a is a function and ξ is a 1-form on M. Introducing the tensor G with components

$$G_{abcd} = k_{[a} g_{b][c} k_{d]} , \tag{4.4}$$

one can replace (4.3) by the equivalent condition

$$\mathscr{L}_k G = 2AG , \tag{4.5}$$

where A is another function on M. It is straightforward to prove:

Theorem 2: The flow on $(M,[g])$ generated by k is conformally geodesic and shear-free if, and only if, $\kappa \wedge \mathscr{L}_k \kappa = 0$ and condition (4.3) or (4.5) holds.

Let g be now chosen in the way described under condition (iii) of Theorem 1. This implies

$$\mathscr{L}_k \kappa = 0 \quad \text{and} \quad A = a . \tag{4.6}$$

By contracting both sides of Eq. (4.3) with k one obtains

$$(2a + k \rfloor \xi)\kappa + (k \rfloor \kappa)\xi = 0 , \tag{4.7}$$

whereas taking the trace leads to

$$\operatorname{div} k = na + k \rfloor \xi , \tag{4.8}$$

where the divergence of k is computed with respect to the volume element defined by the Riemannian metric, $\mathscr{L}_k$ vol $= (\operatorname{div} k)$ vol. It follows from (4.7) and (4.8) that

$$a = m^{-1} \operatorname{div} k , \tag{4.9}$$

where m is $n-1$ or $n-2$ according as the flow is nonnull or null. In the first case, ξ is proportional to κ, and (4.3) reduces to $\mathscr{L}_k g = 2a(g - \kappa \otimes \kappa / k \rfloor \kappa)$; in the second case, they need not be proportional, and $k \rfloor \xi = -2m^{-1} \operatorname{div} k$. In these formulae, m is the number of dimensions of the relevant vector spaces whose conformal structure is preserved by the flow: for $\epsilon \neq 0$, they are the orthogonal spaces $K^{\perp}(x)$; for $\epsilon = 0$, the screen spaces $K^{\perp}(x)/K(x)$.

Remark: A flow of conformal automorphisms, characterized by $\mathscr{L}_k g = 2ag$, is geodesic and shear-free, but the converse is not true.

V. EXPANSION AND VORTICITY

Together with a, the scalar of *expansion*, one can consider

$$b^2 = m^{-1} g^{ab} g^{cd} k_{[a;c]} k_{[b;d]} \tag{5.1}$$

as a measure of the *vorticity* of the congruence associated with the flow. Taking account of (4.6), one obtains

$$k_{[a} k_{b];[c} k_{d]} + k_{[a} k_{c]; [b} k_{d]} = 2a k_{[a} g_{b][c} k_{d]} \tag{5.2}$$

from (4.5), and hence

$$g^{ab} g^{cd} k_{(a;c)} k_{(b;d)} = m a^2 , \tag{5.3}$$

$$g^{ab} g^{cd} g^{ef} k_{(a;c)} k_{[b;e]} k_{[d;f]} = m a b^2 . \tag{5.4}$$

The Ricci identity for the commutator of second covariant derivatives gives

$$k_{a;bc} k^c + k_{a;c} k^c{}_{;b} = R_{cabd} k^c k^d . \tag{5.5}$$

Using (5.3) and (5.4) to simplify the contractions of (5.5) with g^{ab} and $k^{[a;b]}$, respectively, we obtain the real and imaginary parts of

$$\mathscr{L}_k (a + ib) + (a + ib)^2 = m^{-1} R_{cd} k^c k^d . \tag{5.6}$$

This propagation equation for the complex scalar of expansion and vorticity was first obtained by Sachs[16] for the case $n = 4$, $\epsilon = 0$. Raychaudhuri[18] had previously investigated the case $n = 4$, $\epsilon \neq 0$.

VI. CONFORMAL INVARIANCE

It is possible to replace g by $\hat{g}$, the conformally invariant density of (2.10), in all results which do not refer explicitly to a preferred metric. Writing

$$\hat{\kappa} = \hat{k}_a dx^a = \hat{g}(k) , \tag{6.1}$$

for example, we may replace (3.4) by

$$\hat{\kappa} \wedge \mathscr{L}_k \hat{x} = 0 . \tag{6.2}$$

This shows that the flow generated by k is conformally geodesic if, and only if, there is a scalar a such that

$$\mathscr{L}_k \hat{\kappa} = -2mn^{-1} a \hat{\kappa} . \tag{6.3}$$

The coefficient has been chosen to make this equation compatible with (4.9). From (4.5) and (4.6), writing

$$\hat{G}_{abcd} = \hat{k}_{[a} \hat{g}_{b][c} \hat{k}_{d]} , \tag{6.4}$$

we derive

$$\mathscr{L}_k \hat{G} = 2(n-m) n^{-1} a \hat{G} \tag{6.5}$$

as a necessary and sufficient condition for the flow generated by k to be geodesic and shear-free. The vorticity of the flow is described by a 3-form density of weight $4/n$,

$$\Omega = (1/3!) \Omega_{abc} dx^a \wedge dx^b \wedge dx^c = \kappa \wedge d\kappa . \tag{6.6}$$

For a geodesic flow,

$$k \lrcorner \hat{\kappa} b^2 = \tfrac{1}{12} m^{-1} \hat{g}^{ps} \hat{g}^{qt} \hat{g}^{ru} \Omega_{pqr} \Omega_{stu} , \tag{6.7}$$

and, if $\epsilon = 0$,

$$\hat{\kappa}_a \hat{\kappa}_b b^2 = \tfrac{1}{4} m^{-1} g^{qt} g^{ru} \Omega_{aqr} \Omega_{btu} . \tag{6.8}$$

In the null case, (6.2) is equivalent to

$$k \lrcorner \Omega = 0 . \tag{6.9}$$

Thus, a null flow is geodesic if and only if it is orthogonal to its vorticity 3-form. In three dimensions, this is equivalent to the vanishing of the vorticity. For $n > 3$, the condition (6.9) for a null flow to be geodesic is equivalent to the existence of an (n − 4)-form density ω of weight $1 - 4/n$, such that

$$\Omega = k \lrcorner {*\omega} . \tag{6.10}$$

In four dimensions, $\omega = \pm 2b$. For $n > 4$, ω is determined modulo the exterior product of $\hat{\kappa}$ with an arbitrary $(n-5)$-form density.

VII. A SPECIAL CASE: NULL FLOWS IN CONFORMAL SPACETIME

The conformal structure of an oriented four-dimensional space of Lorentz signature induces a natural complex structure on the screen spaces associated with a null vector field k. Indeed, in this case $S(x) = K^{\perp}(x)/K(x)$ is an oriented plane with a conformal structure. Let J be a rotation in that plane through 90°. Clearly $J^2 = -I$, and J defines a complex structure on $S(x)$. The complexified space $\mathbb{C} \otimes S(x)$ may be represented as a direct sum $S^+(x) \oplus S^-(x)$, where

$$S^{\pm}(x) = \{ u \in \mathbb{C} \otimes S(x) : Ju = \pm iu \} . \tag{7.1}$$

Let $K^{\pm}(x)$ be the subspace of $\mathbb{C} \otimes K^{\perp}(x)$ projecting onto $S^{\pm}(x)$ by the canonical map $\mathbb{C} \otimes K^{\perp}(x) \to \mathbb{C} \otimes S(x)$. Clearly,

$$K^+(x) \cap K^-(x) = \mathbb{C} \otimes K(x) , \tag{7.2}$$

$$K^+(x) + K^-(x) = \mathbb{C} \otimes K^{\perp}(x) . \tag{7.3}$$

Each of the spaces $K^+(x)$ and $K^-(x)$ is a maximal, totally null subspace of $\mathbb{C} \otimes T_x M$. Indeed, if $v \in K^+(x)$, say, then, since k is null, the scalar square of v is equal to that of its image $u \in S^+(x)$ under the projection $K^+(x) \to S^+(x)$. Since J is an isometry, the square of u is equal to that of $Ju = iu$. The latter square is opposite to that of u; thus the square of u is zero. Therefore, the scalar product of any two vectors belonging to $K^+(x)$ is zero. $K^+(x)$ is maximal because it is two-dimensional. We remark that one has the exact sequence of complex vector bundles over M

$$0 \to K^+ \to \mathbb{C} \otimes K^{\perp} \to S^- \to 0 . \tag{7.4}$$

The preceding definitions easily lead to:

Theorem 3: The flow generated by a null vector field k on a spacetime is geodesic and shear-free if and only if it preserves the complex vector bundle $K^+ \to M$.

Now consider a nonvanishing null vector field k on M. Its flow need not be geodesic. The subspaces $K^{\pm}(x) \subset \mathbb{C} \otimes K^{\perp}(x)$ may be characterized by means of suitably chosen complex 2-forms. Let $E = \frac{1}{2} E_{ab} dx^a \wedge dx^b \neq 0$ be a form subject to

$$*E = iE, \quad k \lrcorner E = 0 , \tag{7.5}$$

where the dual, in agreement with (2.11), is given by

$$*E_{ab} = \tfrac{1}{2} \hat{g}_{ac} \hat{g}_{bd} \epsilon^{cdef} E_{ef} . \tag{7.6}$$

The complex conjugate form $\overline{E}$ satisfies

$$*\overline{E} = -i\overline{E}, \quad k \lrcorner \overline{E} = 0 . \tag{7.7}$$

At each point $x \in M$, the form E is defined by (7.5) up to a complex factor. The kernel of the map $\mathbb{C} \otimes T_x M \to \mathbb{C} \otimes T_x^* M$ defined by

$$v \mapsto v \lrcorner E \tag{7.8}$$

contains $\mathbb{C} \otimes K(x)$. It is two-dimensional; and since

$$g(u, v) E = g(u) \wedge (v \lrcorner E) + v * \lrcorner (u \lrcorner *E) \tag{7.9}$$

identically, for any 2-form E and vectors u and v, it is totally null. The kernel of (7.8) must therefore coincide with one of the spaces $K^+(x)$ and $K^-(x)$. If E corresponds to K^+, then $\overline{E}$ corresponds to K^-. The total nullity of K^+ finds expression in the equation

$$E_{ap} g^{pq} E_{qb} = 0 . \tag{7.10}$$

For any $g \in [g]$, one may restrict E further by requiring that

$$E_{ap} \overline{E}^{p}{}_{b} = k_a k_b ; \tag{7.11}$$

E is then determined up to a phase factor, and (as a special case of the unscrambling identity[19])

$$4G = E \otimes \overline{E} + \overline{E} \otimes E . \tag{7.12}$$

Writing $D = \mathscr{L}_k E - AE$, one sees that Eq. (4.5) is equivalent to

$$D \otimes \overline{E} + \overline{E} \otimes D = -\overline{D} \otimes E - E \otimes \overline{D} , \tag{7.13}$$

and, therefore, to $D = iBE$, for some real scalar B. Thus the flow generated by k is geodesic and shear-free if and only if

$$\mathscr{L}_k E = eE \tag{7.14}$$

for some complex function e $(= A + iB)$ on M. This provides an alternative proof of Theorem 3. Since (4.3) is the general (real) solution of

$$E_a{}^{p} (\mathscr{L}_k g)_{pq} E^{q}{}_{b} = 0 , \tag{7.15}$$

this too is necessary and sufficient. One can also derive it from (7.14), by taking the Lie derivative of (7.10).

For any vector field ℓ, the 2-form $E_{ap} \nabla_\ell E^{p}{}_{b} dx^a \wedge dx^b$ is self-dual, orthogonal to k, and therefore proportional to E.

Similarly, the self-dual and anti-self-dual parts of $\kappa \wedge \nabla_{\ell} \kappa$ are proportional to E and $\overline{E}$, respectively. Thus there exist 1-forms θ and θ' such that[19]

$$E_{ap} E^{p}{}_{b;c} = E_{ab} \theta'_{c} , \tag{7.16}$$

$$k_b k_{a;c} - k_a k_{b;c} = \overline{E}_{ab} \theta_c - E_{ab} \overline{\theta}_c . \tag{7.17}$$

Contracting the last equation with k, and using (3.5), we have

$$\kappa \wedge \mathscr{L}_k \kappa + (k \lrcorner \theta) \overline{E} + (k \lrcorner \overline{\theta}) E = 0 . \tag{7.18}$$

From (7.5) and (2.7), $k \lrcorner (\theta \wedge E) = (k \lrcorner \theta) E$; from (2.12), writing

$$* \sigma = \theta \wedge E , \tag{7.19}$$

we obtain $*(\sigma \wedge \kappa) = k \lrcorner (\theta \wedge E)$. It thus follows from Theorem 1 that the flow generated by k is conformally geodesic if and only if there exists a scalar c such that

$$\sigma = c \kappa . \tag{7.20}$$

Using (7.5), (7.10), and (7.11) to simplify the contractions of $\overline{E}^{ra}$ with (7.16) and E^{ra} with (7.17), we obtain

$$- E_{ap;q} k^{p} = k_a \theta'_q , \quad E_{ap} k^{p}{}_{;q} = k_a \theta_q . \tag{7.21}$$

From the contraction of E^{qb} with the last equation,

$$E_a{}^{p} (\mathscr{L}_k g)_{pq} E^{q}{}_{b} = 2 k_{(a} \sigma_{b)} . \tag{7.22}$$

Thus $\sigma = 0$ is equivalent to (7.15), a necessary and sufficient condition for the flow to be geodesic and shear-free. From (7.21) and the covariant derivative of (7.5), however, $\theta = \theta'$: from the contraction of (7.16), therefore,

$$i \sigma = *(*E \wedge *dE) . \tag{7.23}$$

One deduces:

Theorem 4: Let k be a null vector field on a conformal spacetime M. The following conditions are equivalent:

(i) the flow generated by k is conformally geodesic and shear-free;

(ii) $\sigma = 0$;

(iii) $K^{+}(x)$ is surface forming, i.e., it admits two-dimensional integral manifolds;

(iv) $\mathscr{L}_k E$ is proportional to E;

(v) there exists a nonvanishing complex function f on M such that $d(fE) = 0$;

(vi) there exists a real nonvanishing 2-form F on M which satisfies Maxwell's equations for empty space, $dF = 0 = d * F$, and the conditions $k \lrcorner F = 0 = k \lrcorner * F$.

Proof: The equivalence of (i), (ii), and (iv) has been established. From the expression (7.23) for σ, the equivalence of (ii), (iii), and (v) follows by the theorem of Frobenius. One demonstrates the equivalence of (v) and (vi) by writing $F = fE + \overline{fE}$.

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