# Natural connections on Stiefel bundles are sourceless gauge fields

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It is shown that the natural connections defined on real and complex Stiefel bundles over Grassmannian manifolds are sourceless gauge fields corresponding to the gauge groups $G = \mathrm{SO}(k)$, $k = 2,3,\cdots$ and $\mathrm{U}(k)$, $k = 1,2,\cdots$, respectively. Stiefel bundles and their connections are important in view of their universality: Any gauge field, with group $G$, defined on a compact manifold may be obtained by embedding the manifold in a Grassmannian of sufficiently high dimension.

## INTRODUCTION

Electromagnetism is a gauge theory[1]; gauge fields are believed to play a role in the description of strong[2] and weak[3] interactions; in a somewhat different sense, gravitation also corresponds to a gauge field.[4,5] Until recently, most of the solutions of the gauge field equations considered by physicists were topologically trivial: They could be continuously deformed into the zero field. A notable exception was the electromagnetic field of a magnetic pole.[6] Solutions of the sourceless Yang–Mills equations which are not homotopic to zero have been found by Belavin, Polyakov, Schwartz, and Tyupkin.[7] They have been given a physical interpretation,[8,9] generalized[10,11] and shown to be associated with a problem in algebraic geometry.[12] Another method for obtaining topologically nontrivial solutions of the Maxwell and Yang–Mills equations has been described in Ref. 13. The method is based on the observation that the magnetic pole of the lowest strength and the pseudoparticle solution of the Yang–Mills equations correspond to the natural connections defined on the Hopf bundles $S_3 \to S_2$ and $S_7 \to S_4$, respectively. In the present paper, we generalize the results on the gauge fields associated with Hopf fibrations to Stiefel manifolds considered as principal bundles over Grassmannians. The structure (gauge) group of these bundles is one of the groups $G = \mathrm{SO}(n)$, $n = 2, 3, \ldots$, $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$, $n = 1, 2, \ldots$ . Each of these bundles has a natural connection[14] and its gauge field (curvature) is sourceless. We give the details of the proof only in the real and complex cases, leaving the quaternionic bundles for further study.

## GAUGE FIELDS AND CONNECTIONS

To fix the terminology, let us recall[15,16] that a (smooth) principal bundle consists of:

(i) a (smooth) fiber bundle $\pi: P \to M$;

(ii) a Lie group $G$ which acts on $P$ smoothly and freely to the right: There is a differentiable map $\delta: P \times G \to P$ such that $\delta_a \circ \delta_b = \delta_{ba}$ and $\delta_a = id_P \longleftrightarrow a = e$, where $\delta_a(p) = \delta(p, a)$, $p \in P$, $a, b \in G$ and $e$ is the unit element of $G$; moreover,

(iii) the action of $G$ is compatible with the fiber bundle structure: $\pi \circ \delta_a = \pi$ and any point in $M$ has a neighborhood $U$ such that $\pi^{-1}(U)$ is isomorphic to $U \times G$.

If $U \subset M$ is open, the $s: U \to P$ is called a (local) section if $\pi \circ s = id_U$; if $U = M$, then such an $s$ is called a global section. A principal bundle $P$ is trivial, i.e., isomorphic to the product $M \times G$, if and only if it admits a global section.

Any homomorphism of Lie groups $h: G \to H$ induces a (derived) homomorphism of Lie algebras, $h': G' \to H'$. For example, if $G = H$ and $\mathrm{ad}_a(b) = aba^{-1}$, then $\mathrm{Ad}_a = ad'_a$: $G' \to G'$ defines the adjoint representation of $G$ in its Lie algebra. The action of $G$ on $P$ induces a homomorphism of Lie algebras $\delta'$: $G' \to$ Lie algebra of vertical vector fields on $P$, given by $\delta'(A) =$ vector field tangent to the curve $t \to \delta_{\exp tA}$, $A \in G'$. Let $T$ denote the tangent functor. A connection on $P$ may be defined by a connection form $\omega$, i.e., by a map $\omega: TP \to G$ which is linear on the fibers of $TP \to P$, equivariant under the action of $G$,

$$\omega \circ T\delta_a = \mathrm{Ad}_{a^{-1}} \circ \omega, \quad a \in G, \tag{1}$$

and such that

$$\omega(\delta'(A)) = A, \quad A \in G'. \tag{2}$$

The vector space $\mathrm{hor}_p P = \{u \in T_p P \mid \omega(u) = 0\}$ is called the horizontal space at $p \in P$; $\mathrm{ver}_p P = \{u \in T_p P \mid T\pi(u) = 0\}$ is the vertical space at $p$, and there is a unique decomposition $u = \mathrm{hor}\, u + \mathrm{ver}\, u$ corresponding to $T_p P = \mathrm{hor}_p P + \mathrm{ver}_p P$. If $\rho: G \to GL(V)$ is a representation of $G$ in the vector space $V$, then the $V$-valued $k$-form $\phi$ on $P$ is said to be of type $\rho$ if $\delta_a^* \phi = \rho_{a^{-1}} \circ \phi$, where $\delta_a^* \phi$ is the pullback of $\phi$ by $\delta_a$. The $k$-form $\mathrm{hor}\, \phi$ defined by $\mathrm{hor}\, \phi(u_1, u_2, \ldots, u_k) = \phi(\mathrm{hor}\, u_1, \mathrm{hor}\, u_2, \ldots, \mathrm{hor}\, u_k)$ is then also of type $\rho$ and $D\phi = \mathrm{hor}\, d\phi$ is a $(k+1)$-form of type $\rho$, called the covariant exterior derivative of $\phi$. For example, the connection 1-form $\omega$ is $G'$ valued, of type Ad, and $\Omega = D\omega = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature form. The $k$-form $\phi$ is called horizontal relative to $\pi$ if $\phi(u_1, u_2, \ldots, u_k) = 0$ for any $u_2, \ldots, u_k \in T_p P$ and $u_1 \in \mathrm{ver}_p P$.

The covariant exterior derivative of a horizontal $V$-

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valued form of type $\rho$ is given by the formula

$$D\phi = d\phi + \rho'(\omega)\Lambda\phi,$$

where $\rho'(\omega): TP \to \mathcal{L}(V)$ is obtained by composition of $\omega$ with the derived map $\rho': G' \to \mathcal{L}(V)$ and the wedge sign implies both the exterior multiplication of forms on $P$ and the evaluation map $\mathcal{L}(V)\times V \to V$. For example, $\Omega$ is a $G'$-valued horizontal 2-form of type Ad, and since $\mathrm{Ad}_A(B) = [A, B]$ $(A, B \in G')$ we have $D\Omega = d\Omega + [\omega, \Omega] = 0$ (the last equality is the "Bianchi identity").

In a gauge theory, the total space $P$ of the bundle is interpreted as the space of phase factors,[17] $M$ is the space–time manifold, $G$ is the gauge group, $\omega$ is the gauge potential, and $\Omega$ is the gauge field.

In theoretical physics, one usually works with local sections of the bundle and the corresponding pullbacks; $s: U \to P$ is then called a local gauge, $\Gamma = s^*\omega$ is the potential and $s^*\Omega$ is the field strength in gauge $s$. The map $\delta_a: P \to P$ is interpreted as a gauge transformation of the first kind. If $S: U \to G$, then $s': U \to P$ defined by $s'(x) = \delta(s(x), S(x))$ is another local section. We say that the local gauge $s'$ is obtained from the local gauge $s$ by the gauge transformation of the second kind $S$. Assuming $G \subset GL(V)$, we may write the relation between $\Gamma$ and $\Gamma' = s'^*\omega$ as

$$\Gamma' = S^{-1}\Gamma S + S^{-1}dS.$$

To construct a gauge theory, one should

(i) choose a gauge group $G$ and consider principal $G$-bundles endowed with connections,

(ii) specify the type of particle(s) coupled to the gauge field: this is done by picking out the representation(s) $\rho: G \to GL(V)$; wavefunctions of particles of type $\rho$ are then given by 0-forms of type $\rho$; $\phi \circ s$ is the wavefunction in gauge $s$;

(iii) make an assumption about the field equations satisfied by the gauge field.

For example, in electromagnetism the gauge group is U(1), all its irreducible representations are of the form $\rho_n: \mathrm{U}(1) \to \mathrm{U}(1)$, where $\rho_n(u) = u^n$, $n \in Z$, $u \in U(1)$. A particle of type $\rho_n$ is simply a particle of electric charge $n$ and the covariant exterior derivative of its wavefunction is

$$D\phi = d\phi + n\omega\phi.$$

Since $U(1) \ni u = \exp iA$, where $A \in R$, the Lie algebra of U(1) is $iR$ and the form $\omega$ is purely imaginary. The Maxwell equations on $P$ are $d\Omega = 0$ and $d^*\Omega = 4\pi^*j$, where $j$ is the 1-form of electric current and the star appearing on the left of a horizontal form denotes its dual with respect to the metric lifted from the base space. In an SU($n$) theory, one often takes $\rho = \mathrm{Ad}$; $\phi$ is then called a Higgs field. In this case, $\Omega$ is assumed to fulfill the field equation

$$D^*\Omega = 4\pi^*j.$$

In the sourceless case, which is the only one considered here,

$$D^*\Omega = 0. \qquad (3)$$

## STIEFEL BUNDLES AND GRASSMANN MANIFOLDS

Stiefel bundles over Grassmann manifolds[18,19] generalize the Hopf fibrations

$$S_{2n-1} \to CP_{n-1}, \qquad (4)$$

$$S_{4n-1} \to HP_{n-1}, \qquad (5)$$

which are known to admit sourceless, topologically nontrivial gauge fields.[13]

Let $F$ be one of the following: the field $R$ of the reals, the field $C$ of complex numbers, or the division algebra $H$ of quaternions. If $z \in F$, then $\bar{z} = z$ for $F = R$ and $\bar{z}$ is the conjugate of $z$ for $z \in C$ or $H$. Consider the right vector space $F^n$ and the scalar product defined by

$$(\mathbf{u}|\mathbf{v}) = \bar{u}_\alpha v_\alpha, \quad \mathbf{u} = (u_\alpha), \quad \mathbf{v} = (v_\alpha), \quad \mathbf{v} = (v_\alpha) \in F^n. \qquad (6)$$

The unit coordinate vectors in $F^n$ are $\epsilon_\alpha$, $\alpha = 1, \ldots, n$ and thus $\mathbf{u} = \sum u_\alpha \epsilon_\alpha$. Here and in sequel, summation is understood over the range indicated by a pair of repeated indices.

Let $\mathrm{U}_n(F)$ be the connected component containing the unit of the (Lie) group of linear transformations $a: F^n \to F^n$, $a\mathbf{u} = (\sum_{\beta=1}^n a_{\alpha\beta}u_\beta)$, preserving the form (6). If $a = (a_{\alpha\beta})$ and $a^+ = (\bar{a}_{\beta\alpha})$ is a transpose conjugate matrix, then (6) is preserved, $(a\mathbf{u}|a\mathbf{v}) = (\mathbf{u}|\mathbf{v})$ provided that $a^+a = I$. Depending on the field, the group $\mathrm{U}_n(F)$ is the group of rotations, the unitary group, or the symplectic group

$$\mathrm{U}_n(F) = \begin{cases} \mathrm{SO}(n) & \text{for } F = R, \\ \mathrm{U}(n) & \text{for } F = C, \\ \mathrm{Sp}(n) & \text{for } F = H, \end{cases}$$

and its real dimension is $\frac{1}{2}n(n+1)\dim_R F - n$.

For $k = 1, 2, \ldots, n$ one defines a $k$-frame $u$ in $F^n$ as the ordered set $(\mathbf{u}_1, \ldots, \mathbf{u}_k) = (\mathbf{u}_i) = u$ of $k$ orthonormal vectors,

$$(\mathbf{u}_i|\mathbf{u}_j) = \delta_{ij}, \quad \mathbf{u}_i \in F^n, \quad i, j = 1, \ldots, k.$$

Each vector $\mathbf{u}_i$ is the $n$-tuple of elements of $F$, $\mathbf{u}_i = (u_{\alpha i})$, $\alpha = 1, \ldots, n$. We put $l = n - k$ and make the set of all $k$-frames in $F^n$ into a manifold. The connected component of $(\epsilon_1, \ldots, \epsilon_k)$ in this manifold is called the Stiefel space $\mathrm{V}_{n,k}(F)$. The group $\mathrm{U}_n(F)$ acts on $\mathrm{V}_{n,k}(F)$ to the left by $(a, u) \to au = u'$, $u'_{\alpha i} = \sum_\beta a_{\alpha\beta}u_{\beta i}$. This action is transitive and the stability group at $(\epsilon_1, \ldots, \epsilon_k)$ is the subgroup $\mathrm{U}_l(F)$ of $\mathrm{U}_n(F)$, consisting of all matrices of the form

$$\left(\begin{array}{c|c} \delta_{ij} & 0 \\ \hline 0 & a_{AB} \end{array}\right),$$

where $(a_{AB}) \in \mathbf{U}_l(F)$, $A, B = k+1, \ldots, n$. Therefore, $\mathbf{V}_{n,k}(F)$ may be identified with the left coset space $\mathrm{U}_n(F)/\mathrm{U}_l(F)$ and, under this identification, the canonical projection

$$\pi_0: U_n(F) \to \mathrm{U}_n(F)/\mathrm{U}_l(F) = \mathrm{V}_{n,k}(F) \qquad (7)$$

maps

$$a = \begin{pmatrix} a_{ij} & a_{iB} \\ a_{Aj} & a_{AB} \end{pmatrix} \in \mathrm{U}_n(F)$$

into the $k$-frame $u=(\mathbf{u}_i)$, where $\mathbf{u}_i=\sum_\alpha \epsilon_\alpha a_{\alpha i}$.

The group $U_k(F)$, which may also be considered as a subgroup of $U_n(F)$, acts on $V_{n,k}(F)$ to the right: $(u,a) \to \delta_a(u)=ua=u'$, $u'_{\alpha i}=\sum_j u_{\alpha j}a_{ji}$, $a=(a_{ij})\in U_k(F)$. The quotient of $V_{n,k}(F)$ by this action is a set $G_{n,k}(F)$ which can be given the structure of a differentiable manifold of real dimension equal to $k\cdot l \dim_R F$. This Grassmann manifold may also be described as the set of all (oriented for $F=R$) $k$-planes through the origin in $F^n$: If $u'$, $u\in V_{n,k}(F)$, then $u$ and $u'$ span the same (oriented) $k$-plane if $u'=ua$ for some $a\in U_k(F)$.

The Stiefel bundle

$$\pi : V_{n,k}(F) \to G_{n,k}(F)$$

with group $U_k(F)$ reduces for $k=1$ to (4) or (5), depending on whether $F=C$ or $H$; moreover, $V_{n,1}(R)=G_{n,1}(R)=S_{n-1}$.

The canonical 1-form on $U_n(F)$

$$\omega=a^*da=(\omega_{\alpha\beta})$$

has values in the Lie algebra of $U_n(F)$, i.e., each of the component forms $\omega_{\alpha\beta}$ $(\alpha,\beta=1,\ldots,n)$ is $F$-valued and $\bar{\omega}_{\alpha\beta}+\omega_{\alpha\beta}=0$. The form $\omega$ is left invariant, right equivariant,

$$\delta_a^*\omega=\mathrm{Ad}_{a^{-1}}\omega, \text{ for any } a\in U_n(F). \tag{8}$$

and satisfies the Maurer–Cartan equations

$$d\omega_{\alpha\beta}+\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=0. \tag{9}$$

The forms $\omega_{\alpha j}$ $(\alpha=1,\ldots,n;\ j=1,\ldots,k)$ are horizontal relative to $\pi_o$. It follows from (8) that the forms $\omega_{ij}$ are invariant under the action of $U_l(F)$ on $U_n(F)$ to the right; therefore, they project to forms on $V_{n,k}(F)$, which will be denoted by the same symbols. Moreover, the forms $(\omega_{ij})$ satisfy on $V_{n,k}(F)$ conditions (1) and (2): They define the natural connection on the Stiefel bundle.[14]

Similarly, the quadratic form $\sum_{\alpha,j}\bar{\omega}_{\alpha j}\omega_{\alpha j}$ is invariant under $U_l(F)$ and defines a Riemannian metric $dl^2$ on $V_{n,k}(F)$,

$$\sum_{\alpha,j}\bar{\omega}_{\alpha j}\omega_{\alpha j}=\pi_o^*\, dl^2.$$

The quadratic form $\sum_{A,j}\bar{\omega}_{Aj}\omega_{Aj}$ on $U_n(F)$ is invariant under the action of both $U_l(F)$ and $U_k(F)$. Therefore, it is the pullback of, or projects to, a Riemannian metric $ds^2$ on $G_{n,k}(F)$

$$\sum_{A,j}\bar{\omega}_{Aj}\omega_{Aj}=(\pi\circ\pi_o)^*\, ds^2 \tag{10}$$

and

$$dl^2=\pi^*(ds)^2+\sum_{i,j}\bar{\omega}_{ij}\omega_{ij} \quad \text{on } V_{n,k}(F).$$

The metric on $V_{n,k}(F)$ is of the type considered in generalized Kaluza–Klein theories.[16] For $k=1$, $dl^2$ reduces to the natural Riemannian metric of a sphere whereas $ds^2$ is the Fubini–Study metric of a projective space ($F=C$ or $H$)[13] or the natural metric of a sphere $(F=R)$. The metrics $dl^2$ and $ds^2$ are positive definite and invariant under the action of $U_n(F)$ on $V_{n,k}(F)$ and $G_{n,k}(F)$, respectively.

It is clear from (10) that $U_n(F)$ considered as a bundle over $G_{n,k}(F)$ may be viewed as a restriction of the bundle of orthonormal frames of $G_{n,k}(F)$, corresponding to the canonical injection $U_k(F)\times U_l(F)\to U_{k+l}(F)$. The form $\delta_{ij}\omega_{AB}+\delta_{AB}\omega_{ij}$ on $U_n(F)$ defines a Levi-Civita connection for $G_{n,k}(F)$. Its curvature form is $\delta_{ij}\Omega_{AB}+\delta_{AB}\Omega_{ij}$, where, by virtue of (9),

$$\Omega_{AB}=d\omega_{AB}+\omega_{AB}+\omega_{AC}\wedge\omega_{CB}=\bar{\omega}_{iA}\wedge\omega_{iB}$$

and

$$\Omega_{ij}=d\omega_{ij}+\omega_{ik}\wedge\omega_{kj}=\bar{\omega}_{Ai}\wedge\omega_{Aj}. \tag{11}$$

The form $\Omega_{ij}$ (or $\Omega_{AB}$) is the pullback to $U_n(F)$ of the curvature form of the natural connection on $V_{n,k}(F)$ (or $V_{n,l}(F)$).

## PROOF OF $D * \Omega = 0$

Consider the following $l=(n-k)$-forms on $U_n(F)$:

$$\psi_{i_1 i_2\cdots i_l}=\frac{1}{l!}\epsilon_{A_1\cdots A_l}\,\omega_{A_1 i_1}\wedge\omega_{A_2 i_2}\wedge\cdots\wedge\omega_{A_l i_l},$$

where $\epsilon_{A_1\cdots A_l}$ is the Levi-Civita symbol. The $\psi$'s are symmetric in the indices $i_1,\ldots,i_l$. These forms are horizontal relative to $\pi_o$. For $F=R$ they are also invariant under the action of $SO(l)$ on $SO(n)$. Therefore, they are pullbacks by $\pi$ of forms on $V_{n,k}(R)$ which will be denoted by the same symbols. For $F=C$, the $2l$-forms

$$\psi_{i_1\cdots i_l}\wedge\psi_{j_1\cdots j_l} \tag{12}$$

are invariant under the action of $U(l)$ on $U(n)$. By using

$$\epsilon_{A_1A_2\cdots A_l}\epsilon^{B_1B_2\cdots B_l}=\delta^{B_1}_{[A_1}\delta^{B_2}_{A_2}\cdots\delta^{B_l}_{A_l]}$$

and Eq. (11), the projections of (12) onto $V_{n,k}(C)$ may be represented as an equivariant homogeneous polynomial of degree $l$ in $\Omega_{ij}$.

Evaluating the covariant exterior derivative of $\psi$ on $V_{n,k}(R)$, using the Maurer-Cartan equation (9) and $\omega_{AB}+\omega_{BA}=0$, one obtains

$$D\psi_{i_1\cdots i_l}=0 \quad \text{on } V_{n,k}(R). \tag{13}$$

It follows directly from the Bianchi identity that covariant exterior derivative of (12) is zero.

The $k\cdot l$ form on $SO(n)$ given by

$$\eta\propto\epsilon_{i_1^1\cdots i_1^k}\cdots\epsilon_{i_l^1\cdots i_l^k}\,\psi_{i_1^1\cdots i_l^1}\wedge\cdots\wedge\psi_{i_1^k\cdots i_l^k} \tag{14}$$

is invariant under both $SO(k)$ and $SO(l)$; therefore, it projects to a nonzero form of maximal degree on $G_{n,k}(R)$ which may be identified with the volume $kl$-form. The form (14) considered on $U(n)$ changes by a phase factor under the action of $U(k)$ and $U(l)$. Therefore, the form $\bar{\eta}\wedge\eta$ projects to a volume element on $G_{n,k}(C)$. In this case, the volume element is an invariant polynomial of degree $k\cdot l$ in $\Omega$.

The volume elements define orientation on the Grassmannian, which, together with the Riemannian metric $ds^2$, enables us to define and compute the dual of any form on $G_{n,k}(F)$ or of any horizontal form on $V_{n,k}(F)$. In particular, for $F=R$ we obtain

$$\begin{aligned}{}^*\Omega_{ij}\propto{}&\epsilon_{ij i_1^3\cdots i_1^k}\epsilon_{i_2^1 i_2^2\cdots i_2^k}\cdots\epsilon_{i_l^1 i_l^2\cdots i_l^k}\\ &\cdot\Omega_{i_1^1 i_2^2}\wedge\Omega_{i_3^1 i_3^2}\wedge\cdots\wedge\Omega_{i_l^1 i_l^2}\wedge\Psi_{i_1^3\cdots i_l^3}\wedge\cdots\wedge\Psi_{i_1^k\cdots i_l^k}.\end{aligned}$$

The Bianchi identities, together with (13), imply that the curvature form $\Omega_{ij}$ is sourceless,

$$D{*}\Omega_{ij}=0. \tag{15}$$

In the complex case, the form $^*\Omega_{ij}$ is of even degree and may be represented as an (equivariant) polynomial of degree $k\cdot l-1$ in $\Omega$. Therefore, Eq. (15) holds as a consequence of the Bianchi identities.

## EXAMPLES

The simplest examples correspond to $F=R$ and $l=1$. In this case we have the fibration

$$\mathrm{SO}(n)=\mathrm{V}_{n,n-1}(R)\to \mathrm{G}_{n,n-1}(R)=S_{n-1}$$

and the corresponding $\mathrm{SO}(n-1)$-gauge field is simply the Levi-Civita connection of the sphere $S_{n-1}$. When standard parametrization is used, the metric of the sphere is $ds^2=\omega_{n1}^2+\omega_{n2}^2+\cdots+\omega_{n\,n-1}^2$, $\omega_{n1}=d\theta_1$, $\omega_{n2}=\sin\theta_1\,d\theta_2,\ldots,\ \omega_{n\,n-1}=\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\,d\theta_{n-1}$ and $\Omega_{ij}=\omega_{ni}\Lambda\omega_{nj}$.

In the complex case the fibration corresponding to $l=1$ is

$$\mathrm{SU}(n)=\mathrm{V}_{n,n-1}(C)\to \mathrm{G}_{n,n-1}(C)=CP_{n-1}$$

with the gauge $\mathrm{U}(n-1)$. For $n=2$ this reduces to the Hopf fibration $S_3\to S_2$. The gauge group is U(1) and the connection corresponds to the magnetic pole of lowest order. For $n=3$ the fibration is $\mathrm{SU}(3)\to CP_2$, the gauge group is $\mathrm{U}(2)=\mathrm{U}(1)\times\mathrm{SU}(2)$ and the connection decomposes into the "electromagnetic instanton" and the solution recently found by C.N. Yang.[20] For instance, with Fubini–Study metric on $CP_2$

$$ds^2=\bar\omega_{31}\omega_{31}+\bar\omega_{32}\omega_{32},$$

$$\omega_{31}=\sigma e^{-i\mu}\{i\sin\theta\cos\phi[(1/\sigma^2)\,d\mu-\sin^2\theta\sin^2\phi\,d\nu]-\cos\theta\cos\phi\,d\theta+\sin\theta\sin\phi\,d\phi\},$$

$$\omega_{32}=\sigma e^{-i\nu}\{i\cos\theta\sin\theta\sin\phi\,d\nu-\sin\phi\,d\theta-\sin\theta\cos\theta\cos\phi\,d\phi\},$$

$$\sigma=(\sin^2\theta\sin^2\phi+\cos^2\theta)^{-1/2}\quad\text{and}\quad \Omega_{ij}=\bar\omega_{3i}\Lambda\omega_{3j}.$$

The case $k=1$ is trivial when $F=R$ and reduces to Hopf fibration (4) for $F=C$. For low dimensions the isomorphism between different groups or Lie algebras leads to a correspondence between different Grassmannians and the solutions of gauge equations.

## CONCLUDING REMARKS

The importance of the Stiefel bundles and of the corresponding natural connections lies in their universality: If $G$ is an orthogonal, unitary, or symplectic group, then any principal $G$-bundle $P$ with connection $\omega$ can be obtained by embedding its base space $M$ in a Grassmann manifold $W$ of sufficiently high dimension.[14] Clearly, the connection on $P$, induced by the embedding $k:M\to W$, in general will not correspond to a sourceless gauge field. An interesting problem is to characterize the embeddings $k$ which lead to sourceless gauge fields. A simple example of such an embedding is known in the case of magnetic poles: For any integer $m$, the embedding.

$$k_m: S_2\to CP_m$$

is given in terms of homogenous coordinates by

$$k_m(z_1,z_2)=(z_1^m,\ \tbinom{m}{1}^{1/2}z_1^{m-1}z_2,\ldots,\tbinom{m}{p}^{1/2}z_1^{m-p}z_2^p,\ldots,z_2^m)$$

pulls back the natural connection on the U(1)-bundle $S_{2m+1}\to CP_m$ to $L(m,1)\to S_2$, where $L(m,1)$ is the lens space.

The method described here is easily generalized to noncompact groups. By replacing the positive definite form (6) with

$$\bar u_1v_1+\cdots+\bar u_pv_p-\bar u_{p+1}v_{p+1}-\cdots-\bar u_{p+q}v_{p+q}$$

one is led to the groups

$$\mathrm{U}_{p,q}(F)=\begin{cases}\mathrm{SO}(p,q) & \text{for } F=R\\ \mathrm{U}(p,q) & \text{for } F=C\\ \mathrm{Sp}(p,q) & \text{for } F=H\end{cases}$$

and fiberings

$$\mathrm{U}_{p,q}(F)\to \mathrm{V}_{p,b;m,n}(F)=\mathrm{U}_{p,q}(F)/\mathrm{U}_{p-m,q-n}(F)\to \mathrm{G}_{p,q;m,n}(F)=\mathrm{U}_{p,q}(F)/\mathrm{U}_{p-m,q-n}(F)\times\mathrm{U}_{m,n}(F).$$

The corresponding Stiefel bundles, with connections which are also sourceless, are defined over an indefinite Grassmann manifold in which metric (10) is replaced by metric with signature

$$[\dim_R F(m(p-m)+n(q-n)),\ \dim_R F(n(p-m)+m(q-n))].$$


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