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THEORETICAL PHYSICS

General Invariance of Lagrangian Structures *)

by

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Summary. The lift of a local diffeomorphism of a manifold X to a prolongation Y of the bundle of frames of X is defined in terms of a functor between appropriate categories. A Lagrangian form defined on a bundle E associated to Y is said to be generally invariant if it is preserved by the lifts of all the local diffeomorphisms of the base space. It is shown that a generally invariant Lagrangian form is completely determined by a function on the typical fibre of E.

1. Introduction. The notion of general invariance has been used and discussed since the advent of Einstein's relativistic theory of gravitation [1]. Hilbert analyzed the variational principles of classical physics and put forward the requirement of general invariance as a fundamental axiom [2]. The notion of invariance of a principle of least action may be conveniently defined when its Lagrangian is considered as a differential form on a fibre bundle [3—5]. In this paper, we develop the notions of differential geometry required to define precisely the concept of general invariance and we prove a theorem on the structure of generally invariant Lagrangians.

All the spaces and maps considered in this paper belong to the category of finite-dimensional, real *differential manifolds* of class C^∞. The subcategory of n-dimensional manifolds is denoted by $\boldsymbol{D}$: $f \in \mathrm{Mor}\, \boldsymbol{D}$ if and only if f is a diffeomorphism between n-dimensional manifolds. For any manifold X of dimension n there is the full subcategory $\boldsymbol{D}_X$ of $\boldsymbol{D}$ of all the local diffeomorphisms of X into itself. The category of *principal bundles* [6] over n-dimensional manifolds is denoted by $\boldsymbol{PB}$. A principal bundle is a triple (X, G, Y) of spaces, together with a pair (π, δ) of maps, such that the Lie group G is the typical fibre of the bundle $\pi: Y \to X$, $\delta: Y \times G \to Y$ defines a free action of G in Y on the right: $\delta_a \circ \delta_b = \delta_{ba}$, where $\delta_a(y) = \delta(y, a)$, $y \in Y$, $a \in G$, and $\pi \circ \delta = \pi \circ pr_1$. A morphism of two principal bundles, (X_1, G_1, Y_1) and (X_2, G_2, Y_2), is a triple (f, g, h) of maps,

$$f: X_1 \to X_2, \quad g: G_1 \to G_2, \quad h: Y_1 \to Y_2,$$

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such that $f \in \operatorname{Mor} \boldsymbol{D}$, g is a morphism of Lie groups, $\delta_2 \circ (h \times g) = h \circ \delta_1$, and $\pi_2 \circ h = f \circ \pi_1$.

The *frame functor* $F: \boldsymbol{D} \to \boldsymbol{PB}$ associates to a manifold X the principal bundle $(X, \mathrm{GL}(n, \boldsymbol{R}), F_0 X)$ of frames, $Ff = (f, \mathrm{id}_{\mathrm{GL}(n,\boldsymbol{R})}, F_0 f)$, where $F_0 f$ is the map of frames induced by the diffeomorphism f. The tangent functor is denoted by T. The *Cartan* (contravariant) *functor* which associates to X the exterior algebra of fields of differential forms over X is denoted by a star. Thus

$$X^* = \bigoplus_{p=0}^{n} X_p^*,$$

where X_p^* is the module of p-forms (X_0^* is simply the algebra of differentiable functions on X). If $f: X \to Y$ and $a \in Y^*$, then $f^* a \in X^*$ is the pull-back of a by f (if $a \in Y_0^*$, then $f^* a = a \circ f$).

If $\pi_E: E \to X$ is a bundle, then a p-form a on E is said to be *horizontal* relative to π_E if

$$u \lrcorner a = 0 \quad \text{for any} \quad u \in TE \quad \text{such that} \quad T\pi_E(u) = 0.$$

Let $\theta_X = (\theta_X^i)$, $i = 1, \ldots, n$, be the canonical, $\boldsymbol{R}^n$-valued 1-form on $F_0 X$ [7]. The n-form

$$\mu_X = \theta_X^1 \wedge \theta_X^2 \wedge \ldots \wedge \theta_X^n$$

is horizontal relative to the natural projection of $F_0 X$ on X. Since the base manifold is usually fixed, it is convenient to write μ instead of μ_X, and this will be done. If δ defines the action of $\mathrm{GL}(n, \boldsymbol{R})$ in $\mathrm{F}_0 X$, then, for any $a \in \mathrm{GL}(n, \boldsymbol{R})$,

(1) $$\delta_a^* \mu = (\det a)^{-1} \mu.$$

Moreover,

(2) $$(F_0 f)^* \mu = \mu$$

for any $f \in \operatorname{Mor} \boldsymbol{D}_X$.

2. Lifting.

Definition. A covariant functor $\tau: \boldsymbol{D} \to \boldsymbol{PB}$ is said to define a *lifting* to the Lie group G if

$$\tau f = (f, \mathrm{id}_G, \tau_0 f), \text{ for any } f \in \operatorname{Mor} \boldsymbol{D},$$

and there exists a natural transformation N from τ to F such that

$$N(X) = (\mathrm{id}_X, g_X, j_X) \quad \text{for any } X \in \operatorname{Ob} \boldsymbol{D}.$$

The isomorphism of bundles $\tau_0 f: \tau_0 X_1 \to \tau_0 X_2$ is called the *lift* of $f: X_1 \to X_2$. A lifting is said to be *transitive* if the lifts act transitively on $Y = \tau_0 X$, i.e., if for any $y_1, y_2 \in Y$ there exists $f \in \operatorname{Mor} \boldsymbol{D}_X$ such that $(\tau_0 f)(y_1) = y_2$. For example, the bundle of holonomic frames of order q is obtained by a transitive lifting to the group $G^q(n)$ [8]. The bundle of affine frames [7] of a manifold is obtained by a non-transitive lifting to the affine group.