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ON THE STRUCTURE OF THE EINSTEIN-CARTAN EQUATIONS (*)

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Summary: The paper contains an account of the first stage of research on the mathematical structure and the physical predictions of a modification, due to Elie Cartan, of Einstein's theory of gravitation. The modification, which consists in considering a curved space-time with torsion and relating the latter to spin, is made plausible by the theorem on the holonomy groups and by the important role of the inhomogeneous Lorentz group in special relativity. On the one hand, curvature and torsion induce, respectively, homogeneous transformations and translations in the tangent spaces of a manifold. On the other, spin and mass are linked to the homogeneous and translational components of the Lie algebra of the group of automorphisms of a Minkowski space.

It is shown that the Einstein equation may be written in two equivalent forms, with either the symmetric or the canonical tensor density of energy and momentum as the source. The Cartan equation determines the linear connection only up to projective transformations; this arbitrariness may be removed by requiring that the connection be metric. Similarly as in Einstein's theory, by covariant exterior differentiation of the equations of the generalized theory of gravity one arrives at a system of relations which hold as an algebraic consequence of the field equations themselves. The Dirac equation in a Riemann-Cartan space-time is written in a manner adapted to the calculus of exterior forms.

1. **Introduction.**

In recent years, there has been a revival of interest in the foundation of Einstein's theory of general relativity. A number of new relativistic theories of gravitation were put forward [1-5], their predictions analyzed

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and compared with those of the older theories and with the available experimental results and observational data. K. S. Thorne and his co-workers have undertaken a systematic study of what they call « metric theories of gravitation » [6-9]. These are theories which may be formulated in terms of a Riemannian geometry in space-time, possibly with supplementary structures added to it. The total stress-energy tensor of matter is assumed to satisfy a differential conservation law determined by the Riemannian linear connection of space-time. J. Ehlers, F.A.E. Pirani and A. Schild show that the Riemannian geometry of Einstein's theory of relativity may be obtained by requiring compatibility between the conformal geometry defined by rays of light and the projective structure of space-time, determined by the trajectories of freely-falling particles [10]. In that analysis, as well as in the « metric theories », the authors assume a linear connection which may be curved but has no torsion.

In 1922 Elie Cartan [11] suggested a simple generalization of Einstein's theory of gravitation. He proposed to consider, as a model of space-time, a four-dimensional differentiable manifold with a metric tensor and a linear connection compatible with the metric but not symmetric, in general. According to Cartan, the torsion tensor of the connection should be related to the density of intrinsic angular momentum [12]. Independently of Cartan, similar ideas were put forward by several authors (for example, see [3], [4] and [5]; the last paper contains other relevant references). The generalization due to Cartan constitutes only a slight departure from Einstein's theory: the field equations in empty space remain inchanged. In our opinion, the Einstein-Cartan theory is the simplest and the most natural modification of the original, Einstein's theory of gravitation. This modification deserves to be analyzed in detail, in precedence over the theories requiring an additional scalar field to describe gravitational phenomena.

The desirability of such an analysis may be related to recent discoveries in astronomy. It is conceivable that torsion may produce observable effects inside those objects which, as the neutron stars, have built-in strong magnetic fields, possibly accompanied by a substantial average value of the density of spin. One is tempted to speculate that intrinsic angular momentum may influence—or even prevent—the occurrence of singularities in gravitational collapse and cosmology. A recent result of W. Kopczyński, on the geometry of a Universe filled with a spherically-symmetric distribution of mass and spin, supports this idea [31] (1).

(1) Cf. also A. Trautman, *Spin and torsion may avert gravitational singularities*, Nature (Phys.), vol. 242, no. 114 (1973) (note added in proof).

For a body with given values of spin and mass, the dimensionless numbers characterizing the order of magnitude of the effects of torsion and of curvature are, respectively,

$$\text{spin}/(\text{radius})^2 \quad \text{and} \quad \text{mass}/\text{radius}\,.$$

(We use a system of units in which the gravitational constant and the velocity of light are equal to 1.) For an electron, the ratio of these two (very small) numbers is of the order of $1/\alpha \approx 137$; the influence of spin on geometry is larger than that of mass. This is no longer so for matter in bulk because mass is essentially additive whereas in most circumstances spins cancel out one another.

A linear connection with torsion has been often used in attempts to construct a unified theory of gravitation and electromagnetism [16], [17]. The Einstein-Cartan theory, as it is understood in this paper, is a purely gravitational theory in the sense that the electromagnetic field is considered here as influencing the geometry rather than being itself part of the geometry. Similarly as in Einstein's theory, the Maxwell stress-energy tensor appears in the equations as a source of the gravitational field.

The following is a heuristic argument to support the Cartan proposal: by the theorems on holonomy, curvature and torsion are related, respectively, to the groups of homogeneous transformations and of translations in the tangent spaces of a manifold. In the approximation of special relativity, the group of inhomogeneous Lorentz transformations and its invariants (mass and spin) play a fundamental role in the description of elementary physical phenomena. In Einstein's theory of general relativity, mass directly influences curvature but spin has no similar dynamical effect. As a result of the absence of torsion, the infinitesimal holonomy groups [18] of the Cartan connection [19] of an Einstein space consist of only homogeneous transformations. By introducing torsion and relating it to spin, one obtains an interesting link between the theory of gravitation and the theory of special relativity [20]. This argument, which is an extension of the ideas of D. W. Sciama [21] has been summarized in the Table.

A very simple example of a space with torsion was given in the same paper where this notion was mentioned for the first time [11]. The example, which motivated Cartan in introducing the term « torsion », may be described as follows. Consider a three-dimensional, oriented Euclidean space E. Given a number $\lambda > 0$, one defines parallel translation of vectors in E as consisting in a helicoidal motion with pitch λ and in agreement with the preferred orientation. This defines on E

Table

Invariance under	Is related by Noether's theorem to	The equations due to	Have as source a vector-density of the quantity appearing in column two. They determine the	The holonomy group associated to the latter consists of
phase transformations	charge	Maxwell	electromagnetic field	phase changes [22]
translations	energy and momentum	Einstein	curvature	Lorentz transformations
the Lorentz group	spin	Cartan	torsion	translations

a new, metric ([2]) linear connection, different from the (flat) connection determined by the affine structure of E. The curvature and torsion forms of the new connection are, respectively,

$$\Omega^{kl} = \lambda^2 \vartheta^l \wedge \vartheta^k \qquad \text{and} \qquad \Theta_i = \lambda \varepsilon_{ikl} \vartheta^k \wedge \vartheta^l$$

where (ϑ^k) is an orthonormal, properly oriented frame in the space of 1-forms belonging to E (cf. Section 2) and ε_{jkl} is the Levi-Civita symbol, $\varepsilon_{jkl} = \varepsilon_{[jkl]}$, $\varepsilon_{123} = 1$. Since in this particular case torsion is associated with a preferred orientation of E, it has been suggested that torsion of space-time may be responsible for the phenomenon of violation of parity occurring in weak interactions [23].

2. Cartan calculus.

This section contains a brief summary of the calculus of tensor-valued exterior forms. A linear connection on a manifold, its curvature and torsion, and the corresponding notion of covariant exterior differentiation are introduced in a manner closely following the original expositions of E. Cartan [12], [24]. In the Appendix, there is presented a modern outline of the relation between curvature, torsion, and the affine group.

([2]) The adjective *metric* is given here the meaning accepted in tensor calculus: a linear connection is metric if the scalar product of any two vectors is unchanged by parallel transport of these vectors.

Let X be an n-dimensional differentiable manifold of class C^∞. All maps and fields on X are also restricted to be of this class. All geometric objects on X, other than forms, will be described by their components with respect to a field (ϑ^k), $k = 1, \ldots, n$, of frames in the cotangent spaces of X [18]. For example, if (x^k) is a system of local coordinates defined on an open set $U \subset X$, then one has a field of (holonomic) frames defined on U by $\vartheta^k = dx^k$. In general, the frames (ϑ^k) are aholonomic, $d\vartheta^k \neq 0$. The metric tensor is written as

$$g_{kl}\vartheta^k \otimes \vartheta^l.$$

A *linear connection* on X is described, with respect to (ϑ^k), by a collection $(\omega^k{}_l)$ of one-forms defining the covariant derivative. Under the replacement of ϑ^k by ϑ'^k, where $\vartheta^k = A^k{}_l \vartheta'^l$ and $(A^k{}_l) = A: X \to \to GL(n, R)$, the connection forms change according to

$$A^k{}_m \omega'^m{}_l = \omega^k{}_m A^m{}_l + dA^k{}_l \tag{1}$$

and the components of the metric become

$$g'_{kl} = g_{mn} A^m{}_k A^n{}_l\,. \tag{2}$$

Let $\sigma: GL(n, R) \to GL(N, R)$ be a homomorphism of Lie groups. For any $a \in GL(n, R)$, $\sigma(a)$ is a non-singular N by N matrix with elements denoted by $\sigma_A^B(a)$; $A, B = 1, \ldots, N$. The derived homomorphism of Lie algebras, $\sigma': \mathscr{L}(R^n) \to \mathscr{L}(R^N)$ may be represented by the matrix $\sigma_k{}^l = (\sigma_{Ak}^{Bl})$, where

$$\sigma_{Ak}^{Bl} = \left.\frac{\partial \sigma_A^B(a)}{\partial a^k{}_l}\right|_{a^m{}_n = \delta^m_n}.$$

It follows from $\sigma_A^C(a)\sigma_C^B(b) = \sigma_A^B(ab)$ that

$$\sigma_k{}^l \sigma_m{}^n - \sigma_m{}^n \sigma_k{}^l = \delta^l_m \sigma_k{}^n - \delta^n_k \sigma_m{}^l. \tag{3}$$

A *p-form of type* σ on X is a law which associates to each field of frames (ϑ^k) a set $\varphi = (\varphi_A)$ of N fields of p-forms, in such a way that to (ϑ'^k) there correspond the fields $\varphi' = (\varphi'_A)$ given by

$$\varphi = (\sigma \circ A)\varphi'$$

where $\sigma \circ A$ is the composition of maps

$$X \xrightarrow{A} GL(n, R) \xrightarrow{\sigma} GL(N, R)\,.$$

If the frames (ϑ'^k) differ « infinitesimally » from (ϑ^k),

$$\vartheta'^k = \vartheta^k + \delta\vartheta^k = \vartheta^k - \alpha^k{}_l \vartheta^l,$$

$$(\alpha^k{}_l) = \alpha: X \to \mathscr{L}(R^n)\,,$$

then $\varphi'_A = \varphi_A + \delta\varphi_A$, where

(4) $$\delta\varphi_A = -\sigma^{Bl}_{Ak}\alpha^k{}_l\varphi_B .$$

Examples of p-forms of type σ.

1) Let id: $GL(n, R) \to GL(n, R)$ be the identity, $\mathrm{id}(a) = a$. The frames (ϑ^k) themselves constitute a 1-form of type id.

2) A 0-form of type id is called a (contravariant) vector field on X.

3) Let 0 denote the trivial representation of $GL(n, R)$, $0(a) = 1$. A p-form φ of type 0, or a scalar-valued p-form, is an ordinary (exterior) differential form of degree p. Its exterior derivative $d\varphi$ is a $(p+1)$-form of type 0. If φ and ψ are scalar-valued forms of degree p and q, respectively, then their exterior product $\varphi\wedge\psi$ is a $(p+q)$-form of type 0 and $d(\varphi\wedge\psi) = d\varphi\wedge\psi + (-1)^p\varphi\wedge d\psi$.

4) The adjoint representation ad: $GL(n, R) \to GL(\mathscr{L}(R^n))$ is defined by $\mathrm{ad}(a)(\alpha) = a\alpha a^{-1}$ (multiplication of matrices, $\alpha \in \mathscr{L}(R^n)$).

5) Let ${}^t a$ be the transpose of the matrix a and $\breve{a} = {}^t a^{-1}$. The representation $\breve{\sigma}$: $GL(n, R) \to GL(N, R)$, where $\breve{\sigma}(a) = \sigma(\breve{a})$ is called contragredient with respect to σ.

6) A pseudoscalar is a 0-form of type π, where $\pi(a) = \mathrm{sgn}\det a$.

7) If σ is a tensorial representation of $GL(n, R)$, then a p-form of type σ (respectively, of type $\pi\otimes\sigma$) is said to be tensor-valued (respectively, pseudotensor-valued). A tensor is a tensor-valued 0-form, etc.

The *covariant exterior derivative* of a p-form $\varphi = (\varphi_A)$ of type σ, with respect to the connection (ω^{kl}), is a $(p+1)$-form of type σ, given by

(5) $$D\varphi_A = d\varphi_A + \sigma^{Bl}_{Ak}\omega^k{}_l\wedge\varphi_B$$

or, for short,

$$D\varphi = d\varphi + \sigma_k{}^l\omega^k{}_l\wedge\varphi$$

where the indices A and B have been suppressed in a manner similar to what is done in the calculus of Dirac spinors. For a tensor field (φ_A), $D\varphi_A = \vartheta^k\nabla_k\varphi_A$ is the usual covariant derivative, whereas for a scalar-valued form φ, $D\varphi$ reduces to the exterior derivative $d\varphi$. If (u^k) is a vector field, then

(6) $$Du^k = du^k + \omega^k{}_l u^l.$$

A vector field (u^k) is said to be *constant* if

$$Du^k = 0;$$

it is called a *radius-vector* field if

$$Du^k + \vartheta^k = 0 .$$

If (u^k) is constant, then, by exterior differentiation of the right-hand side of eqn. (6), one obtains

$$\Omega^k{}_l u^l = 0$$

where

(7) $$\Omega^k{}_l = d\omega^k{}_l + \omega^k{}_m\wedge\omega^m{}_l$$

is a two-form of type ad, called the *curvature form* of $(\omega^k{}_l)$. Similarly, if (u^k) is a radius-vector, then

$$\Omega^k{}_l u^l + \Theta^k = 0$$

where

(8) $$\Theta^k = d\vartheta^k + \omega^k{}_l\wedge\vartheta^l \quad \text{or } D\vartheta^k$$

is a two-form of type id, called the *torsion form* of $(\omega^k{}_l)$. It is easy to prove the following

THEOREM 1: Let X be an n-dimensional manifold with a linear connection. Any point of X has a neighbourhood admitting n linearly independent constant vector fields if and only if $\Omega^k{}_l = 0$. If $\Omega^k{}_l = 0$, then $\Theta^j = 0$ is equivalent to the existence of a radius-vector field defined over some neighbourhood of every point of X.

From the definition of the covariant exterior derivative there follows

(9) $$D^2\varphi = \sigma_k{}^l\Omega^k{}_l\wedge\varphi .$$

If φ is a tensor ($p = 0$), then by differentiating the formula

$$D\varphi = \vartheta^k\nabla_k\varphi$$

one obtains

$$D^2\varphi = \Theta^k\nabla_k\varphi + \vartheta^k\wedge\vartheta^l\nabla_k\nabla_l\varphi$$

or

$$\vartheta^k \wedge \vartheta^l \nabla_{[k} \nabla_{l]} \varphi = \sigma_m{}^n \Omega^m{}_n \varphi - \Theta^m \nabla_m \varphi .$$

Eqn. (9) may be used to derive the *Bianchi identity* for the torsion form,

$$D\Theta^k = \Omega^k{}_l \wedge \vartheta^l. \tag{10}$$

By a direct computation one obtains the other identity

$$D\Omega^k{}_l = 0 . \tag{11}$$

It is often convenient to consider the 1-forms and 0-forms associated to Θ^j and $\Omega^k{}_l$ by the formulae

$$\Theta^j = \tfrac{1}{2}\vartheta^k \wedge Q^j{}_k = \tfrac{1}{2} Q^j{}_{kl} \vartheta^k \wedge \vartheta^l ,$$

$$\Omega^k{}_l = \tfrac{1}{2}\vartheta^m \wedge R^k{}_{lm} = \tfrac{1}{2} R^k{}_{lmn} \vartheta^m \wedge \vartheta^n .$$

(Q^j_{kl}) and $(R^k{}_{lmn})$ are the *torsion tensor* and the *curvature tensor* of the connection, respectively.

If the manifold X is *four-dimensional* and has a *metric tensor*, it is possible to introduce the completely antisymmetric pseudo-tensor η_{ijkl}, where $\eta_{1234} = |\det g_{mn}|^{\frac{1}{2}}$. Together with η_{ijkl}, the forms

$$\eta_{ijk} = \vartheta^l \eta_{ijkl} , \qquad \eta_{ij} = \tfrac{1}{2}\vartheta^k \wedge \eta_{ijk} ,$$

$$\eta_i = \tfrac{1}{3}\vartheta^j \wedge \eta_{ij} , \qquad \eta = \tfrac{1}{4}\vartheta^i \wedge \eta_i$$

span the Grassmann algebra of X and

$$\begin{cases} \vartheta^m \eta_{ijkl} = \delta^m_l \eta_{ijk} - \delta^m_k \eta_{lij} + \delta^m_j \eta_{kli} - \delta^m_i \eta_{jkl} , \\ \vartheta^l \wedge \eta_{ijk} = \delta^l_k \eta_{ij} + \delta^l_j \eta_{ki} + \delta^l_i \eta_{jk} , \\ \vartheta^k \wedge \eta_{ij} = \delta^k_j \eta_i - \delta^k_i \eta_j , \\ \vartheta^j \wedge \eta_i = \delta^j_i \eta . \end{cases} \tag{12}$$

The metric tensor may be used to raise indices,

$$\eta^i = g^{ij}\eta_j , \qquad \eta_i{}^j = g^{jl}\eta_{il} , \qquad \eta^{ij} = g^{ik} g^{jl} \eta_{kl} ,$$

etc. The forms

$$\eta , \quad \eta^i , \quad \eta^{ij} , \quad \eta^{ijk} , \quad \eta^{ijkl}$$

are the *duals* of

$$1 , \quad \vartheta^i , \quad \vartheta^i \wedge \vartheta^j , \quad \vartheta^i \wedge \vartheta^j \wedge \vartheta^k , \quad \vartheta^i \wedge \vartheta^j \wedge \vartheta^k \wedge \vartheta^l ,$$

respectively. According to our definitions, $\eta^{1234} = - |\det g_{mn}|^{-\frac{1}{2}}$. If X has a *metric linear connection*, $Dg_{kl} = 0$, then

$$\begin{cases} D\eta_{ijkl} = 0 , \\ D\eta_{ijk} = \Theta^l \eta_{ijkl} , \\ D\eta_{ij} = \Theta^k \wedge \eta_{ijk} = (Q^k_{ij} - \delta^k_i Q^l_{lj} - \delta^k_j Q^l_{il}) \eta_k , \\ D\eta_i = \Theta^j \wedge \eta_{ij} = Q^j_{ij} \eta . \end{cases} \tag{13}$$

A field of frames (ϑ^i) is *holonomic* if

$$d\vartheta^i = 0 .$$

It is called *harmonic* if it is holonomic and satisfies the *de Donder condition*

$$d\eta^i = 0 .$$

In a Riemannian space, $D\eta^i = 0$, and the de Donder condition reduces to an equation often used by Fock [25]

$$\omega^i{}_j \wedge \eta^j = 0 .$$

Tensor-valued forms are well-suited to describe densities and currents of physical quantities such as charge, energy-momentum and angular momentum. For example, the conservation law of energy and momentum in a Riemannian space-time may be simply written as

$$Dt_k = 0$$

where $t_k = \eta_l t^l{}_k$ is a 3-form associated with the stress-energy tensor $t^l{}_k$. The symmetry of t^{kl} is equivalent to

$$\vartheta^k \wedge t^l = \vartheta^l \wedge t^k .$$

The following lemma will be used in Section 3 and 5:

LEMMA: If $(\lambda_i{}^j)$ is a collection of 1-forms, defined on a four-dimensional manifold with a metric tensor, and

$$\lambda_i{}^k \wedge \eta_k{}^j - \eta_i{}^k \wedge \lambda_k{}^j = 0 ,$$

then there exists a scalar-valued 1-form λ such that $\lambda_i{}^j = \delta^j_i \lambda$.

3. The field equations in empty space ([3]).

Let us consider a four-dimensional manifold X with a metric tensor and a linear connection which, to begin with, are unrelated to each other. The 4-form

$$8\pi K = \tfrac{1}{2}\eta_k{}^l \wedge \Omega^k{}_l$$

is independent of the choice of the frames (ϑ^i) and, therefore, is defined globally on X. On a Riemannian space-time, the form K reduces to the integrand of the variational principle used to derive Einstein's equations.

By varying the metric, the frames, and the linear connection independently of one another, we obtain

$$8\pi\delta K = \tfrac{1}{2}E^{ij}\delta g_{ij} + \delta\vartheta^i \wedge e_i - \tfrac{1}{2}\delta\omega^i{}_j \wedge c_i{}^j + \text{an exact form}$$

where

$$E^{ij} = \tfrac{1}{2}(g^{ij}\eta_l{}^k - g^{ik}\eta_l{}^j - g^{jk}\eta_l{}^i) \wedge \Omega^l{}_k \tag{14}$$

is the (generalized) *Einstein tensor* (-valued 4-form),

$$e_i = \tfrac{1}{2}\eta_{il}{}^k \wedge \Omega^l{}_k \quad \text{and} \quad c_i{}^j = -D\eta_i{}^j. \tag{15}$$

The Einstein tensor is symmetric, even in the non-Riemannian case, whereas $e_i = g_{ij}e^j$ is not, $e^i \wedge \vartheta^j \neq e^j \wedge \vartheta^i$.

According to (1) and (2), a variation of the frames $\delta\vartheta^i = -\alpha^i{}_j\vartheta^j$ induces the following changes in the connection forms and the components of the metric, the connection and the metric themselves being kept fixed,

$$\delta\omega^i{}_j = D\alpha^i{}_j, \qquad \delta g_{ij} = \alpha_{ij} + \alpha_{ji}.$$

These changes do not affect K; the equation $\delta K = 0$ yields the identity

$$E_i{}^j = \vartheta^j \wedge e_i - \tfrac{1}{2}Dc_i{}^j. \tag{16}$$

From the principle of least action, $\delta\int K = 0$, by varying with re-

([3]) Sections 3-7 are based on [26] but some of the quantities are defined here with a different sign than there.

spect to $(g_{ij}, \omega^k{}_l)$ and $(\vartheta^i, \omega^k{}_l)$, one obtains two sets of equations,

$$E^{ij} = 0, \qquad c_k{}^l = 0, \tag{17}$$

and

$$e_i = 0, \qquad c_k{}^l = 0. \tag{18}$$

It follows from Eq. (16) that these two sets are equivalent to each other.

The lagrangian form K is invariant under the «projective transformation» of the connection, i.e. under the replacement of $\omega^i{}_j$ by $\omega^i{}_j + \delta^i_j\lambda$, where λ is any 1-form on X. This implies the identity $c^k{}_k = 0$ and makes it impossible to determine, in a unique manner, the connection from Eqs. (17) or (18). By using the Lemma, one proves

THEOREM 2: For any metric tensor (g_{ij}), the equation $c_k{}^l = 0$ is equivalent to

$$\omega^i{}_j = \gamma^i{}_j + \delta^i_j\lambda$$

where $\gamma^i{}_j$ are the forms of the Riemannian connection of (g_{ij}) and λ is a 1-form.

This leads to

COROLLARY: The following three conditions are equivalent to one another: (*a*) $\omega^i{}_j = \gamma^i{}_j$, (*b*) $c_k{}^l = 0$ and $\Theta^i = 0$, (*c*) $c_k{}^l = 0$ and $Dg_{ij} = 0$.

The equivalence of (*a*) and (*b*) is due to Palatini [27].

The formulae (12) may be used to find the expressions of $E_i{}^j$ and e_i in terms of the components of the curvature tensor:

$$E_i{}^j = \tfrac{1}{2}(\delta^j_i R^{kl}{}_{kl} - R_{ki}{}^{kj} - R^{kj}{}_{ki})\eta, \tag{19}$$

$$e_i = \tfrac{1}{2}(\delta^j_i R^{kl}{}_{kl} - R^{kj}{}_{ki} - R^{jk}{}_{ik})\eta_j. \tag{20}$$

4. A classical field interacting with gravitation.

A classical field, such as the electromagnetic field, may be regarded as a model of a physical system interacting with gravitation. In this case, the equations of motion may be derived from a principle of least action and their formal properties analyzed in detail. To describe the field, let us consider a tensor-valued p-form (φ_A) of type σ and assume a lagrangian 4-form L depending, in a local manner, on g_{ij}, ϑ^k, φ_A

and $D\varphi_A$. Independent variation of the variables leads to

$$\delta L = \tfrac{1}{2} T^{ij} \delta g_{ij} + \delta\vartheta^i \wedge t_i - \tfrac{1}{2} \delta\omega^i{}_j \wedge s_i{}^j + L^A \wedge \delta\varphi_A + \text{an exact form}\,.$$

If all the variations are induced by a mere change of the frames, then $\delta L = 0$ and an argument similar to the one used in the preceding section leads to the identity

$$T_i{}^j = \vartheta^j \wedge t_i - \tfrac{1}{2} D s_i{}^j + \sigma^{Bj}_{Ai} L^A \wedge \varphi_B\,. \tag{21}$$

By varying the total action $\int (K + L)$ with respect to $(\varphi_A, g_{ij}, \omega^k{}_l)$ and $(\varphi_A, \vartheta^i, \omega^k{}_l)$, one obtains two sets of equations:

$$L^A = 0\,, \qquad E^{ij} = -8\pi T^{ij}, \qquad c_k{}^l = -8\pi s_k{}^l \tag{22}$$

and

$$L^A = 0\,, \qquad e_i = -8\pi t_i\,, \qquad c_k{}^l = -8\pi s_k{}^l, \tag{23}$$

which are equivalent to each other by virtue of the identities (16) and (21). This shows that, to write the field equations in the Einstein-Cartan theory, one is free to use either the symmetric, tensor-valued 4-form of energy and momentum (T^{ij}) or the « canonical », asymmetric, vector-valued 3-form (t_i).

The electromagnetic potential should be defined as a scalar-valued 1-form φ so that the field be $F = d\varphi$. The alternative identification of the potential with a covector-valued 0-form (φ_i) would lead to the field $(D\varphi_i) \wedge \vartheta^i$ which is not gauge-invariant in the presence of torsion. Any scalar-valued form leads to $s_i{}^j = 0$ and $T_i{}^j = \vartheta^j \wedge t_i$. Therefore a pure electromagnetic field cannot be the source of torsion, a fact which is hardly surprising if one remembers the non-local character of the spin of a photon. It is amusing to note that the lagrangian of both the massless scalar and the electromagnetic fields can be represented by one formula, $8\pi L = -{*(d\varphi)} \wedge d\varphi$, where star denotes the dual of a form and φ is a 0-form (scalar theory) or a 1-form (electromagnetism).

5. The metric theory.

The relevance of spinors in physics indicates that the linear connection on space-time is compatible with the metric tensor, i.e., that $Dg_{ij} = 0$. Otherwise, there would be no natural lift of the linear connection to a connection on the bundle of spinor frames. By assuming

that the linear connection is metric, as was done by Cartan and the other authors [13-15], it is possible to remove the freedom of projective transformations, inherent in the non-metric theory of section 3. The Lemma of section 2 is useful in the proof of

THEOREM 3: Let (g_{ij}) be a metric tensor and let $(s_k{}^l)$ be a 3-form of type ad, defined on X. Among the linear connections on X, satisfying the Cartan equation

$$c_k{}^l = -8\pi s_k{}^l$$

there is exactly one such that

$$Dg_{ij} = 0 \tag{24}$$

if and only if

$$s_{kl} + s_{lk} = 0\,. \tag{25}$$

The metric conditions (24) and (25), which are assumed to hold from now on, imply

$$D\eta_{ijkl} = 0\,, \qquad \Omega_{ij} + \Omega_{ji} = 0$$

and lead to the following simple form of the *Einstein-Cartan equations*:

$$e_i = \tfrac{1}{2} \eta_{ijk} \wedge \Omega^{jk} = -8\pi t_i\,, \tag{26}$$

$$c_{ij} = -\eta_{ijk} \wedge \Theta^k = -8\pi s_{ij}\,. \tag{27}$$

Introducing the notation

$$t_i = \eta_j t^j{}_i\,, \qquad s_{ij} = \eta_k s^k{}_{ij}$$

$$R^j{}_i = R^{jk}{}_{ik}\,, \qquad R = R^k{}_k$$

eqns. (26) and (27) can be written in terms of components in the form given by Kibble

$$R^j{}_i - \tfrac{1}{2} \delta^j_i R = 8\pi t^j{}_i\,, \tag{28}$$

$$Q^k_{ij} - \delta^k_i Q^l_{lj} - \delta^k_j Q^l_{il} = 8\pi s^k_{ij}\,. \tag{29}$$

The last equation can be solved with respect to the components of the torsion tensor (4)

$$Q^k_{ij} = 8\pi(s^k_{ij} - \tfrac{1}{2}\delta^k_i s^l_{lj} - \tfrac{1}{2}\delta^k_j s^l_{il}) .$$

In the approximation of special relativity, there exists a radius vector $(-x^i)$, $Dx^i = \vartheta^i$, and energy and momentum are conserved, $Dt_i = 0$. Under these assumptions, from eqn. (21) there follows the conservation law of total angular momentum:

$$\text{if } L^A = 0 , \qquad \text{then } D(x^i t^j - x^j t^i + s^{ij}) = 0 .$$

The 3-forms $(x^i t^j - x^j t^i)$ and (s^{ij}) are interpreted as the density of orbital angular momentum and of spin, respectively.

6. The differential identities.

The Bianchi identities (10) and (11) may be used to evaluate the covariant exterior derivatives of the 3-forms e_j and c_{kl} appearing in the Einstein-Cartan equations of the generalized, metric theory of gravitation. It follows directly from eqns. (26) and (27) that

$$De_j = \tfrac{1}{2}\eta_{jklm}\Theta^m \wedge \Omega^{kl}, \tag{30}$$

$$Dc_{kl} = \eta_{kj}\wedge \Omega^j{}_l - \eta_{lj}\wedge\Omega^j{}_k . \tag{31}$$

Eqn. (30) is the generalization, to the Riemann-Cartan space, of the « contracted Bianchi identity » $De_j = 0$ which plays an important role in Einstein's theory of gravitation. The right hand sides of eqns. (30) and (31) may be rearranged to give the formulae

$$De_j = Q^k_j \wedge e_k - \tfrac{1}{2}R^{kl}{}_j \wedge c_{kl} , \tag{32}$$

$$Dc_{kl} = \vartheta_l \wedge e_k - \vartheta_k \wedge e_l . \tag{33}$$

Let σ be a representation of the Lorentz group in R^N and let (σ^{Bl}_{Ak}) be the matrix corresponding to the derived homomorphism of Lie algebras. The matrix $\sigma_{kl} = (\sigma^B_{Akl})$, $\sigma^B_{Akl} = g_{jl}\sigma^{Bj}_{Ak}$ is skew, $\sigma_{kl} + \sigma_{lk} = 0$. In the metric theory, the forms $\omega^{kl} = g^{jl}\omega^k{}_j$ of the connection, referred to a field (ϑ^k) of orthonormal frames are also skew in the pair (k, l). The covariant derivative of a tensor-valued 0-form of type σ,

(4) The corresponding formula in [26] is misprinted.

$\varphi = (\varphi_A)$, is given by (cf. section 2)

$$D\varphi = d\varphi + \omega^{kl}\sigma_{kl}\varphi = \vartheta^j\nabla_j\varphi .$$

The indices $A, B = 1, \ldots, N$ have been suppressed in this formula, which should be interpreted similarly as in spinor calculus.

Let the lagrangian form corresponding to φ be $L = \eta\Lambda(\varphi, \nabla\varphi)$ where Λ is a function on $R^N \times \mathscr{L}(R^4, R^N)$. In this formulation, where everything is referred to orthonormal frames, there is no possibility for considering changes of g_{ij}. A general variation of L is

$$\delta L = \delta\vartheta^j \wedge t_j + \tfrac{1}{2}s_{kl}\wedge\delta\omega^{kl} + L^A\delta\varphi_A + \text{an exact form} ,$$

where

$$t_j = \left(\Lambda\delta^k_j - \frac{\partial\Lambda}{\partial\nabla_k\varphi}\nabla_j\varphi\right)\eta_k , \qquad s_{kl} = -2\frac{\partial\Lambda}{\partial\nabla_j\varphi}\sigma_{kl}\varphi\eta_j \tag{34}$$

and

$$L^A = \eta\frac{\partial\Lambda}{\partial\varphi_A} - D\left(\eta_j\frac{\partial\Lambda}{\partial\nabla_j\varphi}\right).$$

A mere Lorentz rotation of the frames leaves both L and Λ invariant. This invariance results in the equation

$$Ds_{kl} = \vartheta_l\wedge t_k - \vartheta_k\wedge t_l , \qquad \mathrm{mod}\, L^A = 0 \tag{35}$$

proving the symmetry of the tensor T_{kl} which should now be considered as defined by eqn. (21). The covariant exterior derivative of t_j may be evaluated from eqn. (34)

$$Dt_j = Q^k_j\wedge t_k - \tfrac{1}{2}R^{kl}{}_j\wedge s_{kl} , \qquad \mathrm{mod}\, L^A = 0 . \tag{36}$$

A simple comparison of eqns. (23) with (32) (33), (35) and (36) leads to

THEOREM 4: The equations resulting by covariant exterior derivation of the Einstein-Cartan equations are algebraic consequences of the Einstein-Cartan equations themselves and of the equations of motion of matter.

The implications of this result should be compared with, and distinguished from, the point of view of Cartan who required that Dt_j be zero ([28], pp. 21-23).

7. The Dirac equation in a space-time with torsion.

To illustrate the general formalism developed in the preceding sections, we give here a brief account of the Dirac equation appropriate to a Riemann-Cartan space-time. A detailed and different treatment of this subject may be found elsewhere [29], [30].

Let X be a four-dimensional differential manifold endowed with a metric tensor g of hyperbolic signature. A simple-minded, physicist's approach to Dirac spinors on X may be summarized as follows. The Dirac matrices $\gamma_k \in \mathscr{L}(C^4)$ satisfy

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2g_{kl}$$

and $\beta = \beta^\dagger$ is a matrix such that

$$\gamma_k^\dagger = \beta \gamma_k \beta^{-1}$$

where cross denotes hermitian conjugation. The 6 spin matrices $\sigma_{kl} = \frac{1}{8}(\gamma_k \gamma_l - \gamma_l \gamma_k)$ satisfy

$$\sigma_{kl}\gamma_j - \gamma_j \sigma_{kl} = \tfrac{1}{2}(g_{jl}\gamma_k - g_{jk}\gamma_l) \tag{37}$$

$$\sigma_{kl}\gamma_j + \gamma_j \sigma_{kl} = \tfrac{1}{2}\eta_{jklm}\gamma_5 \gamma^m \tag{38}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma^k = g^{kl}\gamma_l$, $\eta_{1234} = 1$, etc. Space-time is assumed to have a global field of orthonormal frames (ϑ^k). One considers only those fields of orthonormal frames (ϑ'^k) which differ « little » from a given field (ϑ^k),

$$\vartheta'^k = \vartheta^k + \delta\vartheta^k = \vartheta^k - \alpha^k{}_l \vartheta^l \quad \text{and} \quad \alpha_{kl} + \alpha_{lk} = 0 \,.$$

A spinor field on X is a law which associates to each of the fields of frames a map from X to C^4 in such a way that if $\psi: X \to C^4$ corresponds to (ϑ^k) and $\psi' = \psi + \delta\psi$ corresponds to (ϑ'^k), then

$$\delta\psi = -\alpha^{kl}\sigma_{kl}\psi \,.$$

Similarly, for the contragredient spinor $\bar\psi = \psi^+\beta$

$$\delta\bar\psi = \bar\psi\sigma_{kl}\alpha^{kl}.$$

Let X be endowed, in addition to the metric tensor, with a metric affine connection described by the collection (ω_{kl}) of 1-forms. The

covariant derivative $D\psi$ of a spinor field ψ is a spinor-valued 1-form

$$D\psi = d\psi + \omega^{kl}\sigma_{kl}\psi \,.$$

The corresponding formula for the derivative of the contragredient spinor is

$$D\bar\psi = d\bar\psi - \bar\psi\sigma_{kl}\omega^{kl}.$$

The covariant exterior derivative of $\iota = \gamma_k \eta^k$ (the dual of $\gamma = \gamma_k \theta^k$),

$$D\iota = \iota \wedge Q^k_k$$

depends on the trace of the torsion tensor. If $*D\psi$ denotes the dual of $D\psi$, then

$$-\iota \wedge D\psi = \gamma \wedge {*D\psi} = \eta\gamma^k \nabla_k \psi \,. \tag{39}$$

The lagrangian form corresponding to a Dirac particle of mass m is

$$L = \frac{i}{2}(\bar\psi\iota \wedge D\psi + D\bar\psi \wedge \iota\psi) + m\eta\bar\psi\psi \,.$$

The Dirac equation obtained by varying the action integral with respect to $\bar\psi$ or ψ,

$$\iota \wedge D\psi - D(\iota\psi) = 2im\eta\psi \tag{40}$$

$$(D\bar\psi)\wedge\iota + D(\bar\psi\iota) = 2im\eta\bar\psi \tag{41}$$

implies the conservation law $dj = 0$, where $j = \bar\psi\iota\psi$ is the current. By varying L with respect to ϑ^j and ω^{kl}, and making use of eqns. (37)-(41), one obtains

$$t_j = \frac{i}{2}({*D\bar\psi}\gamma_j\psi - \bar\psi\gamma_j{*D\psi}) \tag{42}$$

and

$$\begin{aligned} s_{kl} &= i\bar\psi(\iota\sigma_{kl} + \sigma_{kl}\,\iota)\psi \\ &= -\frac{i}{2}\bar\psi\gamma_5\gamma\psi \wedge \vartheta_k \wedge \vartheta_l \,. \end{aligned} \tag{43}$$

APPENDIX

This section contains a brief description of the relation between curvature, torsion and the affine group. Ordinary curvature and torsion may be considered, respectively, as the homogeneous and translational components of the generalized curvature (A.7) associated to the Cartan connection (A.6) defined on the bundle of affine frames [19], [32].

Algebraic preliminary: Let V be an affine space and let $\boldsymbol{V}$ be the corresponding vector space of translations of V. An affine transformation is a bijection $f\colon V \to V$ for which there exists a linear automorphism $sf \in GL(\boldsymbol{V})$ such that, for any $q \in \boldsymbol{V}$, $p \in V$

$$f(q + p) = sf(q) + f(p)\,.$$

The set $GA(V)$ of all affine transformations has the structure of a group, called the *general affine group* of V. For any $q \in \boldsymbol{V}$, the translation tq defined by $tq(p) = q + p$ is an affine transformation. The exact sequence

$$0 \to \boldsymbol{V} \xrightarrow{t} GA(V) \xrightarrow{s} GL(\boldsymbol{V}) \to 1$$

splits; i.e., $GA(V)$ is a semi-direct product of $GL(\boldsymbol{V})$ by $\boldsymbol{V}$. For $V = R^n = \boldsymbol{V}$ we write $GL(\boldsymbol{V}) = GL(n, R)$ and $GA(V) = GA(n, R)$. The latter group may be identified with $GL(n, R) \times R^n$, where the composition of elements is given by

$$(a, q)(a', q') = (aa', q + aq')\,,$$

$a, a' \in GL(n, R)$ and $q, q' \in R^n$. The Lie algebra of $GA(n, R)$ may be identified with $\mathscr{L}(R^n) \times R^n$, and the bracket is

$$[(\alpha, q), (\alpha', q')] = ([\alpha, \alpha'], \alpha q' - \alpha' q)\,, \tag{A.1}$$

where $\alpha, \alpha' \in \mathscr{L}(R^n)$. The adjoint representation of $GA(n, R)$ in its Lie algebra is given by $Ad_{(a,q)}(\alpha, r) = (a\alpha a^{-1}, ar - a\alpha a^{-1} q)$.

An affine frame in V is a pair (e, p), where $p \in V$ and $e = (e_i) \in FL(\boldsymbol{V})$ is a linear frame in $\boldsymbol{V}$. The affine group $GA(n, R)$ acts freely and transitively in the set $FA(V)$ of all affine frames of an n-dimensional affine space V. The action of the group may be defined on the right to be

$$(e, p)(a, q) = (ea, eq + p) \tag{A.2}$$

where $(ea)_i = e_j a^j{}_i$ and $eq = e_i q^i$.

The bundles of frames: The bundle $\pi_L\colon L(X) \to X$ *of linear frames* of an n-dimensional differentiable manifold X is a principal bundle with structure group $GL(n, R)$. The action of $GL(n, R)$ in $L(X)$ is given by

$$\varphi_a(e) = ea$$

where e is a linear frame and $a \in GL(n, R)$. The canonical 1-form ϑ on $L(X)$ is a map

$$\vartheta\colon TL(X) \to R^n$$

linear on the fibres of the tangent bundle $TL(X) \to L(X)$ and defined as follows. If $u \in T_e L(X)$, then

$$e\vartheta(u) = T\pi_L(u)\,,$$

where $T\pi_L$ is the tangent (derived) map of π_L and $e \in FL(T_x X)$. Clearly, for any $a \in GL(n, R)$,

$$\vartheta \circ T\varphi_a = a^{-1}\vartheta\,.$$

The bundle $\pi_A\colon A(X) \to X$ of *affine frames* is a principal bundle with structure group $GA(n, R)$. Its total space is $A(X) = \bigcup_{x \in X} FA(T_x X)$ and the action of the group

$$\tilde{\varphi}_{(a,q)}(e, p) = (e, p)(a, q)$$

is defined by (A.2), $p \in T_x X$, $e \in FL(T_x X)$.
The affine bundle is an extension of the linear bundle: the projection π_A factors through π_L, $\pi_A = \pi_L \circ \pi$, where $\pi(e, p) = e$ and the diagram

$$\begin{array}{ccc} A(X) \times GA(n, R) & \xrightarrow{\tilde{\varphi}} & A(X) \\ {\scriptstyle \pi \times s}\downarrow & & \downarrow{\scriptstyle \pi} \\ L(X) \times GL(n, R) & \xrightarrow{\varphi} & L(X) \end{array}$$

is commutative. On the bundle $A(X)$ there is defined a canonical map

$$(\varrho^i) = \varrho\colon A(X) \to R^n$$

given by the formula

$$e\varrho(e, p) = p\,.$$

In other words, $\varrho^i(e, p)$ is the i-th component of the vector p with respect to the frame $e = (e_i)$.

If $(a, q) \in GA(n, R)$, then

$$\varrho \circ \tilde{\varphi}_{(a,q)} = a^{-1}(\varrho + q) \tag{A.3}$$

where q on the right is a constant map from $A(X)$ to R^n.

A vector field v on X induces an embedding $\bar{v}$ of $L(X)$ in $A(X)$ defined by

$$\bar{v}(e) = \big(e, v \circ \pi_L(e)\big)\,.$$

Clearly, $\pi \circ \bar{v} = id$ and $\varrho \circ \bar{v}\colon L(X) \to R^n$ is the map associating to e the components of the vector $v \circ \pi_L(e)$ with respect to this frame.

Connections in principal bundles: Let $\pi\colon E \to X$ be a principal bundle with a structure group G acting in E on the right. The action of $a \in G$ is given by a map $\varphi_a\colon E \to E$ and $\pi \circ \varphi_a = \pi$. Let G' be the Lie algebra of G and let Ad denote the adjoint representation of G in G', $Ad_a \in GL(G')$. A *connection* on the bundle $\pi\colon E \to X$ is a map

$$\omega\colon TE \to G'$$

linear on the fibres of $TE \to E$, equivariant with respect to G,

$$\omega \circ T\varphi_a = Ad_{a^{-1}} \circ \omega$$

and splitting the exact sequence

$$0 \to G' \times E \xrightarrow{i} TE \to \mathrm{hor}\, E \to 0$$

associated to the principal bundle [33]: if $A \in G'$ and $e \in E$, then

$$\omega \circ i(A, e) = A\,. \tag{A.4}$$

For any $u \in TE$, $\mathrm{hor}\, u = i \circ \omega(u)$ is the horizontal component of u. Let σ be a representation of G in a vector space U. A p-form of type σ

is an U-valued p-form ψ defined on E,

$$\psi\colon \Lambda^p TE \to U$$

equivariant with respect to G, $\psi \circ \Lambda^p T\varphi_a = \sigma_{a^{-1}} \circ \psi$. Its covariant exterior derivative is a $(p+1)$-form of type σ,

$$D\psi = \mathrm{hor}\, d\psi$$

$(\mathrm{hor}\, d\psi)(u_1, \ldots, u_{p+1}) = d\psi(\mathrm{hor}\, u_1, \ldots, \mathrm{hor}\, u_{p+1})\,.$
The connection on $\pi\colon E \to X$ is 1-form of type Ad and its covariant exterior derivative is the 2-form of curvature,

$$\Omega = D\omega = d\omega + [\omega, \omega]\,. \tag{A.5}$$

The bracket $[\omega, \omega]$ is understood as follows: if $u, v \in TE$ then $[\omega, \omega](u, v) = [\omega(u), \omega(v)]$, where the last bracket is evaluated in G'. The Bianchi identity is

$$D\Omega = d\Omega + [\omega, \Omega] = 0\,.$$

The Cartan connection on $A(X)$. Given a linear connection, i.e. a connection $\omega = (\omega^i{}_j)$ on the bundle $\pi_L\colon L(X) \to X$ of linear frames one constructs the Cartan connection $\tilde{\omega}$ on the bundle $\pi_A\colon A(X) \to X$ of affine frames by putting

$$\tilde{\omega} = (\bar{\omega}, \bar{\vartheta} + d\varrho + \bar{\omega}\varrho)\,, \tag{A.6}$$

where $\bar{\omega}$ and $\bar{\vartheta}$ are the pull-backs of ω and ϑ to $A(X)$, $\bar{\omega} = \pi^*\omega = \omega \circ T\pi$, $\bar{\vartheta} = \pi^*\vartheta$, and $\bar{\omega}\varrho$ denotes the evaluation of $\bar{\omega}$ on ϱ, $(\bar{\omega}\varrho)^i = (\pi^*\omega^i{}_j)\varrho^j$.

Clearly,

$$\tilde{\omega}\colon TA(X) \to \mathscr{L}(R^n) \times R^n = GA(n, R)'$$

and (A.3) may be used to prove the equivariance of $\tilde{\omega}$,

$$\tilde{\omega} \circ T\tilde{\varphi}_{(a,q)} = Ad_{(a,q)^{-1}} \circ \tilde{\omega}\,.$$

The splitting property (A.4) of $\tilde{\omega}$ results from a similar property of ω and from

$$\frac{d}{dt}\varrho(e, p + eqt) = q\,, \quad \text{where} \quad t \in R \text{ and } q \in R^n.$$

The curvature of $\tilde{\omega}$ may be evaluated from (A.5) using (A.1),

$$\tilde{\Omega} = (\bar{\Omega}, \bar{\Theta} + \bar{\Omega}_{\varrho}),$$

where $\bar{\Omega}$ and $\bar{\Theta}$ are the pull-backs to $A(X)$ by π of the curvature and the torsion of ω, respectively,

$$\Omega = d\omega + \omega\wedge\omega\,, \qquad \Theta = d\vartheta + \omega\wedge\vartheta\,.$$

Let 0 be the zero vector field on X and $\bar{0}\colon L(X) \to A(X)$ the embedding induced by 0. The pull-backs of $\tilde{\omega}$ and $\tilde{\Omega}$ by $\bar{0}$ are, respectively,

$$\text{(A.7)} \qquad \begin{cases} \bar{0}^*\tilde{\omega} = (\omega, \vartheta)\,, \\ \bar{0}^*\tilde{\Omega} = (\Omega, \Theta)\,. \end{cases}$$

If r is a radius-vector field, then $\bar{r}^*\tilde{\omega} = (\omega, 0)$.

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