the transformation $\underset{l}{g}_{\alpha\beta} \to \underset{l}{g}^*_{\alpha\beta}$ can be regarded as resulting from a co-ordinate transformation [9].

Assuming certain conditions for the co-ordinate system, e. g.,

$$\text{(15)} \qquad \underset{0}{g}^{pq}\underset{l}{g}_{0p;q} = 0, \qquad \underset{0}{g}^{pq}(\underset{l}{g}_{kp;q} - \tfrac{1}{2}\underset{l}{g}_{pq;k}) = 0 \quad (k = 1, 2, 3),$$

we can simplify the expression for $R_{i\beta}$, which becomes, in the case of (15):

$$\text{(16)} \qquad \begin{aligned} \underset{l}{R}_{\alpha 0} &= -\tfrac{1}{2}\underset{0}{g}^{pq}\underset{l}{g}_{\alpha 0;pq} + \underset{l}{R'}_{\alpha 0}, \\ \underset{l}{R}_{ik} &= -\tfrac{1}{2}\underset{0}{g}^{pq}\underset{l}{g}_{ik;pq} - \tfrac{1}{2}\underset{0}{g}^{q}\underset{l}{g}_{00;ik} + \underset{l}{R'}_{ik}. \end{aligned}$$

We conclude with a simple remark which can be useful in applying the method.

Let us take a flat x^ν-space (hence $T_{\alpha\beta} = 0$); this means that there exists a Galilean co-ordinate system in which $g_{\alpha\beta}$ has the form (1). Let us now introduce a non-inertial co-ordinate system $x^{\nu'}$, defined by $\tau = \tau'$ (i. e. $x^0 = x^{0'}$), $x^k = x^k(\tau', x^{k'})$. We have for $g_{\alpha'\beta'}$:

$$g_{0'0'} = g_{\alpha\beta}\, x^\alpha_{|0'}\, x^\beta_{|0'} = g_{00} + \lambda^2 g_{ik}\, x^i_{,0'}\, x^k_{,0'} = 1 - \lambda^2 \delta_{ik}\, x^i_{,0'}\, x^k_{,0'},$$

$$g_{0'k'} = g_{\alpha\beta}\, x^\alpha_{|0'}\, x^\beta_{|k'} = -\lambda \delta_{ik}\, x^i_{,0'}\, x^k_{,k'},$$

$$g_{i'k'} = g_{\alpha\beta}\, x^\alpha_{|i}\, x^\beta_{|k'} = -\delta_{ik}\, x^i_{,i'}\, x^k_{,k'}.$$

Thus, in a general non-inertial co-ordinate system, the metric tensor $g_{\alpha\beta}$ can be written $g_{\alpha\beta} = \underset{0}{g}_{\alpha\beta} + \lambda \underset{1}{g}_{\alpha\beta} + \lambda^2 \underset{2}{g}_{\alpha\beta}$, where $\underset{0}{g}_{\alpha\beta}$ has the form (10),

$$\underset{1}{g}_{\alpha\beta} = \left(\begin{array}{c|c} 0 & \underset{1}{g}_{0k} \\ \hline \underset{1}{g}_{i0} & 0 \end{array} \right), \quad \text{and} \quad \underset{2}{g}_{\alpha\beta} = \left(\begin{array}{c|c} \underset{2}{g}_{00} & 0 \\ \hline 0 & 0 \end{array} \right).$$

I wish to express my thanks to Professor J. Plebański for suggesting the problem, and for many valuable discussions. I am also indebted to Professor L. Infeld for his kind interest in this work.

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BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV. No. 7. 1956

THEORETICAL PHYSICS

Solution of One-Body Problem by the Einstein-Infeld Approximation Method

by

A. TRAUTMAN

Presented by L. INFELD on April 9, 1956

In the previous paper [1] we presented a generalisation of the "new approximation method". In this paper we shall use that method to evaluate the gravitational field of a point mass, using the notation of [1]. We define $T_{\alpha\beta}$ so that $\varkappa = 8\pi k$, where $k = 6.67 \cdot 10^{-8}\text{cm.}^3\ \text{g}^{-1}\ \text{sec.}^{-2}$ (it is perhaps more usual to put $\varkappa = 8\pi\, kc^{-4}$). The energy-momentum tensor for a point mass will be represented by an expression involving the three-dimensional Dirac δ-function. This method of representing singularities was introduced by Infeld [2].

We assume

$$\text{(1)} \qquad \underset{0}{g}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

and denote

$$x^1 = r, \quad x^2 = \vartheta, \quad x^3 = \varphi.$$

Further,

$$\text{(2)} \qquad \underset{0}{T}^0_0 = \frac{1}{c^4}mc^2\delta(\vec{r}) = m\lambda^2\delta(\vec{r}) = \lambda^2 \underset{2}{T}^0_0,$$

$$T^\alpha_\beta = 0 \quad \text{if} \quad \alpha + \beta \neq 0.$$

We also assume that:

(3) *the metric $g_{\alpha\beta}$ is pseudo-Euclidean at infinity;*

(4) $\underset{l}{g}_{22} = \underset{l}{g}_{33} = \underset{l}{g}_{12} = \underset{l}{g}_{13} = \underset{l}{g}_{23} = 0$ and $\underset{l}{g}_{11,2} = \underset{l}{g}_{11,3} = 0$ ("spherical symmetry").

From the field equations (I. 9. 1. $\alpha\beta$) (we shall use the sign "I" when referring to the formulae of [1]), we obtain:

(5) $$\Delta \underset{1}{g}_{00}=0,$$

(6) $$\underset{0}{g}^{mn}(\underset{1}{g}_{0m;kn}-\underset{1}{g}_{0k;mn})=0,$$

(7) $$\underset{0}{g}^{mn}(\underset{1}{g}_{im;kn}-\underset{1}{g}_{ik;mn}+\underset{1}{g}_{nk;mi}-\underset{1}{g}_{mn;ki})-\underset{1}{g}_{00;ik}=0.$$

Equation (5), together with (3), yields $\underset{1}{g}_{00}=0$. Equations (6) and (7) have more than one solution (see (I. 14)), of which we choose the most simple: $\underset{1}{g}_{\alpha k}=0$ (this is equivalent to imposing certain new conditions on the co-ordinate system). A similar procedure will be applied to all homogeneous equations occurring in our further works. From (I. 9. 2. 00) we obtain the equation for $\underset{2}{g}_{00}$:

(8) $$\tfrac{1}{2}\Delta \underset{2}{g}_{00}=8\pi k(\underset{0}{T}_{00}-\tfrac{1}{2}\underset{0}{g}_{00}\underset{2}{T})=4\pi k\underset{0}{g}_{00}\underset{2}{T}^{0}_{0}=4\pi km\delta(\vec{r}).$$

Hence,

(9) $$\underset{2}{g}_{00}=-\frac{2km}{r}\underset{\text{def}}{=}-\psi.$$

Equations (I. 9. 2. 0k) are in our case of the form (6), then $\underset{2}{g}_{0k}=0$.

Equation (I. 9. 2. 11) becomes

$$\underset{2}{R}_{11}=\tfrac{1}{2}\underset{0}{g}^{pq}(2\underset{2}{g}_{1p;1q}-\underset{2}{g}_{11;pq}-\underset{2}{g}_{pq;11})-\tfrac{1}{2}\underset{2}{g}_{00;11}=-4\pi k\underset{0}{g}_{11}\underset{2}{T},$$

or

(10) $$-\frac{1}{r}\underset{2}{g}_{11,1}=-\frac{2km}{r^3}+4\pi km\delta(\vec{r}).$$

Taking into account $r\delta(\vec{r})=0$, and (3), we get from (10):

(11) $$\underset{2}{g}_{11}=-\frac{2km}{r}.$$

The equations for $\underset{3}{g}_{\alpha\beta}$ are identical in form with those for $\underset{1}{g}_{\alpha\beta}$; hence $\underset{3}{g}_{\alpha\beta}=0$. In the fourth-order approximation we have:

(12) $$\underset{4}{R}_{00}=\tfrac{1}{2}\Delta\underset{4}{g}_{00}-\tfrac{1}{2}\underset{2}{g}^{mn}\underset{2}{g}_{00,mn}-(\underset{0}{g}^{mn}\underset{2}{g}^{rs}+\underset{2}{g}^{mn}\underset{0}{g}^{rs})\underset{2}{\Gamma}_{n00}\underset{2}{\Gamma}_{mrs}+\underset{0}{g}^{\mu\nu}\underset{0}{g}^{\sigma\varrho}(\underset{2}{\Gamma}_{\nu0\sigma}\underset{2}{\Gamma}_{\mu\varrho0}-\underset{2}{\Gamma}_{\nu00}\underset{2}{\Gamma}_{\mu\varrho\sigma})=4\pi k\underset{2}{g}_{00}\underset{2}{T}^{0}_{0}=\tfrac{1}{2}\underset{2}{g}_{00}\Delta\underset{2}{g}_{00},$$

from which it follows that

(13) $$\tfrac{1}{2}\Delta\underset{4}{g}_{00}+\tfrac{1}{2}\underset{2}{g}_{11}\Delta\underset{2}{g}_{00}=\tfrac{1}{2}\underset{2}{g}_{00}\Delta\underset{2}{g}_{00},$$

and, in virtue of $\underset{2}{g}_{00}=\underset{2}{g}_{11}$, we obtain

(14) $$\underset{4}{g}_{00}=0.$$

It should be noted that the expressions which cancel out from (13) are of the divergent type $\delta(\vec{r})r^{-1}$.

From (I. 9. 4. 11) we obtain an equation for $\underset{4}{g}_{11}$:

(15) $$\underset{4}{g}_{11,1}=\frac{8k^2m^2}{r^3}-8\pi k^2m^2\delta(\vec{r}).$$

By means of the symbolic formulae

(16) $$\delta(\vec{r})=\frac{\delta(r)}{2\pi r^2},\qquad \delta(r)=\frac{1}{2}\frac{d}{dr}\frac{r}{|r|},$$

we can solve (15):

$$\underset{4}{g}_{11,1}=-\frac{d}{dr}\frac{2k^2m^2}{r^2}-\frac{d}{dr}\left(\frac{2k^2m^2}{r^2}\frac{r}{|r|}\right).$$

Hence,

(17) $$\underset{4}{g}_{11}=-\frac{2k^2m^2}{r^2}-\frac{2k^2m^2}{r^2}\frac{r}{|r|}=-\psi^2.$$

We shall now prove that for $l=2,3,\dots$ we have

(18) $$\underset{2l}{g}_{11}=-\psi^l,\qquad \underset{2l}{g}_{\alpha\beta}=0\quad\text{if}\quad \alpha\beta\neq 1,\qquad \underset{2l-1}{g}_{\alpha\beta}=0.$$

This is true for $l=2$ from the previous results. Let us assume that (18) holds for $l<s$; then we get from (I. 7):

(19) $$\underset{2}{g}^{11}=\psi,\qquad \underset{2l}{g}^{11}=0\quad l=2,3,\dots,s-1.\qquad \underset{2l}{g}^{00}=\psi^l,\quad l=1,2,\dots,s-1.$$

From (I. 9. 2s. 00) we obtain the equation for $\underset{2s}{g}_{00}$:

(20) $$\underset{2s}{R}_{00}=\tfrac{1}{2}\Delta\underset{2s}{g}_{00}+\underset{2s}{R'}_{00}=4\pi k\underset{2s-2}{g}_{00}\underset{2}{T}^{0}_{0}=0.$$

Evaluating $\underset{2s}{R'}_{00}$ and using (18) for $l<s$, we obtain $\underset{2s}{R'}_{00}=0$; thus $\underset{2s}{g}_{00}=0$.

A somewhat troublesome calculus leads to the following equation for $\underset{2s}{g}_{11}$:

(21) $$-\frac{1}{r}\underset{2s}{g}_{11,1}+\frac{s\psi^s}{r^2}=4\pi km\psi^{s-1}\delta(\vec{r}).$$

Solving (21) in a way similar to that of (15), we obtain

(22) $$\underset{2s}{g}_{11}=-\frac{1}{2}\psi^s-\frac{1}{2}\psi^s\frac{|r|}{r}=-\psi^s.$$

The obvious equation $\underset{2s-1}{R'}_{\alpha\beta}=0$ implies $\underset{2s-1}{g}_{\alpha\beta}=0$; thus (18) holds for every $l>1$.

Now we are able to evaluate the metric tensor $g_{\alpha\beta}$:

$$g_{00}=\underset{0}{g}_{00}+\frac{1}{c^2}\underset{2}{g}_{00}=1-\frac{2km}{c^2r},$$

$$g_{11}=\underset{0}{g}_{11}+\frac{1}{c^2}\underset{2}{g}_{11}+\frac{1}{c^4}\underset{4}{g}_{11}+\ldots=-\sum_{l=0}^{\infty}\frac{\psi^l}{c^{2l}}=-\frac{1}{1-\dfrac{2km}{c^2r}}$$

$\left(\text{convergent for } \dfrac{2km}{c^2}<r\right)$,

$$g_{0k}=0,\quad g_{12}=g_{23}=g_{13}=0,\quad g_{22}=\underset{0}{g}_{22},\quad g_{33}=\underset{0}{g}_{33}.$$

In this way is obtained the well-known Schwarzschild metric:

$$ds^2=\left(1-\frac{2km}{c^2r}\right)dx_0^2-r^2(\sin^2\vartheta\, d\varphi^2+d\vartheta^2)-\frac{dr^2}{1-\dfrac{2km}{c^2r}}.$$

The same result can be obtained by expanding $g_{\alpha\beta}$ into a series in $c=\lambda^{-1}$. Let us define $k'=kc^{-4}$ and $T'_{\alpha\beta}=c^4T_{\alpha\beta}$. Thus, the right-hand side of the Einstein equations can be written:

$$8\pi k'(T'_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T').$$

Expanding $T'_{\alpha\beta}$ into powers of c we get

$$T'^0_0=c^4T^0_0=c^2\underset{2}{T}{}^0_0=c^2\underset{2}{T}{}'^0_0.$$

Hence, $\underset{2}{T}{}'^0_0=\underset{2}{T}{}^0_0$. We see that the solution obtained by an expansion in c follows from that obtained in this paper after applying a transformation: $k\to k'$, $c\to c^{-1}$. But the Schwarzschild metric involves k and c only, through a factor $k/c^2\to k'c^2=k/c^2$. The static nature of the field played an essential role in this argument.

I should like to express my thanks to Professors L. Infeld and J. Plebański for their kind interest in this work.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

THEORETICAL PHYSICS

On the Theory of the Electromagnetic Field in Moving Dielectrica

by

J. I. HORVÁTH

Presented by W. RUBINOWICZ on April 24, 1956

1. The relativistic theory of the electromagnetic field in its Hamiltonian formalism (in the case of vanishing four-current density) is usually based on the Lagrangian

(1) $$L'=-\tfrac{1}{4}F_{\mu\nu}G^{\mu\nu},$$

where $F_{\mu\nu}$ and $G_{\mu\nu}$ are Minkowski's antisymmetric field-tensors defined by

$$\mathfrak{E}\overset{\text{def}}{=}\{F_{10},F_{20},F_{30}\},\qquad \mathfrak{B}\overset{\text{def}}{=}\{F_{23},F_{31},F_{12}\},$$

and

$$\mathfrak{D}\overset{\text{def}}{=}\{G_{10},G_{20},G_{30}\},\qquad \mathfrak{H}\overset{\text{def}}{=}\{G_{23},G_{31},G_{12}\}$$

respectively, $\mathfrak{E}$ and $\mathfrak{H}$ being the field vector of the electric and magnetic field, $\mathfrak{D}$ the electric displacement and $\mathfrak{B}$ the magnetic induction *). The tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ are connected by the material equations

$$G_{\mu\nu}v^\nu=\varepsilon F_{\mu\nu}v^\nu,$$

$$G_{\mu\nu}v_\lambda+G_{\nu\lambda}v_\mu+G_{\lambda\mu}v_\nu=1/\mu\,\{F_{\mu\nu}v_\lambda+F_{\nu\lambda}v_\mu+F_{\lambda\mu}v_\nu\},$$

where v^μ is the four-velocity of the ponderable matter fulfilling the condition of normalisation:

(2) $$v^\mu v_\mu=1.$$

Furthermore, the dielectric constant and the magnetic permeability of the ponderable matter are represented by ε and μ respectively. The explicit dependence of $G_{\mu\nu}$ on $F_{\mu\nu}$, ε and μ respectively can be given as follows:

*) Our space is pseudo-Euclidean with the co-ordinates $x^0=ct\equiv t$ (the velocity of light c to be regarded as unity), $x^1=x$, $x^2=y$, $x^3=z$ and with the metrical ground tensor

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=+1,\quad g_{\mu\nu}=0\quad \mu\neq\nu;\quad g=\det g_{\mu\nu}.$$