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THEORETICAL PHYSICS

On the Einstein—Cartan Equations. IV

by

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Summary. It is shown that with every one-parameter group of symmetries of a Riemann—Cartan space-time there is associated a conservation law $dj=0$ for the current of energy, momentum and/or spin.

1. Symmetries of a Riemann—Cartan space. Let (X, g, ω) be a Riemann—Cartan space, i.e., a differential manifold X with a metric tensor g and a linear connection ω compatible with g. The *transposed connection* $\tilde{\omega}$ is defined in terms of its components $\tilde{\omega}^i_j$ relative to a field of frames (θ^i),

$$\tilde{\omega}^i_j=\omega^i_j+Q^i_j,$$

where $Q^i_j=Q^i_{jk}\,\theta^k$ and Q^i_{jk} is the tensor of torsion of the connection ω. The transposed connection is metric if and only if $Q_{ijk}=Q_{[ijk]}$. If ω^i_j and $\tilde{\omega}^i_j$ are written as $\Gamma^i_{jk}\,\theta^k$ and $\tilde{\Gamma}^i_{jk}\,\theta^k$, respectively, then, for a holonomic frame (θ^i),

$$\tilde{\Gamma}^i_{jk}=\Gamma^i_{kj}.$$

The covariant derivative of a vector field v, relative to $\tilde{\omega}$, is

$$\theta^j\,\tilde{\nabla}_j\,v^i=\tilde{D}v^i=Dv^i+v\,\lrcorner\,\Theta^i.$$

Similarly, if (t_i) is a covector-valued p-form, then

$$\tilde{D}t_i=Dt_i-Q^j_i\wedge t_j.$$

A *symmetry* (automorphism) of (X, g, ω) is a diffeomorphism of X which preserves g and ω. Consider a one-parameter group of transformations of X generated by the vector field v. A necessary and sufficient condition for the transformations to be symmetries of (X, g, ω) is that the Lie derivatives of g and ω with respect to v vanish [1]:

$$\tilde{\nabla}^i\,v^j+\tilde{\nabla}^j\,v^i=0, \qquad (1)$$

$$D\tilde{\nabla}_j\,v^i+v\,\lrcorner\,\Omega^i_j=0. \qquad (2)$$

In a Riemannian space, the connections ω and $\tilde{\omega}$ coincide and (2) is a consequence of the Killing equation (1).