42. Comparison of Newtonian and Relativistic Theories of Space-Time

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Abstract. This paper contains a review and comparison of geometrical hypotheses underlying Newtonian and relativistic theories of space and time. The similarity between the Newtonian and the Einstein cosmologies is explained and the field equations for a class of Riemannian space-times with close Newtonian analogues are written explicitly.

Introduction

In lectures and texts on relativity it is customary to emphasize the differences between the Newtonian theory on the one hand and special and general relativity on the other. The theory of general relativity is often said to be beautiful but difficult: its equations are hard to solve and the absence of inertial frames in Einstein's theory complicates the physical interpretation of its results. This is certainly true; however, in this paper, we should like to emphasize the similarities among all theories of space, time, and gravitation, and to show that from the point of view of economy of hypotheses, Newton's theory of gravitation is much more complicated than Einstein's theory. Moreover, already in the Newtonian theory, the notion of inertial frames, when applied to strong gravitational fields, requires an essential modification.

To every physical theory there corresponds a certain mathematical formalism in which the theory is usually expressed. In many cases, the formalism was developed at the same time as the physical bases of the theory were discovered. If one wishes to compare different theories, it is desirable to formulate them in the same mathematical language. Otherwise it is rather difficult to ascertain what are the relationships between the basic assumptions underlying these theories. As stressed by Bondi,[1] there are often hidden, "self-evident" hypotheses that are considered to be no hypotheses at all. The simplicity of a theory should not be judged only on the ground of what its basic assumptions are explicitly said to be.

There are essentially two groups of theories of space-time: the Newtonian and the relativistic ones. All relativistic theories are most naturally expressed in terms of concepts from differential geometry. The Newtonian theory was given a geometric formulation by Cartan,[2] and Friedrichs.[3] Recently, several authors have analyzed in some detail the geometry of space-time according to Newton's theory and the relation of that theory to general relativity.[4-7] It emerges from these analyses that there is a number of geometrical structures common to all theories of space-time. First of all, space-time is always assumed to be a four-dimensional differentiable manifold (continuum). This is true even in multidimensional unified field theories such as those of Kaluza and

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Klein, or Jordan and Thiry. In these theories, space-time is a four-dimensional quotient of a bigger space by some equivalence relation. There were several attempts to replace the physical continuum by a space with discrete topology, but none of them was successful. Even speculations on quantized general relativity take a differentiable manifold as a starting point.

Another feature common to all present physical theories is that they assume the existence of an affine connection in space time. This is closely related to the fact that the fundamental laws of physics have a local character and can be expressed by means of differential equations. Usually, the affine connection is thought to be symmetric. The possibility of a space-time with torsion was considered by Einstein,[8] Schrödinger,[9] and others in connection with attempts to unify gravitation with electromagnetism, and also by Sciama[10] in a different context. Hlavaty[11] has made a profound and exhaustive study of the various possibilities offered by affinely connected spaces with a generalized metric field.

The metric structure of space-time according to the Newtonian theory is rather different from that in relativity. The Newtonian metric is degenerate; clearly, it is the limit as $c \to \infty$, of the relativistic metric. Accordingly, the Newtonian metric has those properties of the relativistic g_{ab} which are preserved by the limiting process. In particular, it is invariant by parallel transport.

The First Law of Dynamics

The Newtonian mechanics is based on the assumption that there exists an absolute time, t, and that the hypersurfaces (spaces) $t = \text{const.}$ are three-dimensional Euclidean. The time t can be taken as one of the coordinates; if (x, y, z, t) is a system of coordinates in space-time, , the motion of a particle can be represented by $x = \xi(t)$, $y = \eta(t)$, $z = \zeta(t)$, i.e., by a curve (world-line) in space-time. Neglecting gravitation, the first law of dynamics may be formulated as follows: there exists a family of privileged motions, called free motions, and a system of coordinates (x, y, z, t), such that the free motions are characterized by

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$$

Coordinate systems whose existence is asserted by the first law are called inertial. A transformation leading from one inertial system to another is called Galilean. Clearly, if we agree to consider the world-lines of free motions as geodesics, the Newtonian space-time becomes endowed with an integrable affine connection. In other words, in the absence of gravitation, the first law says the following: the Newtonian space-time is an affine space whose straight lines correspond to free motions.[12, 13]

The formulation of the first law is not so straightforward when one wants to take gravitation into account. Since the inertial and gravitational masses are always equal, all particles in a given gravitational field move in the same way if their initial positions and velocities are the same. In a gravitational field, there are no free motions in the previous sense.

The best one can do is to remove all nongravitational interactions and to consider free falls as the family of privileged motions. Accordingly, Newton's first law may be rephrased as follows: there exists a family of privileged motions, called free falls, a system of coordinates (x, y, z, t), and a function $\varphi(x, y, z, t)$, such that the free falls are characterized by

$$\frac{d^2x}{dt^2} = -\frac{\partial\varphi}{\partial x}, \quad \frac{d^2y}{dt^2} = -\frac{\partial\varphi}{\partial y}, \quad \frac{d^2z}{dt^2} = -\frac{\partial\varphi}{\partial z}. \tag{1}$$

Clearly, the class of coordinate changes preserving eq. (1) is much larger than the class of Galilean transformations. For example, if a is an arbitrary function of time, then the transformation

$$x' = x + a(t), \; y' = y, \; z' = z, \; \varphi' = \varphi - x\frac{d^2a}{dt^2} \tag{2}$$

preserves eq. (1) but is not Galilean unless $d^2a/dt^2 = 0$. Usually, one considers gravitational fields produced by bounded sources. One then can normalize φ by requiring that it vanish at large distances. This eliminates the possibility of transformations like eq. (2) and restores the privileged role of the Galileo group. This cannot be done, however, when there is a strong gravitational field extending all over space, as in cosmology. In that case, we are faced with the choice of either abandoning the concept of inertial systems altogether or calling inertial all systems in which free falls are characterized by eq. (1). Most authors[1, 15] favor the latter possibility, which has been adopted in this paper. Again, we may call world-lines corresponding to free falls geodesics and thereby introduce an affine connection in space-time.

Geometry of the Newtonian Space-Time

The general geometrical structure of space-time in relativity is very well known.[e.g., 16] The basic assumptions are outlined in the following table.

Comparison of hypotheses underlying Newtonian and relativistic theories of space, time, and gravitation.

Newtonian	Relativistic
Space-time is a four-dimensional differentiable manifold.	
Space-time is endowed with a symmetric affine connection.	
World-lines of free falls are geodetic.	
In every tangent space to the manifold, there is given:	
(a) A non-zero form t_a,	A metric tensor g_{ab}, of signature $+ + + -$.
(b) A symmetric contravariant tensor h^{ab}, of signature $+ + + 0$, such that $h^{ab}t_b = 0$.	

Newtonian	Relativistic
$\nabla_c\ h^{ab} = 0$	$\nabla_c\ g_{ab} = 0$
$\nabla_a\ t_b = 0$	
The last equation implies: $t_a = \partial_a t$, t is an affine parameter along time-like geodesics.	This implies that s, defined by $ds^2 = -g_{ab}\, dx^a\, dx^b$, is an affine parameter along time-like geodesics.
Ideal clocks measure t.	Ideal clocks measure s.
$t_{[e}R^a{}_{b]cd} = 0$	
$h^{ad}R^b{}_{cde} + h^{bd}R^a{}_{edc} = 0$	
Condition for empty space: $R_{ab} = 0$	
Condition for absence of gravitational forces: $R^a{}_{bcd} = 0$.	

In the Newtonian theory the space-time is a differentiable manifold N of class C^∞, homeomorphic to $\mathbf{R}^4$. The Newtonian notion of absolute simultaneity implies the existence of a family T of hypersurfaces in space-time. Distinct elements of T do not intersect, through every event (point of N) there passes an element of T; all these hypersurfaces have the topology of $\mathbf{R}^3$. Let t = const. be the equation of T. Without loss of generality, one can assume that the form $t_a = \partial_a t$ is nowhere zero.* For the moment, the function t is determined only up to $t \rightarrow f(t)$, where f is differentiable and $f' \neq 0$. A vector $\mathbf{v}^a$ tangent to a hypersurface of simultaneity,

$$v^a t_a = 0$$

will be called space-like (one also might call it null); other vectors are called time-like. A regular curve is called time-like if all its tangent vectors are time-like. In this case, t may be used to parametrize the curve.

The family of all free falls determines a symmetric affine connection on N. It follows from the first law of dynamics, eq. (1), that t can be chosen so as to be an affine parameter along all time-like geodesics. This implies

$$\nabla_a t_b = 0 \qquad (3)$$

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* The following notation is used in this paper: local coordinates in space-time are (x^1, x^2, x^3, x^4); Latin indices range and sum from 1 to 4; Greek indices range and sum from 1 to 3; all vectors, tensors, affine connections etc., are represented by their components with respect to the natural bases in tangent spaces; ordinary and covariant derivatives are denoted by ∂_a and ∇_a respectively; and square index-brackets denote antisymmetrization over the indices enclosed. Note that there are differences between these conventions and those employed by the author in a previous work.[5]

and defines t up to linear transformations. Every such t is called the absolute time. Equation (3) may be looked upon as a condition on the connection Γ^a_{bc}; it implies

$$t_a R^a_{bcd} = 0 \tag{4}$$

where R^a_{bcd} is the curvature tensor constructed from Γ^a_{bc}.

Let (y^1, y^2, y^3) be a system of local coordinates in a hypersurface belonging to T. The hypersurface can be represented in a parametric form,

$$x^a = x^a(y^1, y^2, y^3).$$

Let $h^{\alpha\beta}$ denote the components with respect to $(\partial/\partial y^\alpha)$ of the Euclidean metric tensor of the hypersurface. Its components with respect to $(\partial/\partial x^a)$ are

$$h^{ab} = h^{\alpha\beta} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta}.$$

Clearly,

$$h^{ab} t_b = 0 \tag{5}$$

and the canonical form of the matrix (h^{ab}) is diagonal (1, 1, 1, 0). The tensor h^{ab} may be used to define the square of any form and of any space-like vector but not of time-like vectors. According to what was said in the introduction,

$$\nabla_c h^{ab} = 0. \tag{6}$$

The remaining information contained in the first law of dynamics may be expressed by

$$t_{[e} R_{b]}{}^a{}_{cd} = 0 \tag{7}$$

and

$$h^{ad} R^b_{cde} + h^{bd} R^a_{edc} = 0. \tag{8}$$

Alternatively and equivalently, [5] one can assume the existence of a scalar field φ such that

$$\Gamma^a_{bc} = \overset{0}{\Gamma}{}^a_{bc} + t_b\, t_c\, h^{ad}\, \partial_d \varphi \tag{9}$$

where $\overset{0}{\Gamma}{}^a_{bc}$ is an integrable affine connection. The splitting, eq. (9), is not unique. Indeed, if ψ is any solution of

$$t_{[a} \nabla_{b]} \partial_c \psi = 0$$

then

$$\overset{0}{\Gamma}{}^{a}_{bc} - t_b t_c h^{ad} \partial_d \psi \tag{10}$$

is also integrable. It follows from eq. (9) that the curvature tensor is

$$R^{a}_{bcd} = 2t_b \varphi^{a}{}_{[c} t_{d]}, \tag{11}$$

where

$$\varphi^{a}_{b} = h^{ad} \nabla_b \partial_d \varphi \tag{12}$$

and the equation of time-like geodesics may be written as

$$\frac{d^2x^a}{dt^2} + \overset{0}{\Gamma}{}^{a}_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = -h^{ad} \partial_d \varphi. \tag{13}$$

Given a definite splitting of the connection, eq. (9), one can introduce coordinates in N such that

$$\overset{0}{\Gamma}{}^{a}_{bc} \overset{*}{=} 0. \tag{14}$$

By virtue of eq. (3), (5), (6), and (9) this implies

$$\partial_a \partial_b t \overset{*}{=} 0 \text{ and } \partial_c h^{ab} \overset{*}{=} 0.$$

Linear coordinate transformations preserve eq. (14) and may be used to reduce h^{ab} to the canonical form,

$$h^{ab} \overset{*}{=} \left\| \begin{matrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{matrix} \right\|. \tag{15}$$

Because of eq. (5), t becomes a linear function of x^4 only, and one can further specialize the coordinate system by taking

$$t \overset{*}{=} x^4. \tag{16}$$

With x, y, and z instead of x^1, x^2, and x^3, the geodetic equation (13) reduces to eq. (1). This shows that φ may be identified with the gravitational potential and that the special coordinates defined by eq. (14), (15), and (16) are inertial. Clearly, in these coordinates, $h^{ab}\nabla_a\nabla_b\varphi$ reduces to the Laplacian of φ.

The Ricci tensor, $R_{ab} = R^{c}_{abc}$, is

$$R_{ab} = -t_a t_b h^{cd} \nabla_c \partial_d \varphi.$$

Therefore, the Poisson equation of the Newtonian theory of gravitation,

$$\Delta\varphi = 4\pi k\rho, \tag{17}$$

(where ρ is the density of matter and k the gravitational constant) is equivalent to [6]

$$R_{ab} = -4\pi k \rho t_a t_b. \tag{18}$$

It is now clear what the geometrical significance of Galilean transformations is: they preserve eqs. (14) and (15). A necessary and sufficient condition for a vector field ξ^a to be a generator of the Galileo group of transformations is

$$\underset{\xi}{\pounds}\, \overset{0}{\Gamma}{}^a_{bc} = 0 \text{ and } \underset{\xi}{\pounds}\, h^{ab} = 0,$$

where $\pounds$ denotes the Lie derivative. The group generated by vector fields subject only to

$$\underset{\xi}{\pounds}\, h^{ab} = 0$$

is larger than the Galileo group; it contains the "transformations to accelerated frames" of classical mechanics. Note that the difference between these two groups is due to the non-Riemannian character of Newton's space-time and has no counterpart in special relativity.

We should like to emphasize again that it is the connection Γ^a_{bc}, rather than $\overset{0}{\Gamma}{}^a_{bc}$, that can be determined by local physical experiments. In general, the integrable connection $\overset{0}{\Gamma}{}^a_{bc}$ is defined only up to replacements of $\overset{0}{\Gamma}{}^a_{bc}$ by eq. (10), accompanied by $\varphi \to \varphi + \psi$. In special cases, such as that of a gravitational field produced by a bounded system of masses, we can single out a particular solution φ of eq. (18) by a global requirement, and define thereby a preferred integrable connection and the corresponding family of inertial frames. In other words, we can construct inertial frames by performing experiments with free motions far away from the source, where the gravitational field is negligibly weak, and then extending the coordinate system into the region of strong field.

Ether and Electrodynamics

The Newtonian structure described in the preceding section is not sufficient to build a theory of electromagnetism. As is well known, this necessitates the introduction on the manifold N of a new element called the ether. For our purposes, the ether may be defined as a rigging of the hypersurfaces $t = \text{const}$. A rigging associates with every point of a hypersurface a direction not tangent to the hypersurface. In our case, this direction may be thought of as tangent to the world-line of a privileged observer. Therefore, the ether determines a state of absolute rest at every point of N.

For historical reasons and for the sake of simplicity, we shall assume in this section that the basic connection Γ^a_{bc} is integrable. Given an ether on N, let u^a be the vector field tangent to the directions of rigging and normalized so that

$$u^a t_a = 1,$$

and let us introduce the tensor

$$g^{ab} = h^{ab} - u^a u^b / c^2, \tag{19}$$

where c is the velocity of light. If u^a is covariantly constant,

$$\nabla_a u^b = 0, \tag{20}$$

then

$$\partial_{[a} F_{bc]} = 0, \text{ where } F_{ab} = F_{[ab]},$$

and

$$\nabla_b F^{ab} = 0, \text{ where } F^{ab} = g^{ac} g^{bd} F_{cd},$$

are equivalent to the usual Maxwell equations for the vacuum. In the post, various assumptions about u^a, other than eq. (20), were considered. They led to theories according to which the ether was dragged—or partially dragged—by the motion of the medium or by the motion of sources of radiation.

In the optical limit, Maxwell's equations imply the eikonal equation

$$g^{ab} k_a k_b = 0, \; k_a = \partial_a \Psi.$$

A Newtonian observer whose world-line is $x^a = x^a(t)$ ascribes the following value to the speed of light:

$$\frac{|k_a v^a|}{\sqrt{h^{ab} k_a k_b}} = c \frac{|k_a v^a|}{|k_a u^a|}, \tag{21}$$

where $v^a = dx^a/dt$. In general, eq. (21) does not equal c unless $v^a = u^a$. Because of eq. (20), the connection Γ^a_{bc} is metric relative to g^{ab},

$$\nabla_c g^{ab} = 0.$$

Moreover, the matrix g^{ab} is nonsingular. Its inverse, g_{ab}, together with Γ^a_{bc}, defines a flat indefinite Riemannian (Minkowskian) geometry in N. In prerelativistic electrodynamics this geometry coexisted with the Newtonian structure. It has been used to define the Lorentz group[17]: the condition for ξ^a to be a generator of the Lorentz group is

$$\underset{\xi}{\pounds}\, g_{ab} = 0.$$

The essential step taken by Einstein in 1905 consisted in denying any physical significance to the Newtonian structure (t, h^{ab}). In special relativity, the geometry of space-time is fully determined by the Minkowski elements (g_{ab}, Γ^a_{bc}). Accordingly, all equations of physics may contain only these elements in addition to quantities describing the state of the system.* For example, the formula giving the velocity of propagation of electromagnetic waves becomes

$$\frac{|k_a v^a|}{\sqrt{(g^{ab} + v^a v^b/c^2)\, k_a k_b}} = c, \text{ for any } v^a. \tag{22}$$

(in order to obtain a similarity between eq. (21) and eq. (22), the velocity vector of the observer has been normalized so that $g_{ab}v^a v^b = -c^2$).

Cosmology

When one attempts to apply Newtonian mechanics in cosmology, one encounters the following apparent difficulty: assume that the Universe is spatially homogeneous and let $\rho(t)$ be the mean density of matter. A typical solution of Poisson's equation (17) is

$$\varphi = {}^2\!/_3\, \pi k \rho r^2. \tag{23}$$

The corresponding gravitational field, $-\mathrm{grad}\varphi$, seems to contradict the cosmological principle: the particle at $r = 0$ is unaccelerated while all others are accelerated. This difficulty disappears if it is remembered that, in this case, it is impossible to introduce a preferred set of inertial frames defined up to Galilean transformations. The set of all inertial frames is essentially larger and for every galaxy there is one such frame with respect to which the galaxy is at rest. Moreover, there are motions of the substratum compatible with eq. (23) and satisfying the cosmological principle. [1, 14, 15] The homogeneous and isotropic character of the gravitational field corresponding to eq. (23) is best seen in the expression for the curvature tensor, as calculated from eqs. (11) and (12):

$$R^a_{bcd} = {}^8\!/_3\, \pi k \rho\, t_b \delta^a{}_{[c} t_{d]}.$$

The assumption of homogeneity and isotropy leads to the following expression for the velocity field of the substratum, referred to a certain inertial frame,

$$\mathbf{v} = \mathbf{r} R^{-1}\, dR/dt, \tag{24}$$

where R is an arbitrary function of the absolute time. The motion of the substratum provides a natural choice for the ether: the rigging is defined by the tangents to the world-lines of elements of the substratum. As can be shown easily, this assumption leads to an expression for the Doppler shift of light coming from distant galaxies,

$$\frac{\nu_1}{\nu_2} = \frac{R(t_2)}{R(t_1)} \tag{25}$$

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* This statement is often called the "principle of relativity".

which is identical with the corresponding expression obtained in relativistic cosmology[1]. It is not hard to understand the origin of this coincidence: the assumption about the ether is equivalent to writing the eikonal equation with g^{ab} given by eq. (19), h^{ab} by eq. (15) and

$$u^a = (\mathbf{v}, 1). \tag{26}$$

For such a g^{ab}, a straightforward calculation gives

$$g_{ab}\, dx^a dx^b = d\mathbf{r}^2 - 2\mathbf{v} \cdot d\mathbf{r}\, dt - (c^2 - v^2)\, dt^2, \tag{27}$$

where

$$x^a = (\mathbf{r}, t), d\mathbf{r}^2 = dx^2 + dy^2 + dz^2, \text{etc.},$$

and a simple coordinate transformation reduces eq. (27), with $\mathbf{v}$ of the form given in eq. (24), to a Friedmann line-element,

$$R^2 d\mathbf{r}'^2 - c^2 dt^2, \tag{28}$$

which is known to lead to eq. (25) as the formula for the Doppler shift.

A Class of Riemannian Space-Times with Close Newtonian Analogues

In addition to giving the same formula for the Doppler shift, Newtonian and relativistic cosmologies lead to similar equations for the expansion function R(t). This interesting fact was noticed for the first time by Milne and McCrea in 1934.[14, also 1,15] In this section the following problem is considered: what are the physical situations for which the Newtonian and relativistic descriptions are as close as they are in cosmology?

Let $\mathbf{v}(\mathbf{r}, t)$ be a (sufficiently regular) Newtonian velocity field and $\mathbf{r} = \mathbf{F}(\mathbf{r}', t)$ a family of solutions of

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t)$$

satisfying some initial conditions, say, $\mathbf{F}(\mathbf{r}', 0) = \mathbf{r}'$.

The coordinate transformation $\mathbf{r} \to \mathbf{r}'$, with t unchanged, reduces the line-element of eq. (27) to

$$\frac{\partial \mathbf{F}}{\partial x^{\alpha\prime}} \cdot \frac{\partial \mathbf{F}}{\partial x^{\beta\prime}}\, dx^{\alpha\prime}\, dx^{\beta\prime} - c^2 dt^2, \tag{29}$$

where $x^{1\prime}, x^{2\prime}, x^{3\prime}$ are the components of $\mathbf{r}'$. Clearly eq. (28) is a special case of eq. (29).

Consider the Einstein field equations with the cosmological term for a dust of density ρ and four-velocity $u^a/c = -cg^{ab}t_b$,

$$R_{ab} - \tfrac{1}{2}\, g_{ab} R + \frac{\lambda}{c^2}\, g_{ab} = -8\pi k \rho t_a t_b. \tag{30}$$

For a metric of the form of eq. (27), eq. (30) implies

$$\text{curl curl } \mathbf{v} = 0.$$

For the sake of simplicity, all further considerations will be restricted to irrotational motions,

$$\text{curl } \mathbf{v} = 0. \tag{31}$$

The strain tensor then may be written as

$$\partial_\alpha v_\beta = 1/3\ \delta_{\alpha\beta}\theta + \sigma_{\alpha\beta}$$

where

$$\theta = \text{div } \mathbf{v}$$

gives the rate of expansion, and $\sigma_{\alpha\beta}$ describes the velocity of shear. If we denote $u^a\partial_a\alpha = \partial\alpha/\partial t + \mathbf{v}\cdot\text{grad}\,\alpha$ by $\dot{\alpha}$, the remaining field equation (30) assumes the form

$R_{00} - \frac{1}{2} g_{00} R + \lambda$

$$1/2\ \sigma_{\alpha\beta}\sigma^{\alpha\beta} - 1/3\ \theta^2 + \lambda = -8\pi k\rho, \tag{32}$$

$$2\dot{\theta} + \theta^2 + 3/2\ \sigma_{\alpha\beta}\sigma^{\alpha\beta} - 3\lambda = 0, \tag{33}$$

$$\dot{\sigma}_{\alpha\beta} + \theta\sigma_{\alpha\beta} = 0. \tag{34}$$

They imply the equation of continuity,

$$\dot{\rho} + \rho\theta = 0. \tag{35}$$

On the other hand, the Newtonian equations with a cosmological term,

$$\dot{\mathbf{v}} = -\text{grad}\varphi \tag{36}$$

$$\Delta\varphi = 4\pi k\rho - \lambda, \tag{37}$$

$\frac{1}{2}(32) + \frac{1}{2}(33)$

$$\frac{\partial\rho}{\partial t} + \text{div}(\rho\mathbf{v}) = 0, \tag{38}$$

are equivalent to eq. (35) and

$$\dot{\theta} + 1/3\ \theta^2 + \sigma_{\alpha\beta}\,\sigma^{\alpha\beta} - \lambda = -4\pi k\rho. \tag{39}$$

More precisely, if $\mathbf{v}$ is an irrotational vector field which, together with ρ, satisfies eqs. (35) and (39), then there exists a function φ such that eqs. (36) through (38) hold.

It is seen by inspection that the relativistic equations (32) and (33) imply the Newtonian equation (39). Therefore, to any metric equation (27), solution of Einstein's equation (30) with irrotational $\mathbf{v}$ there corresponds an analogous solution of Newton's equations, the functions $\mathbf{v}$ being the same in both cases. The converse is not true: the system of equations

(32) through (34) is essentially stronger than that of equation (39).

As examples of solutions of equations (36) through (38), which lead to Einstein spaces, we give the following:

1. Consider a system of test particles ($\rho = 0$) falling radially towards the center of a spherically symmetric body of mass m. If the velocities of the particles vanish at infinity, then, according to Newtonian mechanics.

$$\mathbf{v} = -\sqrt{\frac{2km}{r}}\,\frac{\mathbf{r}}{r}$$

Substituting this into eq. (27), we obtain the Schwarzschild line-element,

$$r^2\,(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + dr^2 + 2\sqrt{\frac{2km}{r}}\, dr\, dt - (c^2 - \frac{2km}{r})\, dt^2 \;.$$

The co-moving form, eq. (29), of the Schwarzschild metric is

$$r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + (r'/r)\, dr'^2 - c^2\, dt^2,$$

where

$$r = (r'^{3/2} - \tfrac{3}{2}\sqrt{2km}\; t)^{2/3}.$$

2. In a Newtonian world with a cosmic repulsive force ($\lambda > 0$), a possible motion of test particles is given by

$$\mathbf{v} = \sqrt{\lambda/3}\; \mathbf{r} \quad .$$

The corresponding Riemannian metric is that of the de Sitter space.

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