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Robinson manifolds and the shear-free condition

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This paper is a short summary of the talk on *Robinson manifolds and Cauchy–Riemann spaces* given at *Journées Relativistes 2001* in Dublin. It recalls two generalizations of the definition of shear-free congruences of null geodesics. These generalizations are equivalent in dimension 4, but not in higher dimensions, as illustrated on an example described by Ivor Robinson during that conference.

1. Introduction

Shear-free congruences of null geodesics, introduced in a seminal paper by Ivor Robinson[1], play an important role in the description of electromagnetic and gravitational waves; they led to the discovery of new solutions of Einstein's equations[2,3]; their study influenced the development of twistors[4]. Originally, such congruences were considered in Lorentzian space-times where they define three-dimensional Cauchy–Riemann manifolds. Later, the property of being shear-free has been generalized to higher dimensions in at least two different ways; they are being compared in this paper.

The notation used in this paper is standard; for example, the symbols $\otimes$, $\wedge$ and $\lrcorner$ denote the tensor, exterior and interior products, respectively. Manifolds and maps among them are smooth (of class C^∞). The Lie derivative with respect to a vector field k is denoted by L_k; if α is a differential form, then there holds the Cartan formula $L_k\alpha = k\lrcorner \mathrm{d}\alpha + \mathrm{d}(k\lrcorner\alpha)$. The Hodge dual of α is $\star\alpha$.

2. The shear-free condition

Recall the derivation of the condition characterizing shear-free congruences of null geodesics in space-time; see Refs. 1,5 and 6. If $f \neq 0$ is a null electromagnetic field, then the complex 2-form $F = f - \mathrm{i}\star f$ can be written as $F = \kappa \wedge \mu$, where the 1-forms κ and μ are both null and orthogonal to each other, κ is real and μ is genuinely complex, $\mu \wedge \bar{\mu} \neq 0$. One can find a real form λ such $(\kappa, \lambda, \mu, \bar{\mu})$ is a field of null coframes; let $(l, k, \bar{m}, m)$ be the dual field of frames. The forms κ and μ are defined by F only up to transformations $\kappa \mapsto a^{-1}\kappa$ and $\mu \mapsto a\mu + b\kappa$, where a is a nowhere vanishing real function. This freedom can be used to achieve $g(k, l) = 1$;

putting $g(m, \bar{m}) = p^2$, one can write the metric tensor as

$$g = \kappa \otimes \lambda + \lambda \otimes \kappa + p^2(\mu \otimes \bar{\mu} + \bar{\mu} \otimes \mu). \tag{1}$$

Maxwell's equations, $\mathrm{d}F = 0$, imply $L_k F = 0$. The last equation is equivalent to

$$L_k \kappa = q\kappa \quad \text{and} \quad L_k \mu = r\mu + s\kappa, \tag{2}$$

where q, r and s are functions. The first among equations (2) says that the null vector field k is tangent to a congruence of geodesics. Computing the Lie derivative with respect to k of the metric tensor (1), one obtains the full shear-free and geodetic condition,

$$L_k g = \rho g + \kappa \otimes \xi + \xi \otimes \kappa, \tag{3}$$

where ρ is a function and ξ is a real 1-form. Eq. (3) has a simple geometric interpretation: if it is satisfied, then the flow generated by k preserves the conformal geometry of the screen spaces $K^\perp/K$, where $K_x^\perp$ is the subspace of $T_x M$, orthogonal to the null line K_x spanned by the vector k_x. For this reason, Robinson and Trautman[7] proposed to use (3) as the definition of shear-free flows also in the context when k is not necessarily null or the manifold M has more than 4 dimensions.

3. Robinson manifolds

There is another generalization of the shear-free condition to higher dimensions that leads to the notion of a *Robinson manifold*[8,9]. Let now (M, g) be a Lorentzian manifold of dimension $2n$, with g of signature $(2n-1, 1)$. A complex n-form F is said to be null if, at every $x \in M$, the vector space

$$N_x = \{v \in \mathbb{C} \otimes T_x M \mid v \lrcorner F = 0\} \tag{4}$$

is totally null of (maximal) dimension n; the Hodge dual of F is proportional to F. The null n-form can be represented as

$$F = \kappa \wedge \mu_1 \wedge \cdots \wedge \mu_{n-1},$$

where $\kappa = g(k)$ is real. Let $\mathrm{Sec}N$ be the module over $C^\infty(M)$ of sections of the vector bundle $N \to M$, with fiber over x given by (4). If the null form is closed, $\mathrm{d}F = 0$, then the bundle $N \to M$ satisfies the integrability condition

$$[\mathrm{Sec}N, \mathrm{Sec}N] \subset \mathrm{Sec}N. \tag{5}$$

Penrose and Rindler give a spinorial form of the integrability condition and refer to it as 'the essence of the shear-free ray congruence condition', see §7.3 in[10]. The flow generated by k preserves N. If the flow is regular in the sense that the set of its trajectories forms a manifold $\mathcal{M}$ of dimension $2n-1$, then the bundle $N \to M$ descends to a bundle $\mathcal{N} \to \mathcal{M}$, with fibers of complex dimension $n-1$ and defines on $\mathcal{N}$ the structure of a Cauchy–Riemann manifold. A Robinson manifold is now defined, without reference to F, as a Lorentzian manifold M with a bundle $N \subset$

$\mathbb{C} \otimes TM$ such that the fibres of $N \to M$ are totally null of the maximal dimension and (5) holds.

In dimension 4, conditions (3) and (5) are essentially equivalent[8], but not in higher dimensions, as can be seen from the following example given by Robinson. Consider the six-dimensional manifold $M = \mathbb{R}^2 \times \mathbb{C}^2$ with real coordinates r, u and complex coordinates w, z. Let p and q be nowhere vanishing functions of the coordinates, put $2\,\mathrm{d}u\,\mathrm{d}r = \mathrm{d}u \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}u$, etc., and consider the metric tensor

$$g = 2\,\mathrm{d}u\,\mathrm{d}r + p^2 \mathrm{d}w\,\mathrm{d}\bar{w} + q^2 \mathrm{d}z\,\mathrm{d}\bar{z}.$$

The vector fields $k = \partial/\partial r$, $\partial/\partial\bar{w}$ and $\partial/\partial\bar{z}$ span a bundle $N \to M$ with totally null fibres. Let $A(u, w, z) \neq 0$ be a smooth function, holomorphic in w and z. The null form $F = A(u, w, z)\mathrm{d}u \wedge \mathrm{d}w \wedge \mathrm{d}z$ is closed and (4) holds.The congruence generated by k has no twist and so the associated CR manifold $\mathcal{M} = \mathbb{R} \times \mathbb{C}^2$ is trivial. The shear-free condition (3) is not satisfied unless the function p/q is independent of r.

In dimension 4, there is an important class of Robinson manifolds having, as the associated CR space, the 'hyperquadric', in the CR category locally equivalent to $\mathbb{S}_3 \subset \mathbb{C}^2$. The corresponding shear-free congruence of null geodesics is referred to by Penrose[4] as the *Robinson congruence*. The following space-times are in this class: the Gödel universe, the Taub–NUT solution and Hauser's waves of type N. Maxwell's equations for a null electromagnetic field associated with the Robinson congruence involve the celebrated Hans Lewy differential operator; see Refs 6,8 and 9 for further remarks on this subject.

In dimension 6, there is a distinguished class of Robinson manifolds having, as the underlying CR manifold, the five-dimensional space of Penrose's projective null twistors.

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