The exact — in the "infinite approximation" — shape of the function H, which characterizes the limb darkening in the photospheric problem, is given by several authors ([1] p. 125, last column of table XL) in tabular form. Finally we obtain instead of (4) an exact asymptotic formula

$$\frac{L_0}{3\mu_0} = \frac{H(\mu_0)/\sqrt{3}}{(1-\frac{1}{3}\tilde{\omega}_1)\tau_1+2q(\infty)}. \tag{5}$$

When $e^{-\tau_1/\mu_0}$ can be omitted as compared with $L_0/(3\mu_0)$, formula (5) gives directly the fraction of light transmitted diffusely. For the single act of scattering, the net flux with respect to the plane normal to the direction of the incident beam depends only on $\tilde{\omega}_1$; we may therefore expect that formula (5) will give a good approximation also in the case, where the expansion of p is not limited to the two lowest terms only. Some computations made by the writer with the aid of the generalized formula (4) confirm this expectation.

For terrestrial clouds, formula (5) shows in a clear way the dependence of the illumination on $\tilde{\omega}_1$ and τ_1 and can serve as one of the bases for the determination of these parameters by observation. For the average size of water drops in a cloud, $\frac{1}{3}\tilde{\omega}_1$ approaches 1; the smallness of the coefficient of τ_1 in (5) and (4) explains the relative smallness of the variations in illumination by a cloudy sky, in spite of very large variations in the thickness of the clouds; this smallness explains also why, in spite of its enormous surface brightness, the disc of the sun is invisible through a cloud transmitting a considerable fraction (*e.g.* 20%) of incident light.

ASTRONOMICAL OBSERVATORY, UNIVERSITY OF WARSAW

REFERENCES

[1] S. Chandrasekhar, *Radiative Transfer*, Oxford, Clarendon Press 1950.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. III, No. 6, 1955

THEORETICAL PHYSICS

On the Proofs of "Backward" Uniqueness for Some Non-Conservative Fields Describable by Differential Equations of the Hyperbolic Type

by

A. TRAUTMAN

Communicated by A. RUBINOWICZ at the meeting of March 28, 1955

1. The proofs of uniqueness for linear differential equations of the hyperbolic type are based on the assumption of continuity of the physical phenomena described by these equations (e. g. law of conservation of energy, equation of continuity for the probability current). Theorems on the uniqueness of the solution (uniqueness meaning that there exists at most one solution), as usually formulated for hyperbolic type equations or systems thereof, besides their mathematical value, have also a certain physical interpretation. Namely, if the initial conditions are prescribed within a bounded region, then the uniqueness of the solution of Cauchy's problem means that the velocity of propagation of "signals" is finite.

Theorems on the uniqueness of the solution of the problem of initial value were first formulated by Zaremba [1] and Rubinowicz [2], [3]. Their ideas were developed by other authors [4], [5]. Plebański [6] has recently formulated two theorems on uniqueness, valid for a wide class of differential equations in mathematical physics. These theorems will be briefly described in section 3. In the present paper, the results obtained by Plebański have been extended to some equations, such that the equation of continuity $\Gamma^\nu{}_{,\nu}+Q=0$ follows from them as a consequence (see section 4). This makes possible the proof of "backward uniqueness" for Maxwell's macroscopic equations and for the equations of the scalar fields (see section 5).

2. In the present paper, the usual notations of the theory of relativity will be used, repeated upper and lower index meaning a sum from 0 to 3 (Greek indices) or from 1 to 3 (Latin indices), x^0 — time co-ordinate; x^1, x^2, x^3 — space co-ordinates. $\varphi_{,\alpha}$ designates $\frac{\partial\varphi}{\partial x^\alpha}$ and the symbol $d_k\tau$ in the integrand denotes integration over a k-dimensional continuum.

FA denotes the boundary of the region A. We shall deal with systems of linear partial differential equations ("the field equations") such that the characteristic quadratic form

(1) $$A[\vec{\xi}] = a^{\alpha\beta}\xi_\alpha\xi_\beta,$$

where $a^{\alpha\beta} = a^{\alpha\beta}(x^\nu)$ satisfies, in the region under consideration, the following conditions:

(2.1) $a^{00} > 0,$ (2.2) $a^{0k} = 0,$ (2.3) $a^{\alpha\beta} = a^{\beta\alpha},$

(2.4) $a^{kl}\xi_k\xi_l < 0,$ *if* $\vec{\xi} \neq 0.$

Let R denote a bounded region of the three-dimensional continuum. We assume for simplicity that R lies on a hyperplane $x^0 = 0$. Let us consider a set of characteristic cones (we understand the characteristic cone as a surface formed by a set of bicharacteristics passing through a definite point) with vertices on a two-dimensional surface FR. We denote by Ω the set of such points x^ν of a four-dimensional continuum, so that each of them can be connected with an arbitrary point in R by means of a line which will not cross any of the characteristic cones. The set of points belonging to Ω so that $0 < x^0 < x^0_k$ (or $x^0_k < x^0 < 0$, if $x^0_k < 0$), is denoted by $\Omega(x^0_k)$. $R(x^0_k)$ is the common part of Ω and the hyperplane $x^0 = x^0_k$. Further: $T(x^0_k) = F\Omega(x^0_k) - R - R(x^0_k)$. We chose the direction of the normal unit vector n_ν so as to have $n_0 > 0$.

3. The following lemma was shown by Plebański: if I^ν is a vector field such that

(3) $$I^0 \geqslant 0, \quad a_{\alpha\beta}I^\alpha I^\beta \geqslant 0; \quad \text{where} \quad (a_{\alpha\beta}) = (a^{\alpha\beta})^{-1},$$

then

(4) $$\lambda = I^\alpha n_\alpha \geqslant 0 \quad \text{on} \quad R = R(0),\ R(x^0_k),\ T(x^0_k).$$

If an equation of continuity

(5) $$I^\alpha_{,\alpha} + Q = 0$$

follows from the equations of the field, then integrating both sides of (5) over $\Omega(x^0_k)$ we obtain:

(6) $$\varepsilon \int\limits_{\Omega(x^0_k)} Q\, d_4\tau - \int\limits_{R(0)} I^0 d_3\tau + \int\limits_{R(x^0_k)} I^0 d_3\tau + \int\limits_{T(x^0_k)} \lambda\, d_3\tau = 0,$$

where $\varepsilon = \operatorname{sgn} x^0_k$. If $I^0|_R = 0$, we obtain from (6) with respect to (4):

$$I^0 = 0 \quad \text{in} \quad R(x^0_k), \quad \text{if} \quad Q \geqslant 0, \quad x^0_k > 0,$$

$$I^0 = 0 \quad \text{in} \quad R(x^0_k), \quad \text{if} \quad Q \leqslant 0, \quad x^0_k < 0.$$

If vanishing of I^0 is equivalent to vanishing of the corresponding functions

of the field, then the theorem on uniqueness follows from the above two lemmata in the known way. In the case of $Q = 0$ ("conservative field") the uniqueness of the solution can be proved in the whole region Ω, and in particular, in Ω^- (i. e. in $\Omega(x^0_k)$ for any $x^0_k < 0$) which is the case of "backward" uniqueness.

4. The above theorems do not determine the uniqueness if the sign of Q is not constant; they also give no information concerning the behaviour of the field in $\Omega(x^0_k)$ for $x^0_k > 0$ provided $I^0|_R = 0$ in the case of $Q \leqslant 0$, and they do not determine the "backward" uniqueness in the case of $Q \geqslant 0$, which is the case of Maxwell's equations for conducting media. We shall now prove a theorem from which the "backward" uniqueness for Maxwell's macroscopic equations follows as one of the consequences.

Theorem. *If a vector field I^ν, continuous in Ω, satisfies the conditions* (3) *and* (5) *and there exists such a constant M, that for each $x^\nu \in \Omega^0$ there is*

(7) $$|Q| \leqslant M I^0,$$

then the vanishing of I^0 on R implies the vanishing of I^0 in Ω.

We shall prove this theorem for $x^0_k > 0$ and $x^0_k < 0$ simultaneously. From (6) we get

(8) $$0 \leqslant E(x^0_k) = \int\limits_{R(x^0_k)} I^0 d_3\tau = \int\limits_{R(0)} I^0 d_3\tau - \varepsilon \int\limits_{\Omega(x^0_k)} Q\, d_4\tau - \int\limits_{T(x^0_k)} \lambda\, d_3\tau.$$

Taking into account (4) we obtain

(9) $$0 \leqslant E(x^0_k) \leqslant -\varepsilon \int\limits_{\Omega(x^0_k)} Q\, d_4\tau + \int\limits_{R(0)} I^0 d_3\tau \leqslant \int\limits_{\Omega(x^0_k)} |Q|\, d_4\tau + E(0),$$

hence by (7):

(10) $$0 \leqslant E(x^0_k) \leqslant E(0) + \int\limits_{\Omega(x^0_k)} M I^0 d_4\tau = E(0) + \varepsilon M \int\limits_0^{x^0_k} E(x^0)\, dx^0.$$

Denoting

$$\psi(x^0_k) = e^{-\varepsilon M x^0_k} \int\limits_0^{x^0_k} E(x^0)\, dx^0$$

we obtain:

$$\psi'(x^0_k) = -\varepsilon M \psi + e^{-\varepsilon M x^0_k} E(x^0_k).$$

Thus (10) becomes

(11) $$\psi'(x^0_k) \leqslant E(0)\, e^{-\varepsilon M x^0_k}.$$

Now let x^0_n be an arbitrary number so that $\operatorname{sgn} x^0_n = \operatorname{sgn} x^0_k$. Integrating both sides of inequality (11) over the interval $(0, x_n^0)$, if $x_n^0 > 0$, or over $(x_n^0, 0)$ if $x_n^0 < 0$, we obtain:

$$\varepsilon[\psi(x_n^0) - \psi(0)] \leqslant -\frac{E(0)}{M}(e^{-\varepsilon M x_n^0} - 1)$$

and, since $\psi(0) = 0$, we get:

$$(12) \qquad \varepsilon \int_0^{x_n^0} E(x^0)\,dx^0 \leqslant \frac{E(0)}{M}(e^{\varepsilon M x_n^0} - 1).$$

According to (12), and bearing in mind that $\varepsilon x_n^0 = |x_n^0|$, we obtain from (10) for an arbitrary x_k^0:

$$(13) \qquad 0 \leqslant \int_{R(x_k^0)} I^0 d_3\tau \leqslant E(0) + E(0)(e^{\varepsilon M x_k^0} - 1) = e^{M|x_k^0|} \int_R I^0 d_3\tau.$$

Putting $I^0|_R = 0$ into (13), we obtain proof of our theorem.

If I^0 is interpreted as the density of the field energy, then (13) states that the changes of energy in Ω are at most of an exponential nature.

Conclusion: if

1^0 the functions $\varphi^A (A = 1, 2, \ldots, N)$ satisfy linear differential equations with the characteristic quadratic form $A[\vec{\xi}] = a^{\alpha\beta}\xi_\alpha\xi_\beta$,

2^0 the form $A[\vec{\xi}]$ has the properties (2.1)–(2.4),

3^0 the field equations lead to the equation of continuity (5),

4^0 the vector field I^γ is continuous and satisfies (3),

5^0 there exists a constant M satisfying (7) inside Ω,

6^0 vanishing of I^0 is equivalent to vanishing of φ^A,

there exists at most one solution of the field equations in Ω satisfying on R the prescribed initial conditions.

Proof. If two solutions φ_1^A, φ_2^A existed, then in view of the linearity of the equations, $\varphi_*^A = \varphi_1^A - \varphi_2^A$ would also be a solution satisfying the initial homogeneous conditions $\varphi_*^A|_R = 0$. However, the vector I_*^α corresponding to the field φ_*^A would vanish in R, and thus, by virtue of the theorem which has been proved, it would also vanish in Ω. Thus it follows that $\varphi_*^A \equiv 0$ i. e. $\varphi_1^A = \varphi_2^A$ inside Ω.

5. Examples.

a. Maxwell's equations.

From the macroscopic equations of the electromagnetic field we can obtain a conservation principle of the form

$$(14) \qquad \operatorname{div}\vec{S} + \frac{\partial W}{\partial t} = -\sigma E^2 = -Q,$$

where $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$, $W = \frac{1}{8\pi}(\varepsilon E^2 + \mu H^2)$; ε, μ, σ are positive functions of the point x^k, and $\varepsilon \geqslant 1$. The components of $\vec{E}$ and $\vec{H}$ satisfy the wave equation with the characteristic form $\frac{\varepsilon\mu}{c^2}\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2$ which is of type (2). Denoting $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, $(I^0, I^1, I^2, I^3) = (W, \vec{S})$ we can write (14) in the form of (5). It can easily be proved that vector I^α satisfies the conditions (3). Assuming $M = 8\pi\cdot\sup\sigma$, we find (in view of $\varepsilon \geqslant 1$) that condition (7) is satisfied. Assumptions 1^0–6^0 are thus fulfilled and it follows that there exists at most one electromagnetic field in Ω taking prescribed values in a bounded region R.

b. Scalar field equation.

Let us consider the equation

$$(15) \qquad K[u] = (a^{\alpha\beta}u_{,\alpha})_{,\beta} + b^\alpha u_\alpha + cu = 0,$$

where

$$(16) \quad \begin{cases} a^{\alpha\beta} & \text{are differentiable functions satisfying (2),} \\ c \geqslant 0 & \text{inside } \Omega, \\ b^\alpha & \text{are functions continuous in } \Omega. \end{cases}$$

If $b^\alpha = 0$, (15) is the Euler-Lagrange equation of the following problem:

$$\delta \int_\Omega L\,d_4\tau = 0, \qquad \text{where} \quad L = a^{\alpha\beta}u_{,\alpha}u_{,\beta} - cu^2.$$

The impulse-energy tensor

$$T_\alpha^\beta = \delta_\alpha^\beta L - u_{,\alpha}\frac{\partial L}{\partial u_{,\beta}} = \delta_\alpha^\beta L - 2a^{\mu\beta}u_{,\mu}u_{,\alpha}$$

satisfies the equation

$$T_{\alpha,\beta}^\beta = 2u_{,\alpha}b^\nu u_{,\nu}.$$

Defining $I^\beta = -T_0^\beta$, we obtain: $I^\beta_{,\beta} = -2u_{,0}b^\nu u_{,\nu} = -Q$ and

$$(17) \qquad I^0 = a^{00}u^2_{,0} - a^{mn}u_{,m}u_n + cu^2 \geqslant 0.$$

If, further, $(a_{\alpha\beta}) = (a^{\alpha\beta})^{-1}$, then

$$(18) \qquad a_{\alpha\beta}I^\alpha I^\beta = a_{00}L^2 + 4cu^2u^2_{,0} \geqslant 0.$$

Denoting the upper bound of the function $2|b^0| + |b^1| + |b^2| + |b^3|$ in $\overline{\Omega}$ by M_1 we get:

$$(19) \qquad |Q| = |2u_{,0}b^\nu u_{,\nu}| \leqslant M_1\sum_{\alpha=0}^{3}u^2_{,\alpha}.$$

Let $\varkappa$ denote the smallest lower bound of the moduli of characteristic roots of the form $A[\vec{\xi}]$. In view of (2.1) and (2.4), it must be that $\varkappa > 0$, and

$$(20) \qquad a^{00}\xi_0^2 - a^{mn}\xi_m\xi_n \geqslant \varkappa\sum_{\alpha=0}^{3}\xi_\alpha^2.$$

Taking into account (17), (19) and (20), we get:

$$(21)\qquad |Q|\leqslant \frac{M_1}{\varkappa}(a^{00}u^2_{,0}-a^{mn}u_{,m}u_{,n})\leqslant \frac{M_1}{\varkappa}I^0=MI^0.$$

It can be seen from equations (16)–(21) that in the case of equation (15) the assumptions 1^0-5^0 of the conclusion on uniqueness are satisfied. Further, vanishing of I^0 is equivalent to vanishing of $u_{,a}$. Thus we easily obtain a theorem on the uniqueness of the solutions of the equation: $K[u]=f(x^r)$, with Cauchy-type conditions for u prescribed on R.

c. It should be noted that the third of the assumptions (16), which makes estimation (19) possible, is of essential importance; e. g. the equation

$$u_{tt}-\Delta u-\frac{u}{\sin t}+u=0$$

has a non-trivial solution $u=\cos t-1$, vanishing together with its partial derivatives on the hyperplane $t=0$. This example shows also that the assumption $|Q|\leqslant MI^0$ is essential for the truth of the theorem.

The author is greatly indebted to G. Plebański for his help while this paper was in course of preparation.

INSTITUTE OF THEORETICAL PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] S. Zaremba, Rendic. Acc. Lincei, ser. 5, **14** (1915), 904.
[2] A. Rubinowicz, Monatsh. f. Math. u. Phys. **30** (1920), 65.
[3] A. Rubinowicz, Phys. Z. **27** (1926), 707.
[4] Fiedrichs a. Lewy, Math. Ann. **98** (1927), 192.
[5] S. L. Sobolev, Matiem. sbornik, **5** (1) (1939), 71.
[6] J. Plebański, Acta Phys. Pol. **12** (1953), 230.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. III, No. 6, 1955

THEORETICAL PHYSICS

Note on Coulomb Effects in Stripping Reactions

by
J. SAWICKI

Presented by L. INFELD on April 16, 1955

Coulomb corrections to the Butler cross-section for the (d, p) and (d, n) reactions were recently [1] discussed with the help of the "zero-range" Horowitz-Messiah approximation for the $n-p$ interaction. The integrations in the matrix elements were performed with the help of complicated numerical computations.

It appears that on applying the H.-M. approximation and the asymptotical exponential form of the wave function of the captured particle in the particular case of the S-state in the one-particle model, one can calculate analytically the Born approximation matrix element for the (d, p) or (d, n) reaction with the plane waves replaced by the Coulomb wave functions.

Let us consider first the (d, p) reaction. The usual Born approximation c. m. differential cross-section may be written in the form:

$$(1)\qquad \sigma(\theta)=\frac{M_d^*M_p^*}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{1}{3(2I_i+1)}\sum_{\substack{\mu_p\mu_d\\ \mu_i\mu_f}}{}'\Big|\int d\sigma_n\,d\sigma_p\,d\xi\,d_{(3)}\bar{r}_n\,\chi_f^*(\xi,\bar{r}_n\,\sigma_n)\,\chi_t(\xi)\,e^{i\bar{k}_d\bar{r}_n}\cdot$$
$$\cdot e^{-i\bar{k}'_p\bar{r}_n}\cdot\int d_{(3)}\bar{\varrho}\,e^{-i\bar{k}\bar{\varrho}}\,V_{np}(\varrho)\psi(\varrho)S_{\mu d}(\sigma_n,\sigma_p)\Big|^2,$$

where M_d^x, M_p^x are the deuteron and proton reduced masses; k_d, k_p their respective momenta; μ_p, μ_d, μ_i, μ_f — magnetic numbers; I_i — total angular momentum quantum number of the initial nucleus; σ_n, σ_p — spin variables of the neutron and the proton; $\bar{r}_n$ — vectors from the initial nucleus (i) to the neutron and $\bar{\varrho}$ — the vector from the neutron to the proton; $\bar{k}'_p=\frac{M_i}{M_f}\bar{k}_p$ and $\bar{K}=\frac{\bar{k}_d}{2}-\bar{k}_p$; χ_i and χ_f are the internal wave functions of the initial and final nuclei; $V_{np}(\varrho)$ is the n-p interaction and $\psi(\varrho)\cdot S_{\mu_d}(\sigma_n,\sigma_p)$ — the deuteron internal wave function.