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OPTICAL STRUCTURES IN RELATIVITY

Notes for lecture given on Nov. 24, 2015, at the Conference
100 years of General Relativity organized by PoToR in Warsaw, Poland

Informal review, mainly historical

Explain title: why optical?

What to call a vector $k \neq 0$ such that $g_{\mu\nu}k^\mu k^\nu = 0$?

mathematicians: *isotropic* (only in dimension 2 null directions are isotropic: they are preserved by rotations)

physicists: *null* (OK in English, but in many languages – such as French – there is confusion: nul = zero)

Élie Cartan: *optique* (excellent: null vectors lie on the light cone, but it did not catch on).

In a short paper of 1922 (no equations), Cartan wrote

...Il existe en chaque point quatre directions *optiques* (c'est-à-dire annulant le $\mathrm{d}s^2$) privilégiées...Dans le cas du $\mathrm{d}s^2$ d'une seule masse attirante ($\mathrm{d}s^2$ de Schwarzschild), ces quatre directions optiques privilégiées se réduisent à deux (doubles)...

From that last sentence one sees that Cartan had the premonition of the Petrov-Penrose classification of Weyl tensors.

There is also a vague remark about the parallel transport of optical directions (shear-free property?)

Null (optical) electromagnetic field: $f = (\mathbf{E}, \mathbf{B}) \neq 0$

$$\mathbf{E} \cdot \mathbf{B} = 0 \; \& \; \mathbf{E}^2 - \mathbf{B}^2 = 0 \Leftrightarrow \exists \text{ optical } k \text{ such that } T^{\mu\nu} = k^\mu k^\nu$$

Such an f is said to be associated with k; algebra: given null k there is an associated f.

If Maxwell's eqs satisfied, then div $T = 0$ implies k is geodetic (Mariot 1954). But the geodetic condition is not sufficient for the existence of an optical solution of Maxwell's eqs. (example: consider k for cylindrical waves)

Ivor Robinson (in late 1950s presented at seminars in England, published 1961) has shown:

there is a solution $f \neq 0$ of Maxwell's eqs associated with an optical k on a Lorentz mfld M iff k is geodetic and shear-free, i.e. the flow of k preserves the conformal structure of the plane bundle

$$K^{\perp}/K \to M,$$

of 'screen spaces', where $K \subset TM$ is the line bundle of null directions, $K_x = \mathbb{R}k(x)$.

This is expressed by: there exist function ρ and 1-form μ such that there holds the **Robinson equation** (a weakened form of the Killing eq.):

$$\mathcal{L}(k)g = \rho g + 2\lambda.\mu \qquad \text{(Rob)}$$

where $\lambda = g(k)$ and $\mathcal{L}(k)$ is the Lie derivative in the direction of k. (Notational convention: $2\lambda.\mu = \lambda \otimes \mu + \mu \otimes \lambda$.)

Call (M, g, K) an **optical space-time** if the Robinson eq. is satisfied for a section $k \neq 0$ of the line bundle K of null directions.

Many early solutions of Einstein's eqs, such as Schwarzschild (1916) and conformally flat space-time have been recognized to be optical.

An early fundamental result is the **Goldberg–Sachs theorem** (1962):

An empty space-time is optical iff its Riemann tensor is algebraically special i.e. there is k such that $k_{[\mu}R_{\nu]\rho\sigma\tau}k^\rho k^\sigma = 0$.

More refined definition in terms of the spinor repr. of the Weyl tensor $\Psi_{ABCD} = \varphi_{(A}\psi_B\chi_C\omega_{D)}$: type N, D, II, etc.

There have been doubts concerning the physical relevance of the optical space-times: they constitute a very 'small subset' of the set of all solutions. But

they occur in the **asymptotic region** (peeling off: Riemann tensor $R = N/r + III/r^2 + ...$)

there are surprising properties of the **gyromagnetic ratio** of the charged solutions (Kerr-Newman, charged Taub-NUT...): double of the classical value; do they indicate connections between gravitation and quantum mechanics?

there is the unexpected relevance of the **Kerr metric as the generic black hole**

It is easy to construct k satisfying the Robinson condition in Minkowski space.

The simplest: if $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$, then $\lambda = \mathrm{d}u$, $u = t - z$ leads to the plane wave $f = \lambda \wedge (a(u)\,\mathrm{d}x + b(u)\,\mathrm{d}y)$.

There are analogous **plane(fronted) gravitational waves** found by H. W. Brinkmann already in 1925; rediscovered several times.

Einstein and Nathan Rosen around 1937 found the plane waves, in a different coordinate system, inducing singularities in the components of the metric tensor; interesting history of the attempt to publish the Einstein–Rosen paper in Phys.Rev. (Kennefick 2005).

In 1936 and 1937 Leopold Infeld was with Einstein in Princeton and, under his influence acquired conviction that gravitational radiation did not exist. This had an influence on research in Warsaw in the years 1955-68. At the suggestion of Jerzy Plebański, Andrzej Trautman was preparing his Ph. D. thesis on gravitational waves. Rose Michalska–Trautman convinced Infeld of the existence of grav. waves (joint paper, Ann. Physics 1969)

Another remarkable solution found by Kurt Gödel (1949); first 'twisting': $\lambda \wedge \mathrm{d}\lambda \neq 0$, global and homogeneous, with closed time-like world lines.

In the 1950s there started research on optical solutions of Einstein's equations making explicit use of the conditions resulting from the Robinson equation.

Robinson and Trautman (1960) non-twisting, expanding: div $k \neq 0$ (among them Schwarzschild and waves with spherical fronts; they have a simple electromagnetic analog); $R = N/r + III/r^2 + D/r^3$ exactly.

Wolfgang Kundt (1961) non-twisting and non-expanding (among them plane-fronted waves)

In 1962, at the GRG Conference in Poland, IR and AT gave the metric tensor for a general twisting and shear-free k, but the first twisting solutions of Einstein's eqs were found by

Newman, Unti and Tamburino (NUT 1963; also found, from symmetry, by A. H. Taub 1951)

Roy Kerr (1963)

Later: Plebański and Demiański (1976) essential generalization of NUT and charged Kerr

There are several good reviews and a book by D. Kramer, H. Stephani, M. MacCallum and E. Herlt (1980)

The **Robinson optical, twisting congruence** has been constructed in Minkowski space: the metric

$$g = 2\lambda\,\mathrm{d}r - (r^2+1)(\mathrm{d}x^2 + \mathrm{d}y^2),$$

where

$$\lambda = g(k) = \mathrm{d}u + x\,\mathrm{d}y - y\,\mathrm{d}x, \quad k = \partial/\partial r,$$

is flat, the null vector field k is optical, $\mathcal{L}(k)g = -2r(\mathrm{d}x^2 + \mathrm{d}y^2)$, and twisting,

$$\lambda \wedge \mathrm{d}\lambda = 2\,\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y \neq 0.$$

Let $\Phi = f - \mathrm{i}\star f$, then $\star\Phi = \mathrm{i}\,\Phi$ and Maxwell's eqs are $\mathrm{d}\Phi = 0$. Put

$$\Phi = A(x,y,r,u)\lambda \wedge (\mathrm{d}x + \mathrm{i}\,\mathrm{d}y), \quad \text{then} \quad \star\,\Phi = \mathrm{i}\,\Phi$$

and Maxwell's eqs reduce to $\partial A/\partial r = 0$ and $Z(A) = 0$, where

$$Z = \frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y} - \mathrm{i}(x + \mathrm{i}\,y)\frac{\partial}{\partial u}$$

is a differential operator on $\mathbb{R}^3$ introduced by Hans Lewy (1957) who showed, to the surprise of many mathematicians, that there are smooth (of class C^∞) functions $h : \mathbb{R}^3 \to \mathbb{C}$ such that the differential equation

$$Z(f) = h$$

has no solution, even locally, for $f : \mathbb{R}^3 \to \mathbb{C}$. (By the Cauchy-Kowalewski theorem, there are such solutions if h is analytic.)

The Robinson congruence, generated by $k = \partial/\partial r$ played a role in the development of **twist**ors; those **twist**ing congruences provided the 'missing dimension' in the space $\mathbb{C}P_3$ of projective twistors (5 dim are provided by the set of null lines in Minkowski space); see Penrose on *The origins of twistor theory* (1987).

In much of the work on solutions of Eintein's eqs for optical space-times there appeared, in a natural manner, complex numbers; this has been recognized as an indication of the role played there by

Cauchy–Riemann structures

They appeared explicitly, for the first time in relativity, in the work of Penrose (1983). The five-dimensional submanifold L of $\mathbb{C}P_3$,

$$L = \{\mathrm{dir}\, Z \in \mathbb{C}P_3 \mid |Z_1|^2 + |Z_2|^2 - |Z_3|^2 - |Z_4|^2 = 0\}, \quad Z = (Z_1, Z_2, Z_3, Z_4)$$

has such a natural Cauchy–Riemann structure obtained by its embedding in the complex manifold $\mathbb{C}P_3$. Penrose conjectured that manifolds with a Cauchy–Riemann structure that cannot be so embedded may play a role in a quantum theory of gravitation.

Observation (Warsaw group): provided suitable regularity conditions are satisfied, with any optical space-time, there is an associated 3-dimensional C-R manifold L (and conversely).

Sketch of proof: the set of curves defined by the flow ϕ_t of k is a 3-manifold $L = M/\phi$.The bundle $K^\perp/K$ over M descends to a bundle

$$TL \supset H \to L$$

with 2-dim. fibres that have a conformal structure. With a choice of orientation on L, this conformal structure is equivalent to a complex structure in the fibres: by definition, this is a Cauchy–Riemann structure on L. The manifold L with H is as close to being a complex manifold as a 3-dim. manifold can be. The form $\lambda = g(k)$ on M descends to a form on L, also denoted by λ and $H = \ker \lambda$.

Method of constructing C-R spaces: embed 3-dim. manifold L in $\mathbb{C}^2$ and put $H = TL \cap \mathbb{C}^2$ (but not all C–R spaces can be embedded; J. Lewandowski, P. Nurowski and J. Tafel (1990) have shown that if the C-R structure defines a solution of Einstein's eqs, then it is embeddable)

The Robinson congruence is associated with a C-R structure on

$$\mathbb{S}_3 \subset \mathbb{C}^2$$

(the most symmetric such structure in 3 dimensions: its group of symmetries is SU$(2, 1)$, as may be seen by projectivising the equation of the sphere to $|z_1|^2 + |z_2|^2 - |z_3|^2 = 0$.)

In the proper Riemannian case, $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, in the neutral, $\mathfrak{so}(2,2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and in the complex case, the Riemann tensor decomposes and one can impose conditions on its two components separately. This has been done first by J. Plebański and I. Robinson (1976), and extended by S.Hacyan, B. Broda, K. Rózga, M. Przanowski, P. Nurowski… giving rise to research on 'heavenly' spaces and equations.

Analogous structures in other signatures and dimensions

Complex numbers provide links to other signatures.

Recall that a vector subspace W of a vector space (V, g) is said to be totally null if every element of W is null; if W is complex $2n$-dimensional, then it has *maximal totally null* (mtn) subspaces that are n-dimensional.

Definition: A N-structure on a Riemannian manifold (M, g) of even dimension $\geqslant 4$ is a complex vector subbundle N of the complexified tangent bundle $C \otimes TM$ such that, for every $x \in M$, the fiber N_x is *mtn* and the following integrability condition is satisfied: let $\operatorname{Sec} N$ be the vector space of sections of the bundle N (its elements are vector fields), then

$$[\operatorname{Sec} N, \operatorname{Sec} N] \subset \operatorname{Sec} N. \tag{int}$$

If (M, g) is proper Riemannian, then an N-structure on M is equivalent to that of an *Hermite* manifold; the orthogonal complex structure J on M is defined as

$$J(v) = \mathrm{i}\, v \quad \text{and} \quad J(\bar{v}) = -\,\mathrm{i}\, \bar{v} \quad \text{for} \quad v \in N.$$

If (M, g) is a space-time so that g has signature $(1, 2n-1)$, $n = 2, 3, \ldots$ then

$$K = N \cap \bar{N}$$

is a real line bundle of null vectors and the integrability condition (int) generalizes the Robinson equation (Rob).

Partial results on the Goldberg-Sachs theorem in $\geqslant 6$ dimensions obtained by M. Ortaggio, V. Pravda and A. Pravdová (2013).

Summary

The study of congruences of shear-free null geodesics has led to large classes of interesting solutions of Einstein's equations such as Kerr's and gravitational waves.

The 'optical' space-time geometry underlying these solutions has interesting links to Cauchy–Riemann spaces and the question of their embeddability in complex manifolds.

Optical, Lorentzian geometries are analogous to the Hermitean geometries in proper Riemannian manifolds.