Double covers of pseudo-orthogonal groups

Andrzej Trautman
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski
Hoża 69, PL-00681 Warszawa

Abstract. For every pair (m,n) of non-negative integers one defines $\mathsf{E}_{m,n}$ to be the group of equivalence classes of central extensions of the pseudo-orthogonal group $\mathsf{O}_{m,n}$ by $\mathbb{Z}_2$. The isomorphism $k : \mathsf{E}_{m,n} \to \mathsf{H}^2(B\mathsf{O}_{m,n}, \mathbb{Z}_2)$ is used to show that $\mathsf{E}_{m,n}$ is isomorphic to the group $\mathbb{Z}_2^{l(m,n)}$, where $l(0,0) = 0$, $l(1,0) = 1$, $l(m,0) = 2$, $l(1,1) = 3$, $l(1,n) = 4$ and $l(m,n) = 5$ for $m, n > 1$. If M is a manifold with a metric tensor g of signature (m,n) and f is a smooth map from M to the classifying space $B\mathsf{O}_{m,n}$ inducing the principal $\mathsf{O}_{m,n}$-bundle P of orthonormal frames defined by g, then the bundle P can be reduced to an element H of $\mathsf{E}_{m,n}$—i.e. to a double cover of $\mathsf{O}_{m,n}$—if, and only if, the element $f^*k(H)$ of $\mathsf{H}^2(M, \mathbb{Z}_2)$ vanishes. This generalizes the classical topological condition for the existence of a pin structure on a pseudo-Riemannian manifold. The set of all $32 = 2^5$ inequivalent double covers of $\mathsf{O}_m \times \mathsf{O}_n$, the maximal compact subgroup of $\mathsf{O}_{m,n}$, $m, n > 1$, is described explicitly.

Keywords: Pin and spin groups, extensions of pseudo-orthogonal groups by $\mathbb{Z}_2$, generalized pin structures, topological obstructions.

2000 MSC: Primary 11E88, 55R40; Secondary 20C35, 22E43.

Dedicated to Professor Richard Delanghe on the occasion of his 60th birthday

Published in Clifford analysis and its applications (Prague, 2000), 377388, NATO Sci. Ser. II Math. Phys. Chem., 25, Kluwer Acad. Publ., Dordrecht, 2001.

1. Introduction

Ever since the discovery of the spin of the electron, spinors have played a major role in theoretical physics; see the chapters by Jost and van der Waerden in Fierz and Weisskopf (1960) for the early history of the subject and an account of the major role of Pauli. At first, spinors were considered only in the context of flat spaces; they were defined in terms of suitable representations of the groups SU_2 and $\mathsf{SL}_2(\mathbb{C})$, providing the unique non-trivial double covers of SO_3 and of the connected component of the Lorentz group, respectively. The discovery of parity non-conservation in weak interactions (see the chapter by Wu in *loc. cit.*) induced an increased interest in the behaviour of spinors under reflections. Mathematicians have coined the name of 'pin groups' to denote double covers of the full orthogonal groups that reduce to spin groups upon restriction to the connected component (Atiyah et al., 1964). The development of general relativity forced physicists to consider spinor fields on pseudo-Riemannian manifolds (see Schrödinger (1932) and the references to earlier work given there). A precise definition of 'spin structures' on manifolds was possible only after the notion of a fibre bundle had been introduced; Haefliger (1956) found the topological obstruction to the existence of a spin structure on an orientable, Riemannian manifold and Karoubi (1968) extended this result to the non-orientable and pseudo-Riemannian cases.

In the late 1950s, Shirokov (1960) (see also the references to earlier papers given there) pointed out that the full Lorentz group $\mathsf{O}_{1,3}$ may have 8 inequivalent double covers. His argument, later taken up and extended to $\mathsf{O}_{m,n}$ by Dąbrowski (1988), is as follows. Consider a double cover H — i.e. a central extension by $\mathbb{Z}_2$ — of $\mathsf{O}_{1,3}$. Every element of the Lorentz group is covered by two elements, say h and $-h$ of H. In particular, let $\pm s$ and $\pm t$ be elements of H covering commuting space and time reflections, respectively. Since the square of a reflection is the identity, one has $s^2 = a1$, $t^2 = b1$ and $(st)^2 = c1$, where 1 is the unit of H and $a, b, c \in \{+, -\}$. Since there are 8 different combinations of the signs a, b and c, one expects to have 8 inequivalent double covers. For example, $a = b = c = +$ may be realized as the trivial extension $\mathsf{O}_{1,3} \times \mathbb{Z}_2$, whereas the two cases when $a \neq b$ and $c = +$ correspond to the two pin groups.

Shirokov discussed the possible relevance of the different covers of the Lorentz group to the description of elementary particles. Similar ideas, in a modern setting and in connection with strings, have been put forward by Carlip and DeWitt-Morette (1988) and DeWitt-Morette and DeWitt (1990). Chamblin (1994) determined the topological obstructions to the reductions of an $\mathsf{O}_{m,n}$-bundle to the 8 double covers considered by Dąbrowski. None of these authors have shown that there are precisely 8 such double covers. The only double covers that have been described explicitly, besides the trivial one, are the two corresponding to $\mathsf{Pin}_{m,n}$ and $\mathsf{Pin}_{n,m}$.

In this paper, I determine the group $\mathsf{E}_{m,n}$ consisting of inequivalent central extensions by $\mathbb{Z}_2$ of the group $\mathsf{O}_{m,n}$ for $m, n \geqslant 0$. The order of this group is the number of inequivalent double covers of $\mathsf{O}_{m,n}$. It turns out that the Lorentz group $\mathsf{O}_{1,3}$ has 16 such double covers; only 8 among them are 'Cliffordian' in the sense of being generated, as elements of $\mathsf{E}_{1,3}$, by the pin groups that are subsets of the complex Clifford algebra $\mathsf{Cliff}_4(\mathbb{C})$. For $m, n > 1$, the group $\mathsf{O}_{m,n}$ has as many as 32 inequivalent double covers. The group $\mathsf{O}_{1,1}$ has 8 double covers; all of them are Cliffordian. The isomorphism $\mathsf{E}_{m,n} \to \mathsf{H}^2(B\mathsf{O}_{m,n}, \mathbb{Z}_2)$ gives rise to a simple and general formulation of the topological condition for the existence of a generalized pin structure, i.e. of a reduction of the bundle of orthonormal frames of a pseudo-Riemannian manifold to a double cover of $\mathsf{O}_{m,n}$.

2. Notation

For every $n \in \mathbb{N}$, one has the orthogonal group O_n, its connected component SO_n, the spin group Spin_n, two pin groups Pin_n^+ and Pin_n^-: if $u \in \mathbb{R}^n \cap \mathsf{Pin}_n^\pm$, then $u^2 = \pm 1$ and one says that u is a unit vector. For $m, n \in \mathbb{N}$, one has the group $\mathsf{O}_{m,n} \subset \mathsf{GL}_{m+n}(\mathbb{R})$ of automorphisms of the quadratic form

$$x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+n}^2. \tag{1}$$

It is understood that O_0 and $\mathsf{O}_{0,0}$ are both the trivial group. For $mn \neq 0$, the connected component $\mathsf{SO}_{m,n}^0$ of the group $\mathsf{O}_{m,n}$ is a proper subgroup of $\mathsf{SO}_{m,n} = \{a \in \mathsf{O}_{m,n} | \det a = 1\}$; a similar notation is used for the spin and pin groups. The real Clifford algebra associated with the quadratic form (1) is denoted by $\mathsf{Cliff}_{m,n}$. The set $\mathbb{Z}_2 = \{0, 1\}$ is considered, depending on the context, either as a group (with respect to addition mod 2) or as a ring and $\operatorname{span} X$ denotes the linear span over $\mathbb{Z}_2$ of the elements of the set X.

3. Generalities on central extensions of topological groups

Recall that a *central extension* of a topological group G by an Abelian discrete group A is an exact sequence of continuous group homomorphisms,

$$A \xrightarrow{i} H \xrightarrow{p} G, \tag{2}$$

such that i is injective, p is surjective and $i(A)$ is contained in the centre of H; see, e.g., Ch. I § 6 in Bourbaki (1970) for the algebraic aspect.

Another extension

$$A \xrightarrow{i'} H' \xrightarrow{p'} G$$

is said to be *equivalent* to (2) whenever there is an isomorphism of topological groups $j : H \to H'$ such that $j \circ i = i'$ and $p' \circ j = p$. There is always the extension

$$A \to G \times A \xrightarrow{\mathrm{pr}_1} G. \tag{3}$$

An extension equivalent to (3) is said to be *trivial*.

It is often convenient to abuse the language by saying that the group H, appearing in (2), is the extension. This is a real abuse: for example, the dihedral group D_4 and the group $\mathbb{Z}_2 \times \mathbb{Z}_4$ provide each 3 inequivalent extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$. This being kept in mind, it is possible to use the simplified notation without running into trouble. The trivial extension is then denoted by $\mathbf{O}$.

Given extensions by A of two groups,

$$A \xrightarrow{i_\alpha} H_\alpha \xrightarrow{p_\alpha} G_\alpha, \quad \alpha = 1, 2,$$

one defines an extension of their direct product,

$$A \xrightarrow{i} H_1 \diamond H_2 \xrightarrow{p} G_1 \times G_2, \tag{4}$$

by putting $H_1 \diamond H_2 = (H_1 \times H_2)/A$, $i(a) = [(1, a)]$, $p([(h_1, h_2)]) = (p_1(h_1), p_2(h_2))$, etc.

The set $\mathsf{E}(G, A)$ of equivalence classes of such extensions of G by A has the structure of an Abelian group; to describe it, consider two extensions

$$A \xrightarrow{i_\alpha} H_\alpha \xrightarrow{p_\alpha} G, \quad \alpha = 1, 2.$$

One defines their *sum* to be the extension $A \xrightarrow{i} H \xrightarrow{p} G$ given as follows. Let $H' = \{(a_1, a_2) \in H_1 \times H_2 | p_1(a_1) = p_2(a_2)\}$. The injection $A \to H'$, $a \mapsto (a, a^{-1})$, makes A into a normal subgroup of H'; let H be the resulting quotient group: $[(a_1, a_2)] = [(a'_1, a'_2)] \in H$ whenever there is $a \in A$ such that $a'_1 = aa_1$ and $a'_2 = a^{-1}a_2$. The map $p : H \to G$ given by $p([a_1, a_2]) = p_1(a_1)$ is a surjective homomorphism and its kernel is the group $\{[(a, 1)] = i(a) \in H | a \in A\}$, isomorphic to A; therefore $H \in \mathsf{E}(G, A)$. This extension is written as $H_1 + H_2$; one easily checks that $H_2 + H_1$ is an extension equivalent to $H_1 + H_2$, that so defined sum of (equivalence classes of) extensions is associative and the trivial extension is the neutral element of the group $\mathsf{E}(G, A)$.

4. Cohomology of classifying spaces

How can one find the group $\mathsf{E}(G, A)$? There is a well-known isomorphism of the group of all central extensions of G by A with the algebraic (in the sense of Eilenberg and Mac Lane (1942)) second cohomology group $\mathsf{H}^2_{\mathrm{alg}}(G, A)$ defined without any reference to the topology of G; see, e.g., Ch. IV in Mac Lane (1963). With a topological group G there is associated its *classifying space* BG whose singular cohomology with coefficients in A, relatively easy to compute, is closely related to the Eilenberg-Mac Lane cohomology of G and to $\mathsf{E}(G, A)$.

Recall that, for every *topological* group G, there is a *universal* principal G-bundle $EG \to BG$ such that every principal G-bundle over a paracompact space M is obtained as the bundle induced by a map from M to the classifying space BG. Two maps from M to BG induce the same bundle if, and only if, they are homotopy equivalent (Husemoller, 1966).

For a topological group G one can consider the group G^δ that has the same underlying set of elements, but is endowed with the discrete topology. The natural homomorphism $G^\delta \to G$ is continuous and lifts to a continuous map $BG^\delta \to BG$. Following a suggestion of Friedlander, Milnor (1983) conjectured that, for a Lie group G and a finite Abelian group A, this map induces an isomorphism of the corresponding cohomology rings with coefficients in A.

Jackowski (2000) informs me that, independently of Milnor's conjecture, for a locally simply connected G—therefore, in particular, for a Lie group— and a discrete A, there holds the isomorphism

$$\mathsf{E}(G, A) \cong \mathsf{H}^2(BG, A). \tag{5}$$

This paper is based on the validity of the isomorphism of groups (5) for $G = \mathsf{O}_{m,n}$ and $A = \mathbb{Z}_2$.

5. Cohomology of the classifying spaces of pseudo-orthogonal groups

From now on assume $A = \mathbb{Z}_2$ so that (2) reads $\mathbb{Z}_2 \to H \to G$ and, for every extension H of G, one has $H + H = \mathbf{O}$. The composition of elements in G and H is denoted multiplicatively, one writes $i(0) = 1$, $i(1) = -1$ and $\mathsf{H}^*(BG)$ instead of $\mathsf{H}^*(BG, \mathbb{Z}_2)$. One refers now to the extensions as 'double covers'.

The cohomology rings mod 2 of the classifying spaces of the groups O_n and SO_n are well-known; see, e.g., Milnor and Stasheff (1974). These rings are polynomials over $\mathbb{Z}_2$ in the Stiefel-Whitney classes,

$$\mathsf{H}^*(B\mathsf{O}_n) = \mathbb{Z}_2[w_1, w_2, \dots, w_n]$$

and

$$\mathsf{H}^*(B\mathsf{SO}_n) = \mathbb{Z}_2[w_2, \dots, w_n],$$

where $\deg w_k = k$. In particular, the second cohomology groups are $\mathsf{H}^2(B\mathsf{O}_1) = \operatorname{span}\{w_1^2\}$ and

$$\mathsf{H}^2(B\mathsf{O}_n) = \operatorname{span}\{w_2, w_1^2\} \quad \text{for } n > 1.$$

Since $\mathsf{O}_{n'} \times \mathsf{O}_{n''}$ is a maximal compact subgroup of $\mathsf{O}_{n',n''}$, the quotient $\mathsf{O}_{n',n''}/(\mathsf{O}_{n'} \times \mathsf{O}_{n''})$ is contractible and so the spaces $B(\mathsf{O}_{n'} \times \mathsf{O}_{n''})$ and $B\mathsf{O}_{n',n''}$ are homotopy equivalent. Using now the Künneth theorem one obtains

$$\mathsf{H}^*(B\mathsf{O}_{n',n''}) = \mathsf{H}^*(B\mathsf{O}_{n'}) \otimes_{\mathbb{Z}_2} \mathsf{H}^*(B\mathsf{O}_{n''}).$$

This implies, in a self-explanatory notation,

$$\mathsf{H}^2(B\mathsf{O}_{1,1}) = \mathrm{span}\,\{{w'_1}^2, {w''_1}^2, w'_1 w''_1\},$$

$$\mathsf{H}^2(B\mathsf{O}_{1,n''}) = \mathrm{span}\,\{w''_2, {w'_1}^2, {w''_1}^2, w'_1 w''_1\} \quad \text{for } n'' > 1,$$

and

$$\text{(6)} \quad \mathsf{H}^2(B\mathsf{O}_{n',n''}) = \mathrm{span}\,\{w'_2, w''_2, {w'_1}^2, {w''_1}^2, w'_1 w''_1\} \quad \text{for } n', n'' > 1.$$

Therefore, the groups O_n, $\mathsf{O}_{1,1}$, $\mathsf{O}_{1,n}$ and $\mathsf{O}_{m,n}$ have (for $m, n > 1$) 4, 8, 16 and 32 double covers, respectively. In particular, the set of extensions of $\mathsf{O}_{1,1}$ is in a natural, bijective correspondence with the set of extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathsf{O}_1 \times \mathsf{O}_1$ by $\mathbb{Z}_2$.

Similarly,

$$\mathsf{H}^2(B\mathsf{SO}^0_{1,n''}) = \mathrm{span}\,\{w''_2\} \quad \text{for } n'' > 1,$$

$$\mathsf{H}^2(B\mathsf{SO}_{n',n''}) = \mathrm{span}\,\{w'_2, w''_2, {w'_1}^2\} \quad \text{for } n', n'' > 1$$

and

$$\mathsf{H}^2(B\mathsf{SO}^0_{n',n''}) = \mathrm{span}\,\{w'_2, w''_2\} \quad \text{for } n', n'' > 1.$$

Clearly, by symmetry, $w'_2 + w''_2$ corresponds to the cover of $\mathsf{SO}^0_{n',n''}$ by $\mathsf{Spin}^0_{n',n''}$. But it is not evident what corresponds to w'_2. This problem is at the core of the difficulties in constructing explicitly more than 8 double covers of the Lorentz group $\mathsf{O}_{1,n}$.

The groups $\mathsf{E}(\mathsf{O}_n, \mathbb{Z}_2)$ and $\mathsf{E}(\mathsf{O}_{m,n}, \mathbb{Z}_2)$ are from now on denoted by E_n and $\mathsf{E}_{m,n}$, respectively.

6. The isomorphism $\mathsf{E}_{m,n} \to \mathsf{H}^2(B\mathsf{O}_{m,n})$

What is the correspondence between the double covers of $\mathsf{O}_{n',n''}$ and the elements of (6)? By inspection of the obstructions to the classical pin structures (Karoubi, 1968), one expects that it can be described as follows. The group $\mathsf{O}_{n',n''}$ is known to be generated by reflections in hyperplanes orthogonal to non-isotropic vectors (Cartan-Dieudonné). Every reflection is an idempotent; reflections in hyperplanes associated with orthogonal vectors commute. Therefore, the square of an element of H covering a reflection is either 1 or -1. Two elements of H that cover two commuting and distinct reflections either commute or anticommute. Let $V = \mathbb{R}^{n'+n''} \subset \mathsf{Cliff}_{n',n''}$. Consider the spaces of unit vectors

$$U' = \{u \in V | u^2 = 1\} \quad \text{and} \quad U'' = \{u \in V | u^2 = -1\}.$$

Reflections associated with elements of U' (resp., U'') are called time (resp., space) reflections. Let

$$k(H) = \lambda' w_2' + \lambda'' w_2'' + \mu w_1' w_1'' + \nu' {w_1'}^2 + \nu'' {w_1''}^2, \tag{7}$$

where $\lambda', \lambda'', \mu, \nu', \nu'' \in \mathbb{Z}_2$, be the element of (6) corresponding to the double cover H.

PROPOSITION 1. *The isomorphism $k : \mathsf{E}_{n',n''} \to \mathsf{H}^2(B\mathsf{O}_{n',n''})$ is as follows:*

$\lambda' = 0$ (resp., $\lambda' = 1$) if every two elements of H covering reflections associated with orthogonal elements of U' commute (resp., anticommute); similarly for λ'' and U'';

$\mu = 0$ (resp., $\mu = 1$) if every two elements of H covering reflections associated with orthogonal elements, one in U' and the other in U'', commute (resp., anticommute);

$\nu' = 0$ (resp., $\nu' = 1$) if the square of every element of H covering a reflection associated with an element of U' is 1 (resp., -1); similarly for ν'' and U''.

Moreover, to characterize the double cover of $\mathsf{SO}^0_{n',n''}$, obtained by restriction of the one corresponding to (7), one puts $\mu = \nu' = \nu'' = 0$ in (7).

Sketch of proof. To justify the interpretation of ν', ν'' look at double covers of O_1. Similarly, the significance of μ is obtained from $\mathsf{O}_{1,1}$. By considering two extensions H_1 and H_2, and two pairs of elements $(u_1, u_2) \in H_1 \times H_2$ and $(u_1', u_2') \in H_1 \times H_2$ such that $p_1(u_1) = p_2(u_2)$ and $p_1(u_1') = p_2(u_2')$ are reflections in hyperplanes determined by orthogonal vectors, one easily checks that if $u_1 u_1' + u_1' u_1 = 0$ and $u_2 u_2' + u_2' u_2 = 0$, then the elements $[(u_1, u_2)]$ and $[(u_1', u_2')]$ of $H_1 + H_2$ commute, etc. $\square$

7. Construction of double covers

7.1. The 4 double covers of O_n

This case is the simplest and best known: there are two generators of the group E_n,

$$\mathsf{Pin}_n^+ \text{ corresponding to } w_2 \quad \text{and} \quad \mathsf{Pin}_n^- \text{ corresponding to } w_2 + w_1^2.$$

Their sum $\mathsf{Pin}_n^+ + \mathsf{Pin}_n^-$ is a non-trivial double cover that trivializes upon restriction to SO_n. It corresponds to w_1^2.

It is worth-while to describe in some detail the extension

$$\mathbb{Z}_2 \to \mathsf{Pin}_n^+ + \mathsf{Pin}_n^- \to \mathsf{O}_n; \tag{8}$$

all sums of double covers of the groups $\mathsf{O}_{m,n}$ are constructed in a similar manner; I refer to them as being given *explicitly* in terms of the summands. Consider

$$\mathbb{Z}_2 \to \mathsf{Pin}_n^\pm \xrightarrow{p_\pm} \mathsf{O}_n$$

and put $H' = \{(a,b) \in \mathsf{Pin}_n^+ \times \mathsf{Pin}_n^- | p_+(a) = p_-(b)\}$, as in Section 3. The group $\mathsf{Pin}_n^+ + \mathsf{Pin}_n^-$ is the quotient of H' by the equivalence relation $(a,b) \equiv (a',b') \Leftrightarrow a = a'$ and $b = b'$ or $a = -a'$ and $b = -b'$. Let $[(a,b)]$ denote the corresponding equivalence class. The injection $i : \mathbb{Z}_2 \to \mathsf{Pin}_n^+ + \mathsf{Pin}_n^-$ is given by $i(1) = [(1,-1)]$. For every unit vector u one has $[(u,u)]^2 = [(1,-1)]$. Therefore, the extension (8) does not split. If u and v are orthogonal unit vectors, then the elements $[(u,u)]$ and $[(v,v)]$ commute.

7.2. Explicit construction of 8 double covers of $\mathsf{O}_{m,n}$ for $m,n \geqslant 1$

For every $m,n \geqslant 1$ one constructs 8 inequivalent double covers by giving a set of 3 generators in the group $\mathsf{E}_{m,n}$.

Let $\mathrm{i} = \sqrt{-1}$. There are 4 pin groups, defined as subsets of the *complexified* Clifford algebra $\mathsf{Cliff}_{n'+n''}(\mathbb{C}) = \mathbb{C} \otimes \mathsf{Cliff}_{n',n''}$, generated multiplicatively:

$$\mathsf{Pin}_{n',n''}^{\nu',\nu''} \quad \text{is generated by} \quad \mathrm{i}^{\nu'} U' \cup \mathrm{i}^{1+\nu''} U'', \tag{9}$$

where $\nu', \nu'' \in \mathbb{Z}_2$. In the traditional notation, one usually writes $\mathsf{Pin}_{n',n''}$ instead of $\mathsf{Pin}_{n',n''}^{0,1}$ and $\mathsf{Pin}_{n'',n'}$ instead of $\mathsf{Pin}_{n',n''}^{1,0}$. In the positive definite case, $U'' = \varnothing$, one has $\mathsf{Pin}_n^+ = \mathsf{Pin}_{n,0}^{0,*}$ and $\mathsf{Pin}_n^- = \mathsf{Pin}_{n,0}^{1,*}$; there is no need for the elaborate notation. The epimorphism $p : \mathsf{Pin}_{n',n''}^{\nu',\nu''} \to \mathsf{O}_{n',n''}$

is as in the classical case, $p(a)v = \alpha(a)va^{-1}$, where $v \in V$ and α is the main (grading) automorphism of $\mathsf{Cliff}_{n'+n''}(\mathbb{C})$. The 'new' groups $\mathsf{Pin}^{0,0}_{n',n''}$ and $\mathsf{Pin}^{1,1}_{n',n''}$ also provide double covers of $\mathsf{O}_{n',n''}$. (They should not be confused with the compact groups $\mathsf{Pin}^{+}_{n'+n''}$ and $\mathsf{Pin}^{-}_{n'+n''}$.) No two extensions among the four are equivalent, but only 3 among them are independent, as elements of the group $\mathsf{E}_{n',n''}$; the following relation is easy to check:

$$\mathsf{Pin}^{0,0}_{n',n''} + \mathsf{Pin}^{1,0}_{n',n''} + \mathsf{Pin}^{0,1}_{n',n''} + \mathsf{Pin}^{1,1}_{n',n''} = \mathbf{O}.$$

It is convenient to introduce the 'total 2nd Stiefel-Whitney class',

$$w_2 = w_2' + w_2'' + w_1' w_1''.$$

It follows from Prop. 1 that there is the following correspondence between the double covers of $\mathsf{O}_{n',n''}$ described in (9) and the elements of $\mathsf{H}^2(B\mathsf{O}_{n',n''})$ given by (6):

$$k(\mathsf{Pin}^{\nu',\nu''}_{n',n''}) = w_2 + \nu' {w_1'}^2 + \nu'' {w_1''}^2. \tag{10}$$

To describe all the 32 double covers of $\mathsf{O}_{n',n''}$, $n', n'' > 1$, one would need two more independent generators, for example, those corresponding to w_2' and w_2''. For the (generalized) Lorentz group $\mathsf{O}_{1,n}$, $n > 1$, one extra generator suffices. I do not know how to construct them.

Among the 32 double covers only the 8 corresponding to

$$w_2 + \bar{\mu} w_1' w_1'' + \nu' {w_1'}^2 + \nu'' {w_1''}^2, \quad \text{where } \bar{\mu}, \nu', \nu'' \in \mathbb{Z}_2,$$

are 'spinorial' in the sense that, restricted to the connected component, they reduce to the classical double cover $\mathsf{Spin}^0_{n',n''} \to \mathsf{SO}^0_{n',n''}$. Among those 8, there are the 4 given in (10); I do not know how to describe the remaining 4 which are characterized by $\bar{\mu} = 1$.

There are also 8 double covers (corresponding to $\mu w_1' w_1'' + \nu' {w_1'}^2 + \nu'' {w_1''}^2$) that trivialize upon restriction to the connected component. Among them, 4 (those with $\mu = 0$) are generated from (10).

The following table summarizes the relations between the notation of Dąbrowski (1988) and Chamblin (1994), briefly described here in Section 1, and the present one. It covers only the cases when $\lambda' = \lambda'' = 1$. Here Q is the quaternion group. The dimensions n' and n'' are omitted.

a	b	c	μ	ν'	ν''	group H	its subgroup generated by $\{-1, s, t\}$
$+$	$+$	$+$	0	0	0		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$+$	$-$	$-$	0	1	0		$\mathbb{Z}_2 \times \mathbb{Z}_4$
$-$	$+$	$-$	0	0	1		$\mathbb{Z}_2 \times \mathbb{Z}_4$
$-$	$-$	$+$	0	1	1		$\mathbb{Z}_2 \times \mathbb{Z}_4$
$+$	$+$	$-$	1	0	0	$\mathsf{Pin}^{0,0}$	D_4
$+$	$-$	$+$	1	1	0	$\mathsf{Pin}^{0,1}$	D_4
$-$	$+$	$+$	1	0	1	$\mathsf{Pin}^{1,0}$	D_4
$-$	$-$	$-$	1	1	1	$\mathsf{Pin}^{1,1}$	Q

7.3. All the double covers of $\mathsf{O}_m \times \mathsf{O}_n$

It is easy to describe explicitly all the 32 double covers of the group $\mathsf{O}_{n'} \times \mathsf{O}_{n''}$, $n', n'' > 1$. Recall first that there are 4 double covers of O_n,

$$H_n^0 = \mathsf{O}_n \times \mathbb{Z}_2, \quad H_n^1 = \mathsf{Pin}_n^+,$$

$$H_n^2 = \mathsf{Pin}_n^+ + \mathsf{Pin}_n^-, \quad H_n^3 = \mathsf{Pin}_n^-.$$

The 16 groups (cf. (4)) $H_{n'}^\alpha \diamond H_{n''}^\beta$, $\alpha, \beta = 0, \ldots, 3$, correspond to the elements of the form (7) with $\mu = 0$.

To describe the remaining ones, one uses a graded ('supersymmetric') construction that appears already in Karoubi (1968). Every double cover $p : H_n^\alpha \to \mathsf{O}_n$ is $\mathbb{Z}_2$-graded by putting, for $h \in H_n^\alpha$, $\det p(h) = (-1)^{\deg h}$. In the Cartesian product $H_{n'}^\alpha \times H_{n''}^\beta$ one defines a group structure by

$$(h_1', h_1'')(h_2', h_2'') = (h_1' h_2', (-1)^{\deg h_1'' \deg h_2'} h_1'' h_2''), \tag{11}$$

where $h_1', h_2' \in H_{n'}^\alpha$, $h_1'', h_2'' \in H_{n''}^\beta$. The groups

$$H_{n'}^\alpha \diamond^{\mathrm{gr}} H_{n''}^\beta = (H_{n'}^\alpha \times H_{n''}^\beta)/\mathbb{Z}_2, \quad \alpha, \beta = 0, \ldots, 3,$$

with a composition induced from (11), provide the 16 double covers with $\mu = 1$.

7.4. Complex double covers

In the complex domain, it is natural to consider the conformal group,

$$\mathsf{CO}_n(\mathbb{C}) = \{A \in \mathsf{GL}_n(\mathbb{C}) | AA^t = \lambda \mathrm{id},\ \lambda \in \mathbb{C} \setminus \{0\}\},$$

and the corresponding Clifford (=conformal spin) group $\mathsf{CPin}_n(\mathbb{C})$ generated multiplicatively in $\mathsf{Cliff}_n(\mathbb{C})$ by all non-isotropic elements of $\mathbb{C}^n$. There is the exact sequence

$$\mathsf{CPin}_n(\mathbb{C}) \xrightarrow{\rho} \mathsf{CO}_n(\mathbb{C}) \to 1,$$

where $\rho(a)v = \alpha(a)v\beta(a), v \in \mathbb{C}^n$. The kernel of ρ is of order 4 (see, e.g., (Robinson and Trautman, 1993); here α and β are the grading automorphism and the main anti-automorphism of $\mathsf{Cliff}_n(\mathbb{C})$, respectively; $\beta(ab) = \beta(b)\beta(a)$, etc.).

One can define a *complex* double cover of $\mathsf{O}_{n',n''}$ as a double cover $p : H \to \mathsf{O}_{n',n''}$ such that there exists a cover $p_\mathbb{C} : H_\mathbb{C} \to \mathsf{CO}_{n'+n''}(\mathbb{C})$ and a monomorphism j such that $p_\mathbb{C} \circ j = \mathrm{inj} \circ p$, where inj is the injection of $\mathsf{O}_{n',n''}$ into $\mathsf{CO}_{n'+n''}(\mathbb{C})$.

The 4 extensions of the compact group O_n are complex.

Conjecture. A double cover $p : H \to \mathsf{O}_{n',n''}$ is complex if, and only if,

$$k(H) = \lambda w_2 + \nu' {w_1'}^2 + \nu'' {w_1''}^2, \quad \text{where } \lambda, \nu', \nu'' \in \mathbb{Z}_2.$$

In other words, the conjecture says that, for every $n', n'' \geqslant 1$,there are 8 complex double covers of $\mathsf{O}_{n',n''}$, generated be $\mathsf{Pin}^{\nu',\nu''}_{n',n''}$.

8. Topological obstructions

Given a pseudo-Riemannian manifold M with a metric tensor of signature (m, n) and a double cover $H \to \mathsf{O}_{m,n}$, one can consider the reduction of the bundle P of all orthonormal frames of M to the group H. The corresponding topological obstruction can be determined in a way similar to the one used for (s)pin structures; see § 26.5 in Borel and Hirzebruch (1959), Milnor (1963), Prop. 1.1.26 in Karoubi (1968), and § 3.5 in Ward and Wells Jr (1990). In fact, from an obstruction theory argument (Spanier, 1966) one obtains

PROPOSITION 2. *Let $f : M \to B\mathsf{O}_{m,n}$ be the map inducing the bundle $P \to M$. The pull-back $f^*k(H)$ is an element of $\mathsf{H}^2(M, \mathbb{Z}_2)$ that vanishes if, and only if, there is a reduction of P to H.*

As an application of Prop. 2, consider the real projective space $\mathbb{R}P_5$. It is orientable, but its second Stiefel-Whitney class is $\neq 0$; therefore, it has no spin structure, but its bundle of all orthonormal frames can be reduced to the group $\mathsf{Pin}^+_5 + \mathsf{Pin}^-_5$. A less trivial example would be provided by a non-orientable Riemannian space with $w_2 \neq 0$ and $w_1^2 = 0$.

Acknowledgments

I thank Professor Stefan Jackowski for enlightening discussions on the subject of group extensions and cohomology. Sections 4 and 5 have been written under his influence, but I am solely responsible for the shortcomings of the present text.

Work on this paper was supported in part by the Polish Committee for Scientific Research (KBN) under grant no. 2 P03B 060 17.

References

Atiyah, M. F., R. Bott, and A. Shapiro: 1964, 'Clifford modules'. *Topology* **3** Suppl. 1, 3–38.

Borel, A. and F. Hirzebruch: 1959, 'Characteristic classes and homogeneous spaces, II'. *Amer. J. Math.* **81**, 315–382.

Bourbaki, N.: 1970, *Éléments de Mathématiques: Algèbre, Ch. 1 à 3.* Paris: Diffusion C.C.L.S.

Carlip, S. and C. DeWitt-Morette: 1988, 'Where the sign of the metric makes a difference'. *Phys. Rev. Lett.* **60**, 1599–1601.

Chamblin, A.: 1994, 'On the obstructions to non-Cliffordian pin structures'. *Comm. Math. Phys.* **164**, 65–85.

Dąbrowski, L.: 1988, *Group Actions on Spinors.* Napoli: Bibliopolis.

DeWitt-Morette, C. and B. S. DeWitt: 1990, 'Pin groups in physics'. *Phys. Rev. D* **41**, 1901–07.

Eilenberg, S. and S. Mac Lane: 1942, 'Group extensions and homology'. *Ann. Math.* **43**, 757–831.

Fierz, M. and V. F. Weisskopf: 1960, *Theoretical Physics in the Twentieth Century: A Memorial Volume to Wolfgang Pauli.* New York: Interscience.

Haefliger, A.: 1956, 'Sur l'extension du groupe structural d'un espace fibré'. *C. R. Acad. Sci. (Paris)* **243**, 558–560.

Husemoller, D.: 1966, *Fibre Bundles.* New York: McGraw-Hill.

Jackowski, S.: 2000, private communication.

Karoubi, M.: 1968, 'Algèbres de Clifford et K-théorie'. *Ann. Sci. Éc. Norm. Sup., 4ème sér.* **1**, 161–270.

Mac Lane, S.: 1963, *Homology.* Berlin: Springer.

Milnor, J.: 1963, 'Spin structures on manifolds'. *Enseign. Math.* **9**, 198–203.

Milnor, J.: 1983, 'On the homology of Lie groups made discrete'. *Comment. Math. Helv.* **58**, 72–85. MR **85b:**57050 by E. Friedlander.

Milnor, J. W. and J. D. Stasheff: 1974, *Characteristic Classes.* Princeton: Princeton University Press.

Robinson, I. and A. Trautman: 1993, 'The conformal geometry of complex quadrics and the fractional-linear form of Möbius transformations'. *J. Math. Phys.* **34**, 5391–5406.

Schrödinger, E.: 1932, 'Diracsches Elektron im Schwerefeld I'. *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.* **XI**, 105–28.

Shirokov, Y. M.: 1960, 'Space and time reflections in relativistic theory'. *Nuclear Physics* **15**, 1–12.

Spanier, E. H.: 1966, *Algebraic Topology.* New York: McGraw-Hill.

Ward, R. S. and R. O. Wells, Jr.: 1990, *Twistor Geometry and Field Theory*. Cambridge: Cambridge University Press.