Published in:
The Geometric Universe
Science, Geometry and the Work of Roger Penrose
Conference at St. John's College, Oxford, June 25–29 1996
Edited by S. A. Hugget, L, J. Mason, K. P. Tod, S. T. Tsou and N. M. J. Woodhouse
Oxford University Press, Oxford 1998

PYTHAGOREAN SPINORS
AND
PENROSE TWISTORS

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1 Introduction

Besides the major applications of spinors and twistors to equations of mathematical physics, there are minor results, where these objects play an auxiliary role or bring a new light on otherwise well-known facts. One example of such a result is the twistor-inspired derivation, based on the use of the conformal compactification of a (pseudo-) Euclidean space, of the fractional-linear form of Möbius transformations. Even simpler is the remark that the solution, attributed to Euclid, of the Pythagorean equation, has a spinorial interpretation: it is equivalent to the statement that a null vector in $\mathbb{Z}^3$, considered as a subset of $\mathbb{R}^3$ with a scalar product of signature $(1, 2)$, is the (tensor) square of an integer-valued spinor. In this short article, I expand the latter observation and give a summary of the rudiments of twistor notions associated with higher-dimensional spaces; this account is included here not because of its novelty, but in the hope that there may be some interest in a presentation by an outsider. Only 'global' twistors are considered here; I gave a brief account of my view of 'local' twistors in (Trautman 1993).

Originally, Roger Penrose intended twistor spaces to be associated with, or serve as replacements for, the 4-dimensional, Lorentzian space-times. His belief in the privileged and exceptional role of four dimensions, apparent in twistor theory, was strikingly confirmed by the discoveries of exotic differential structures on $\mathbb{R}^4$, and of the Donaldson and Seiberg-Witten invariants. There are, however, interesting generalizations of twistor ideas to other dimensions and signatures; especially to proper Riemannian 3- and 4-manifolds. As is often the case with important ideas, the original notion of twistor has been generalized in many ways; only some of them are briefly presented below.

2 Pythagorean spinors

If p and q are integers, then the triple of integers (x, y, z), given by

$$x = p^2 - q^2, \quad y = 2pq, \quad z = p^2 + q^2, \tag{1}$$

is *Pythagorean*: it satisfies the equation $x^2 + y^2 = z^2$. If (x, y, z) is Pythagorean, then at least one of the numbers x and y is even; moreover, if $t \in \mathbb{Z}$, then (y, x, z) and (tx, ty, tz) are also Pythagorean. I say that a Pythagorean triple (x, y, z) is *standard* if $z > 0$ and either the triple (x, y, z) is relatively prime (*rp*) and y is even or $(x/2, y/2, z/2)$ is a triple of *rp* integers and $y/2$ is odd. For example, the triples $(-1, 0, 1)$ and $(8, 6, 10)$ are standard, but $(4, 3, 5)$ and $(6, 8, 10)$ are not. Every Pythagorean triple can be written as (tx, ty, tz), where $t \in \mathbb{Z}$, the integers (x, y, z) are *rp* and $z > 0$; if y is even, then (x, y, z) is standard; if y is odd, then $(2x, 2y, 2z)$ is standard.

Proposition 1. *If (x, y, z) is a standard Pythagorean triple, then there is a pair (p, q) of relatively prime integers such that* (1) *holds.*

In other words: there is a bijection between the set of directions in $\mathbb{Z}^2$ and the set of 'null directions' in $\mathbb{Z}^3$.

Proof. Note that $z > 0$, y even and $y^2 = (z + x)(z - x)$ imply $z + x = 2m \geq 0$ and $z - x = 2n \geq 0$, where m and n are integers. If $y = 2r$, then the

Pythagorean equation is equivalent to $r^2 = mn$. If the triple (x, y, z) is *rp*, then so is the triple (m, n, r). If r is odd and the integers x, y and z are all even, but have no divisor > 2, then the triple (m, n, r) is also *rp*. If (m, n, r) is *rp*, then $r^2 = mn$ implies that both m and n are squares. □

Recall the classical lemma (Sierpiński 1987):

Lemma 1. If p and q are integers, then a necessary and sufficient condition for the existence of integers u and v, such that $pu + qv = 1$, is that p and q be relatively prime.

It leads to

Proposition 2. *The group $\mathsf{SL}_2(\mathbb{Z})$ acts transitively on the set $P \subset \mathbb{Z}^2$ of integer-valued 'spinors' with relatively prime components.*

In other words: the group $\mathsf{SL}_2(\mathbb{Z})$ acts transitively on the set of directions in $\mathbb{Z}^2$.

Proof. Indeed, let $a, b, c, d \in \mathbb{Z}$ and consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

The matrix A is in $\mathsf{SL}_2(\mathbb{Z})$ iff $ad - bc = 1$; it acts in $\mathbb{Z}^2$ by sending $\begin{pmatrix} p \\ q \end{pmatrix}$ to $\begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}$. If $\begin{pmatrix} p \\ q \end{pmatrix} \in P$, then there are integers u and v such that $pu + qv = 1$. Putting $(u', v') = (u, v)A^{-1}$ one obtains $p'u' + q'v' = 1$; therefore, $\mathsf{SL}_2(\mathbb{Z})$ acts in P. This action is transitive: if $\begin{pmatrix} p \\ q \end{pmatrix} \in P$ and $pu + qv = 1$, then the matrix $\begin{pmatrix} p & -v \\ q & u \end{pmatrix}$ sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} p \\ q \end{pmatrix}$. □

The stabilizer of an element of P is a subgroup of $\mathsf{SL}_2(\mathbb{Z})$ isomorphic to $\mathbb{Z}$.

Recall that the group $\mathsf{SL}_2(\mathbb{R})$ is the connected component of the group $\mathsf{Spin}_{1,2}(\mathbb{R})$: there is the exact sequence of homomorphisms of groups,

$$1 \to \mathbb{Z}_2 \to \mathsf{SL}_2(\mathbb{R}) \xrightarrow{\rho} \mathsf{SO}^{\circ}_{1,2}(\mathbb{R}) \to 1.$$

If $(x, y, z) \in \mathbb{R}^3$ is represented by the matrix $\begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} (p \;\; q)$ and

$$v = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} J, \quad \text{where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

so that $v^2 = (x^2+y^2-z^2)I$, then $\rho(A)v = AvA^{-1}$. By restriction, one obtains the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathsf{SL}_2(\mathbb{Z}) \xrightarrow{\rho} \mathsf{G} \to 1.$$

The group $\mathsf{G} \subset \mathsf{SO}^{\circ}_{1,2}(\mathbb{R})$ which, by definition, is the image of $\mathsf{SL}_2(\mathbb{Z})$ by ρ, is a group of matrices with entries that are either integer or half-integer. For example,

$$\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}.$$

The group G acts on the set $\{(x, y, z) \in \mathbb{Z}^3 : x + z \text{ is even}\}$. It is an easy exercise to find the subgroup of $\mathsf{SL}_2(\mathbb{Z})$ that covers the subgroup of G containing all matrices with integer elements.

3 Projective quadrics and twistors

Global twistors, described in the first paper on the subject (Penrose 1967), are associated with *projective quadrics*, i.e. with conformal compactifications of (pseudo-) Euclidean spaces. Most of the time, the name 'projective quadric' is shortened to 'quadric'. Let V be an m-dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$ with a non-degenerate quadratic form g. In the vector space $W = V \oplus K^2$ one introduces the quadratic form h by putting $h(w) = g(v) + \lambda\mu$, where $w = (v, \lambda, \mu) \in W$, and defines the quadric $\mathsf{Q}(h)$ to be the subset of the projective space $\mathsf{P}(W)$ consisting of all null directions, $\mathsf{Q}(h) = \{\operatorname{dir} w : w \in W, w \neq 0, h(w) = 0\}$. The quadric inherits from the quadratic space (W, h) a (locally flat) *conformal structure*. The map $V \to \mathsf{Q}(h)$, given by $v \mapsto \operatorname{dir}(v, -g(v), 1)$, is a conformal embedding with an image open and dense in $\mathsf{Q}(h)$; the complement of the image is the 'null cone at infinity'. The group $\mathsf{Spin}(h)$ acts on $\mathsf{Q}(h)$ by sending $\operatorname{dir} w$ to $\operatorname{dir}(AwA^{-1})$, where $w \in W$ and $A \in \mathsf{Spin}(h)$. This action is conformal and transitive. The Clifford algebra $\mathsf{Cl}(h)$, associated with the quadratic space (W, h), is isomorphic to the algebra $\mathsf{Cl}(g) \otimes \mathsf{End}K^2$; an isomorphism is induced by the Clifford map

$$(v, \lambda, \mu) \mapsto \begin{pmatrix} v & \lambda \\ \mu & -v \end{pmatrix} = X. \tag{4}$$

Therefore, every $A \in \mathsf{Spin}(h) \subset \mathsf{Cl}(h)$ can be written in the form (2), where a, b, c and d are now suitable elements of $\mathsf{Cl}(g)$. Let $a \mapsto {}^t a$ be the antiautomorphism of the algebra $\mathsf{Cl}(g)$ such that ${}^t 1 = 1$ and ${}^t v = v$ for every $v \in V$. There holds

Proposition 3. *If $A \in \mathsf{Spin}(h)$, then the space*

$$V_A = \{v \in V : cv + d \text{ is invertible in } \mathsf{Cl}(g)\}$$

is open and dense in V; the map

$$f_A : V_A \to V_{A^{-1}} \text{ defined by } f_A(v) = (av + b)(cv + d)^{-1}$$

is a conformal diffeomorphism, $g(\mathrm{d}v) = {}^t(cv + d)(cv + d)g(\mathrm{d}f_A(v))$.

This result goes back to Th. Vahlen; see (Robinson and Trautman 1993) and the references given there. The Clifford algebra $\mathsf{Cl}(g)$ has an irreducible Dirac (m even) or Pauli (m odd) representation γ in a complex, $2^{[\frac{1}{2}m]}$-dimensional vector space S of spinors. The Dirac representation, restricted to the even Clifford algebra, decomposes into the sum of two Weyl representations in the spaces of spinors of opposite 'chirality'. A representation γ of $\mathsf{Cl}(g)$ in S extends to a representation δ of $\mathsf{Cl}(h)$ in $S \oplus S$. Namely, if $a \in \mathsf{Cl}(g)$ and $b \in \mathsf{End}K^2 \subseteq \mathsf{End}\mathbb{C}^2$, then

$$\delta : \mathsf{Cl}(h) \to \mathsf{End}(S \oplus S) = (\mathsf{End}S) \otimes \mathsf{End}\mathbb{C}^2 \text{ is given by } \delta(a \otimes b) = \gamma(a) \otimes b. \tag{5}$$

3.1 The complex case

3.1.1 Complex projective quadrics of dimension m

Assume $K = \mathbb{C}$ so that $V = \mathbb{C}^m$. The corresponding complex quadric Q_m is, in the words of Kobayashi and Ochiai (1982), 'a holomorphic analogue of a sphere'. It has no complex-bilinear Riemannian structure; its complex conformal structure supports a unique conformal spin structure which can be described as follows. Let Cl_m denote the Clifford algebra associated with $(\mathbb{C}^m, g)$. The conformal spin (Clifford) group is defined here as the subset Cpin_m of Cl_m consisting of products of all *even* sequences of non-null vectors in V. This group acts on vectors by sending, for every $a \in \mathsf{Cpin}_m$, the vector v to $\rho(a)v = av\,{}^t\!a$. There is the exact sequence of group homomorphisms,

$$1 \to \mathsf{K}_m \to \mathsf{Cpin}_m \xrightarrow{\rho} \mathsf{CO}_m \to 1,$$

where CO_m is the connected component of the conformal group (understood here as the group of rotations and dilations). If m is odd, then the kernel K_m of ρ is $\mathbb{Z}_2 = \{1, -1\}$. For m even, ρ gives a four-fold cover and $\mathsf{K}_m = \{1, -1, \eta, -\eta\}$, where $\eta \in \mathsf{Cl}_m$ is a volume element normalized so that ${}^t\eta\eta = -1$. The group K_m is isomorphic to $\mathbb{Z}_4$ for $m \equiv 0 \bmod 4$ and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $m \equiv 2 \bmod 4$. The spin group is $\mathsf{Spin}(g) = \mathsf{Spin}_m = \{a \in \mathsf{Cpin}_m : {}^t a a = 1\}$. The group Spin_{m+2} acts transitively on Q_m. The image of the null vector $w_\infty = (0,1,0) \in W$ by (4) is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathsf{Cl}_{m+2}$. The stabilizer of $\operatorname{dir} w_\infty \in \mathsf{Q}_m$ is the semi-direct product H_m of Cpin_m by $\mathbb{C}^m$ given explicitly by

$$\mathsf{H}_m = \{ \begin{pmatrix} a & v \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathsf{Cpin}_m,\ v \in \mathbb{C}^m \} \subset \mathsf{Spin}_{m+2}\,.$$

The groups Cpin_m and $\mathbb{C}^m$ are thus made into subgroups of Spin_{m+2}. Let PO_m be the quotient of the complex special orthogonal group SO_m by its centre: $\mathsf{PO}_{2n+1} = \mathsf{SO}_{2n+1}$ and $\mathsf{PO}_{2n} = \mathsf{SO}_{2n}/\mathbb{Z}_2$. There is the commutative diagram

$$\begin{array}{ccc} \mathsf{Cpin}_m & \longrightarrow & \mathsf{CO}_m \\ \big\downarrow & & \big\downarrow \\ \mathsf{Spin}_{m+2} & \longrightarrow & \mathsf{PO}_{m+2} \end{array}$$

of group homomorphisms: the vertical arrows are injective and the horizontal ones are 4:1 or 2:1 depending on whether m is even or odd. The map $\mathbb{C}^m \to \mathsf{Spin}_{m+2}$ descends to a monomorphism of groups, $\mathbb{C}^m \to \mathsf{PO}_{m+2}$. With these observations in mind, one can formulate

Proposition 4. (i) *The conformal spin structure on* Q_m *is given by the principal bundle maps*

$$\begin{array}{ccccc} \mathsf{Cpin}_m & \longrightarrow & \mathsf{CO}_m & & \\ \big\downarrow & & \big\downarrow & & \\ \mathsf{Spin}_{m+2}/\mathbb{C}^m & \longrightarrow & \mathsf{PO}_{m+2}/\mathbb{C}^m & \longrightarrow & \mathsf{Q}_m\,. \end{array}$$

(ii) *The associated bundle of spinors,*

$$(\mathsf{Spin}_{m+2}/\mathbb{C}^m) \times_{\mathsf{Cpin}_m} S \to \mathsf{Q}_m,$$

corresponding to the representation $\gamma : \mathsf{Cpin}_m \to \mathsf{GL}(S)$*, is isomorphic to the bundle* $\Sigma \to \mathsf{Q}_m$*, where*

$$\Sigma = \{(\mathrm{dir} w, \Phi) \in \mathsf{Q}_m \times S \oplus S : \delta(w)\Phi = 0\}$$

and δ *is as in* (5)*.*
(iii) *The Maurer-Cartan form* $A^{-1}\mathrm{d}A$ *defines a flat Cartan connection on the* H_m*-bundle* $\varpi : \mathsf{Spin}_{m+2} \to \mathsf{Q}_m$*.*

A proof of (i) is in (Robinson and Trautman 1993). The map ϖ : $\mathsf{Spin}_{m+2} \to \mathsf{Q}_m$ is given by $\varpi(A) = \mathrm{dir}(Aw_\infty A^{-1})$. Part (ii) generalizes a similar observation made by Manin (1981) for $m = 4$. I learned of this generalization from Harnad; the isomorphism in question is given by $[(A\mathbb{C}^m, \varphi)] \mapsto (\varpi(A), \delta(A)(\varphi, 0))$. Part (iii) follows from the definition of a Cartan connection; see (Friedrich 1977) and the references given there.

If $\Phi \in S \oplus S$ is a non-zero spinor, then the vector space $\{w \in W : \delta(w)\Phi = 0\}$ is totally null; if it is maximal (*mtn*), then Φ is said to be *pure*. If $m = 2n$ is even, then a pure spinor is Weyl (chiral) and the $(n+1)$-vector formed from a linear basis spanning the corresponding *mtn* is either self-dual or antiself-dual. The projective *twistor space* T_m for Q_m is the manifold of directions of pure spinors associated with (W, h). For m even, it has two components, T_m^+ and T_m^-. If one puts $m = 2n$ (m even) or $m = 2n - 1$ (m odd), then $\dim \mathsf{T}_m = \frac{1}{2}n(n+1)$. In particular, each of the spaces T_3, T_4^+ and T_4^- is diffeomorphic to $\mathbb{C}\mathsf{P}_3$. A global twistor $\mathrm{dir}\Phi \in \mathsf{T}_m$ is identified with the *mtn* space of vectors annihilating Φ. This space descends to a totally null geodesic submanifold of Q_m of the maximal dimension $[\frac{1}{2}m]$. The dimensions of Q_m and T_m coincide only for $m = 3$ (minitwistors; cf. the papers by K. P. Tod in (Mason *et al.* 1995); see also (Ward 1996) and the papers by N. J. Hitchin referred to there) and $m = 6$ (in this case, the three spaces Q_6, T_6^+ and T_6^- are diffeomorphic to each other; this coincidence reflects triality; cf. the papers by L. P. Hughston in (Mason *et al.* 1995)). The flag manifold for Q_m is defined as the 'projectivized' bundle of pure spinors, $\mathsf{F}_m = \{(\mathrm{dir} w, \mathrm{dir}\Phi) \in \mathsf{Q}_m \times \mathsf{T}_m : \delta(w)\Phi = 0\}$. The two natural projections define the double fibration $\mathsf{Q}_m \leftarrow \mathsf{F}_m \to \mathsf{T}_m$ which underlies the *Penrose correspondence* (Wells 1979). For m even, F_m has two connected components and there are two such double fibrations.

3.1.2 The case of four dimensions

Instead of representing W as $V \oplus \mathbb{C}^2$, one uses, in this case, the identification of $\mathbb{C}^6$ with $\bigwedge^2 \mathbb{C}^4$. Let $\mathbb{T}$ be the complex, four-dimensional vector space of *Penrose twistors*; $\mathbb{T}$ is assumed to be endowed with a volume element $\varepsilon \in \bigwedge^4 \mathbb{T}^*$, $\varepsilon \neq 0$. A frame $(e_\alpha)_{\alpha=1,\dots,4}$ in $\mathbb{T}$ is said to be *unimodular* if $\varepsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4$, where (e^α) is the frame in $\mathbb{T}^*$, dual to (e_α). From now on, only unimodular frames are used. The six-dimensional vector space $W = \bigwedge^2 \mathbb{T}$ has a quadratic form h—the *Pfaffian*—defined by $\frac{1}{2} w \wedge w = h(w) e_1 \wedge e_2 \wedge e_3 \wedge e_4$. The volume element defines also the Hodge map $\star : \bigwedge \mathbb{T} \to \bigwedge \mathbb{T}^*$, such that $\star(1_{\wedge \mathbb{T}}) = \varepsilon$. If $w = \frac{1}{2} w^{\alpha\beta} e_\alpha \wedge e_\beta$, then $\star w = \frac{1}{2} \star w_{\alpha\beta} e^\alpha \wedge e^\beta$, where $\star w_{12} = w^{34}$, etc. If $w \in W$ is considered as a linear map $\mathbb{T}^* \to \mathbb{T}$ and $\star w$ as a linear map $\mathbb{T} \to \mathbb{T}^*$, then

$$w \circ \star w = -h(w) \mathrm{id}_{\mathbb{T}} \quad \text{and} \quad \star\, w \circ w = -h(w) \mathrm{id}_{\mathbb{T}^*}. \tag{6}$$

In the notation with indices, these equations read $w^{\alpha\gamma} \star w_{\gamma\beta} = -\delta^\alpha_\beta (w^{12} w^{34} + w^{13} w^{42} + w^{14} w^{23})$. The *Klein quadric* is $\mathsf{Q}_4 = \{\mathrm{dir} w : w \in W, w \neq 0, w \wedge w = 0\}$. By (6), the linear map $W \to \mathsf{End}(\mathbb{T} \oplus \mathbb{T}^*)$ given by

$$w \mapsto \begin{pmatrix} 0 & w \\ \star w & 0 \end{pmatrix} \tag{7}$$

has the Clifford property and yields a faithful and irreducible representation of $\mathsf{Cl}(h) = \mathsf{Cl}_6$ in $\mathbb{T} \oplus \mathbb{T}^*$. With respect to this representation, the elements of $\mathbb{T}$ and $\mathbb{T}^*$ are Weyl spinors of opposite chirality; using the notation of §3.1.1 one can put $\mathsf{T}_4^+ = \mathsf{P}(\mathbb{T})$ and $\mathsf{T}_4^- = \mathsf{P}(\mathbb{T}^*)$. The projective twistor $\mathrm{dir}\Phi$, $0 \neq \Phi \in \mathbb{T}$, is identified with the *mtn* 3-space $\{w \in W : w \wedge \Phi = 0\}$; this space projects to a totally null, geodesic, self-dual 2-dimensional submanifold of Q_4: $\alpha(\Phi) = \{\mathrm{dir}(\Phi \wedge \Phi') : \Phi' \in \mathbb{T}, \Phi \wedge \Phi' \neq 0\}$. As a complex manifold, $\alpha(\Phi)$ is $\mathbb{C}\mathsf{P}_2$. If $\Phi, \Phi' \in \mathbb{T}$ and $\Phi \wedge \Phi' \neq 0$, then $\mathrm{dir}(\Phi \wedge \Phi') \in \mathsf{Q}_4$ is the intersection of $\alpha(\Phi)$ and $\alpha(\Phi')$. Similarly, if $\Psi \in \mathbb{T}^*$, $\Psi \neq 0$, then there is the submanifold of Q_4: $\beta(\Psi) = \{\mathrm{dir}(\Phi \wedge \Phi') : \Phi, \Phi' \in \mathbb{T}, \Phi \wedge \Phi' \neq 0, \langle \Phi, \Psi \rangle = \langle \Phi', \Psi \rangle = 0\}$. The submanifolds $\alpha(\Phi)$ and $\beta(\Psi)$ intersect along a null geodesic iff $\langle \Phi, \Psi \rangle = 0$; as a complex manifold, such a null geodesic is $\mathbb{C}\mathsf{P}_1$; two distinct points $\mathrm{dir} w$, $\mathrm{dir} w' \in \mathsf{Q}_4$ lie on such a null geodesic iff $w \wedge w' = 0$; see §9.3 in (Penrose and Rindler 1986) and (Penrose 1996).

The group $\mathsf{Spin}(h) = \mathsf{Spin}_6$ is isomorphic to $\mathsf{SL}_4 = \mathsf{SL}(\mathbb{T})$ embedded in

$\mathsf{Cl}(h)$ by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \tag{8}$$

where $A^* \in \mathsf{SL}(\mathbb{T}^*)$ is the transpose of $A \in \mathsf{SL}(\mathbb{T})$. The element A acts in W by sending w to AwA^*, as may be checked from (7), (8) and the equation $\star(AwA^*) = (\det A)(A^{*-1} \star wA^{-1})$ valid for every $w \in W$ and $A \in \mathsf{GL}(\mathbb{T})$. A frame (e_α) in $\mathbb{T}$ can be used to construct a 'null frame' $(w_a)_{a=0,1,\dots,4,\infty}$ in W by putting (say): $w_0 = e_3 \wedge e_4$, $w_1 = e_1 \wedge e_3$, $w_2 = e_1 \wedge e_4$, $w_3 = e_2 \wedge e_3$, $w_4 = e_2 \wedge e_4$ and $w_\infty = e_1 \wedge e_2$. For $z = (z^\mu) \in \mathbb{C}^4$, put $w(z) = w_0 + z^\mu w_\mu + (z_1 z_4 - z_2 z_3) w_\infty$; then for every z one has $w(z) \neq 0$ and $w(z) \wedge w(z) = 0$; the map $z \mapsto \operatorname{dir} w(z)$ is a conformal embedding of $V = \mathbb{C}^4$ in Q_4. Put $S = \operatorname{span}\{e_1, e_2\}$ and $S' = \operatorname{span}\{e_3, e_4\}$; the direction of w_∞ (equivalently: the plane S) is preserved by the subgroup

$$\mathsf{H}_4 = \{ \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} : a \in \mathsf{GL}(S),\ b \in \mathsf{GL}(S'),\ \det a \det b = 1 \ \text{ and } \ v \in \mathsf{Hom}(S', S) \}$$

of SL_4 so that Cpin_4 is isomorphic to $\{(a, b) \in \mathsf{GL}_2 \times \mathsf{GL}_2 : \det a \det b = 1\}$.

3.2 The real case

Assume now $K = \mathbb{R}$ and let (k, l), $k + l = m$, be the signature of g. The real quadric $\mathsf{Q}_{k,l}$ is diffeomorphic to $(\mathbb{S}_k \times \mathbb{S}_l)/\mathbb{Z}_2$. In particular, $\mathsf{Q}_{k,0} = \mathbb{S}_k$; a proper real quadric, i.e. one with $kl \neq 0$, is orientable iff $k + l$ is even (Cahen *et al.* 1993). An essential difference between the complex and the real case is that, in the latter, the conformal structure on the quadric is generated by a pseudo-Riemannian metric. One can consider spin or pin structures corresponding to such a metric. (S)pin structures on real quadrics have been determined and a method for finding the spectrum of the Dirac operator given in (Cahen *et al.* 1995). There is neither room nor need to describe here the construction of the twistor spaces associated with the real quadrics. The most important case of $\mathsf{Q}_{1,3}$ is fully treated in the works of Penrose and his school. Instead, I describe here the *real twistors* on $\mathsf{Q}_{1,2}$ that could have been discovered by Euclid, had he followed the 'spinorial method' of solving the Pythagorean equation, outlined in §2.

3.2.1 Real twistors on $\mathsf{Q}_{1,2}$

Let $\lambda, \mu \in \mathbb{R}$ and let v be as in (3). The matrix X, given by (4), can be now considered as an endomorphism of $\mathbb{U}$, a four-dimensional vector space of *real* twistors. The antisymmetric matrix

$$\omega = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} : \mathbb{U} \to \mathbb{U}^*$$

is a symplectic 2-form on $\mathbb{U}$ and $\varepsilon = \frac{1}{2}\omega \wedge \omega$ is the corresponding volume 4-form. It follows from $X^* = \omega \circ X \circ \omega^{-1}$ that the map $X \circ \omega^{-1} : \mathbb{U}^* \to \mathbb{U}$ is antisymmetric. Since $X^2 = (x^2+y^2-z^2+\lambda\mu)\mathrm{id}_{\mathbb{U}}$, if the vector $(x, y, z, \lambda, \mu) \in \mathbb{R}^6$ is null, then the bivector $X \circ \omega^{-1}$ is of rank ≤ 2 and there are twistors $\varPhi, \varPsi \in \mathbb{U}$ such that $X \circ \omega^{-1} = \varPhi \wedge \varPsi$. Moreover, $\mathrm{tr} X = 0$ implies $\omega(\varPhi, \varPsi) = 0$. Conversely, given a four-dimensional real symplectic space $(\mathbb{U}, \omega)$, the vector space $W = \{w \in \wedge^2\mathbb{U} : \mathrm{tr}(w \circ \omega) = 0\}$ is five-dimensional and the restriction of the Pfaffian to W is a quadratic form of signature (2,3). Therefore, the quadric $\mathsf{Q}_{1,2}$ can be identified with the set of null directions in W, or, equivalently, with the set of *lagrangian planes* in $\mathbb{U}$. A real twistor $\varPhi \in \mathbb{U}$ defines the null geodesic $\gamma(\varPhi) = \{\mathrm{dir}(\varPhi \wedge \varPsi) : \varPsi \in \mathbb{U}, \omega(\varPhi, \varPsi) = 0\}$ on $\mathsf{Q}_{1,2}$. If $\varPhi \wedge \varPsi \neq 0$ and $\omega(\varPhi, \varPsi) = 0$, then $\gamma(\varPhi) \cap \gamma(\varPsi) = \mathrm{dir}(\varPhi \wedge \varPsi)$. Two distinct points of $\mathsf{Q}_{1,2}$ lie on one null geodesic iff the corresponding lagrangian planes intersect along a line. For the material of this paragraph, see Note 1 to Chapter 6 in (Woodhouse 1980) and §7.2 in (Penrose and Rindler 1986).

Acknowledgments

This research was sponsored by the Polish Committee on Scientific Research (KBN) under Grant No. 2 P302 112 7. The author is grateful to the Mathematical Institute at Oxford for the hospitality and financial support extended to him during the conference on *Geometric issues in the foundations of science.*

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