GEOMETRIC ASPECTS OF SPINORS

A Short Review

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1. Introduction: a Touch of History

Spinors may be considered to appear, for the first time, in slightly disguised form, in the work of L. Euler (1770) and O. Rodrigues (1840), who discovered a rational representation of rotations. The latter wrote equations equivalent to

$$Z' = UZU^\dagger/(1 + \tfrac{1}{4}\,(m^2 + n^2 + p^2)), \tag{1}$$

where $U^\dagger$ is the Hermitian conjugate of the 2 by 2 matrix

$$U = I + \tfrac{1}{2}\,\mathrm{i}\,(m\sigma_x + n\sigma_y + p\sigma_z),$$

I and the sigmas are the unit and the Pauli matrices, respectively,

$$Z = x\sigma_x + y\sigma_y + z\sigma_z, \tag{2}$$

and similarly for Z'.

Spinors are even more explicit in the work of W. R. Hamilton (1844), A. Cayley (1845), W. K. Clifford (1878) and R. O. Lipschitz (1880). Their ideas led to many developments presented also at the two previous conferences on *Clifford algebras and their applications in mathematical physics* (Canterbury 1985 and Montpellier 1989). Élie Cartan (1913) discovered what are now called spinor representations of the complex Lie algebras $so(n)$, $n > 2$. According to B. L. van der Waerden (1960), the name *spinor* is due to P. Ehrenfest who suggested, on a visit to Göttingen, to develop a spinor analysis analogous to tensor calculus (Van der Waerden, 1929).

In atomic and particle physics, spinor-valued functions are used to describe the quantum-mechanical behaviour of fermions. Most of the time, there appear complex, four-component Dirac spinors over

Minkowski space-time. Two-component Weyl spinors are used in the context of parity violating weak interactions.

Weyl (reduced, semi– or half–)spinors are used by Roger Penrose and his school (Penrose and Rindler, 1984 and 1986) in the theory of general relativity and twistors. As emphasized by P. Budinich, Weyl spinors occur there because they are simple (pure) in the sense of Cartan (Chevalley) and not because of their relevance in the theory of the neutrino. Incidentally, the germ of the notion of a pure spinor can be found in a paper by J. W. Givens (1937).

2. The Square Root Ideas

2.1. The Clifford Idea

If Z is as in (2), then $Z^2 = (x^2+y^2+z^2)\,I$ so that Z is a *linear form of the square root of the quadratic form* $x^2+y^2+z^2$. Clifford generalized this and similar observations; to fix the notation and terminology, I state the relevant definitions and theorems.

Let V be an n-dimensional vector space over the field $F = \mathbf{R}$ or $\mathbf{C}$; and let $g : V \times V \to F$ be a bilinear and symmetric map; if it is, moreover, non-degenerate, then it is a scalar product in V and defines an isomorphism $g : V \to V^*$. Recall that, denoting by $\mathbf{T}(V) = \bigoplus \bigotimes^k V$, where $\bigotimes^0 V = F$, the tensor algebra of V and by $\mathbf{I}(V,g)$ its bilateral ideal generated by all elements of the form $v \otimes v - g(v,v)$, one defines the *Clifford algebra* as $\mathbf{Cl}(V,g) = \mathbf{T}(V)/\mathbf{I}(V,g)$ and finds it to be associative, with unit element, and to have the following universal property: if $\mathcal{A}$ is an algebra over F, with unit element $1_{\mathcal{A}}$, and $f : V \to \mathcal{A}$ is a linear map such that

$$f(v)^2 = g(v,v)\,1_{\mathcal{A}} \text{ for every } v \in V,$$

then there is a homomorphism of algebras with units $\tilde{f} : \mathbf{Cl}(V,g) \to \mathcal{A}$ extending f. In particular, $\mathbf{Cl}(V,0) = \bigwedge V$ is the exterior algebra of V. For $v \in V$, let $e(v)$ and $c(v) : \bigwedge V \to \bigwedge V$ be the exterior multiplication by v and the contraction with $g(v) \in V^*$, respectively. The map $f : V \to \operatorname{End} \bigwedge V$ given by $f(v) = e(v) + c(v)$ has the properties described above, where now $\mathcal{A} = \operatorname{End} \bigwedge V$. The homomorphism $\tilde{f} : \mathbf{Cl}\,(V,g) \to \operatorname{End} \bigwedge V$ is injective and the map $\mathbf{Cl}\,(V,g) \to \bigwedge V$ given by
$a \mapsto \langle \tilde{f}(a), 1_{\wedge V} \rangle$ is an isomorphism of vector spaces. Since it is natural, one can identify these two vector spaces; this will be done in the sequel without further comments; the Clifford product of $v \in V$ by $a \in \bigwedge V$ is

then given by the Chevalley-Kähler formula (Chevalley, 1954; Kähler, 1962),

$$va = e(v)a + c(v)a.$$

2.2. The Cartan Idea

The Pythagorean equation

$$x^2 + z^2 = y^2$$

is equivalent to each of the statements:
(i) (x, y, z) is a null (isotropic, optical, light-like) vector in $\mathbf{R}^3$ with a scalar product of signature $(2, 1)$;
(ii) one has $\det X = 0$, where $X = x\sigma_x + \mathrm{i}y\sigma_y + z\sigma_z$ is a real matrix;
(iii) there exists a *spinor* $\varphi \in \mathbf{R}^2$ such that

$$BX = \varphi \otimes \varphi, \tag{3}$$

where $B = \sigma_y/i$ is a matrix such that

$$X^t = -BXB^{-1}.$$

Informally, equation (3) can be interpreted to mean that the *spinor* φ *is a square root of the null vector* (x, y, z), represented by the symmetric matrix BX. Cartan (1938) found a generalization of this observation to vector spaces of any dimension, with a scalar product, admitting totally null subspaces of maximal dimension.

2.3. The 'Double–valuedness' of the Spinor Representations

2.3.1. *The Algebraic Aspect*

Let V and g be as in §2.1. If $u \in V$ is non-null, $u^2 = g(u, u) \neq 0$, then u is invertible, as an element of $\mathbf{Cl}(V, g)$, and the map $v \mapsto -uvu^{-1}$, where $v \in V$, is a reflection in the hyperplane orthogonal to u. The vector u can be normalized, but there is always an *ambiguity of sign*: the vectors u and $-u$ represent the same reflection. The group $\mathbf{Pin}(V, g)$ is defined as the subset of $\mathbf{Cl}(V, g)$ consisting of products of all finite sequences of unit vectors; if $a \in \mathbf{Pin}(V, g)$ and α is the *main automorphism* of the Clifford algebra, i.e. such that $\alpha(v) = -v$ for $v \in V$, then the linear map $v \mapsto \rho(a)v = \alpha(a)va^{-1}$ is an orthogonal transformation of V and there is the exact sequence of group homomorphisms

$$1 \to \mathbf{Z}_2 \to \mathbf{Pin}(V, g) \to \mathbf{O}(V, g) \to 1. \tag{4}$$

The main automorphism α defines the $\mathbf{Z}_2$−grading of the Clifford algebra,

$$\mathbf{Cl}\,(V,g) = \mathbf{Cl}_0(V,g) \oplus \mathbf{Cl}_1(V,g).$$

In general, the group $\mathbf{Pin}\,(V,g)$ has two distinguished subgroups:

$$\mathbf{Spin}\,(V,g) = \mathbf{Pin}(V,g) \cap \mathbf{Cl}_0(V,g)$$

and its connected component $\mathbf{Spin}_0(V,g)$. There is an exact sequence similar to (4),

$$1 \to \mathbf{Z}_2 \to \mathbf{Spin}(V,g) \to \mathbf{SO}(V,g) \to 1. \tag{5}$$

If V is a complex vector space, $V = \mathbf{C}^n$, and g is non-degenerate, then one writes $\mathbf{Cl}\,(n,\mathbf{C})$ and $\mathbf{Pin}\,(n,\mathbf{C})$ instead of $\mathbf{Cl}\,(V,g)$ and $\mathbf{Pin}\,(V,g)$, respectively. A similar notation is used for the groups $\mathbf{Spin}(n,\mathbf{C})$ which are connected for $n > 1$. If g is a scalar product of signature (s,t) in $\mathbf{R}^{s+t}$, then one writes $\mathbf{Cl}\,(s,t)$ instead of $\mathbf{Cl}\,(V,g)$, etc. The groups $\mathbf{Spin}(n,0)$ and $\mathbf{Spin}(0,n)$ are isomorphic and abbreviated as $\mathbf{Spin}(n)$; similarly for $\mathbf{Spin}_0$. One has $\mathbf{Spin}(n) = \mathbf{Spin}_0(n)$ for $n > 1$.

2.3.2. *The Topological Aspect*

The kernel of ρ, $\mathbf{Z}_2 = \{1,-1\}$, appearing in (4) and (5), has an algebraic origin, described in the preceding paragraph. The case when $V = \mathbf{R}^2$ and g is of signature (1,1), is exceptional: the special Lorentz group $\mathbf{SO}_0(1,1)$ and the group $\mathbf{Spin}_0(1,1)$ are both isomorphic to the additive group $\mathbf{R}$. The generic case is illustrated by $s = 3$ and $t = 0$. On the basis of (1) one sees that the map $Z \mapsto Z'$ represents a rotation by the angle $\omega = 2\arctan\frac{1}{2}\sqrt{m^2+n^2+p^2}$ around an axis defined by the vector (m,n,p). Introducing the 'Cayley–Klein parameters', $a = (1+\frac{1}{2}\,\mathrm{i}p)\cos\frac{1}{2}\omega$, $\;b = \frac{1}{2}(\mathrm{i}m - n)\cos\frac{1}{2}\omega$, and allowing them to assume all values compatible with $|a|^2+|b|^2 = 1$, one obtains $\mathbf{Spin}(3) = \mathbf{SU}(2)$ and

$$\rho\begin{pmatrix} e^{\frac{1}{2}\mathrm{i}\omega} & 0 \\ 0 & e^{-\frac{1}{2}\mathrm{i}\omega} \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One sees that the restriction of ρ to the circle $\mathbf{U}(1) \subset \mathbf{SU}(2)$, defined by $0 \le \omega \le 4\pi$, is the 'squaring map', $z \mapsto z^2$, $z \in \mathbf{U}(1)$. This property of ρ is general in the following sense: if s or $t > 1$, then there are the exact sequences

$$1 \to \mathbf{Z}_2 \to \mathbf{Spin}_0(s,t) \to \mathbf{SO}_0(s,t) \to 1 \tag{6}$$

and

$$1 \to \pi_1(\mathbf{Spin}_0(s,t)) \to \pi_1(\mathbf{SO}_0(s,t)) \to \mathbf{Z}_2 \to 1. \tag{7}$$

The latter sequence defines a non-trivial $\mathbf{Z}_2-$grading of the fundamental group $\pi_1(\mathbf{SO}_0(s,t))$. Incidentally, there is a little 'experiment' closely related to the Hopf map $\mathbf{SU}(2) \to \mathbf{SO}(3) \to \mathbf{S}_2$. Take two coins of one ecu each and put them on a table. Holding one coin, roll the other around it so that their rims touch and there is no slipping. Children, who perform the experiment for the first time, are astonished when they realize that the moving coin rotates by 720^0 after completing one turn around the fixed coin.

3. Pure Spinors

3.1. Representations of Clifford Algebras Associated with Even–dimensional Vector Spaces

In this section, one considers only even-dimensional vector spaces with a scalar product. These assumptions imply that the corresponding Clifford algebras are simple and, as such, have only one, up to equivalence, faithful and irreducible representation.

3.1.1. *Complex Spaces*

Let $(e_\mu), \mu = 1, \ldots, n$, be an orthonormal basis in the complex vector space $W = \mathbf{C}^n$, where n is even, $n = 2m$. One chooses the basis so that $g_{\mu\nu} = g(e_\mu, e_\nu)$ is given by $g_{\mu\nu} = 0$ for $\mu \neq \nu$ and $g_{\nu\nu} = (-1)^{\nu+1}$. The corresponding volume element $\eta = e_1 \ldots e_n \in \mathbf{Cl}\,(n)$ satisfies $\eta^2 = 1$ and can be used to define the Hodge dual $*a$ of $a \in \bigwedge W$ by means of the Kähler formula $*a = \eta a$ so that $**a = a$.

Recall (see e.g. (Benn and Tucker, 1987; Budinich and Trautman, 1988a) and the references given there) that, for every $m = 1, 2, \ldots$, there is a faithful and irreducible representation γ of the Clifford algebra $\mathbf{Cl}\,(2m)$ in a complex, 2^m- dimensional vector space S of 'Dirac spinors',

$$\gamma : \mathbf{Cl}\,(2m) \to \mathrm{End}\, S. \tag{8}$$

The automorphisms of S defined by $\gamma_\mu = \gamma(e_\mu)$ are called 'Dirac matrices' and

$$\Gamma = \gamma(\eta) = \gamma_1 \ldots \gamma_{2m}$$

is the 'helicity' automorphism, satisfying $\Gamma\gamma_\mu + \gamma_\mu\Gamma = 0$ and $\Gamma^2 = I$. If φ is an eigenvector of Γ,

$$\Gamma\varphi = (-1)^{\lambda(\varphi)}\varphi, \quad \text{where} \quad \lambda(\varphi) = 0 \;\text{ or }\; 1, \tag{9}$$

then it is called a 'Weyl spinor'. The transposed matrices γ_μ^t correspond to the contragredient representation of $\mathbf{Cl}\,(2m)$ in the dual space

S^* . From the simplicity of $\mathbf{Cl}\,(2m)$ there follows the existence of an isomorphism $B : S \to S^*$ such that

$$\gamma_\mu^t = (-1)^m B\gamma_\mu B^{-1}, \quad \Gamma^t = (-1)^m B\Gamma B^{-1} \quad \text{and} \quad B^t = (-1)^{\frac{1}{2}m(m+1)}B. \tag{10}$$

Since Γ anticommutes with the Dirac matrices, the sign in the first equation above can be changed by replacing B with $B\Gamma$. Following Cartan, it is chosen in such a way that the equation holds also for the matrix $\gamma_{2m+1} = \Gamma$, appearing in the representation of the algebra $\mathbf{Cl}\,(2m+1)$.

3.1.2. *Real Spaces*

Let g be a scalar product of signature $(2p+\varepsilon, 2q+\varepsilon)$ in the real vector space $V = \mathbf{R}^{2m}$, where $m = p+q+\varepsilon$ and $\varepsilon = 0$ or 1. Since $\mathbf{C}\otimes\mathbf{Cl}(2p+\varepsilon, 2q+\varepsilon) = \mathbf{Cl}(2m)$, the Dirac representation (8) gives, by restriction, a representation

$$\gamma : \mathbf{Cl}\,(2p+\varepsilon, 2q+\varepsilon) \to \mathrm{End}\, S, \tag{11}$$

where the same letter γ is used for the restricted representation. Since now the Clifford algebra is defined over the reals, the complex conjugate $\bar{\gamma}$ of the representation (11) is also a representation; it follows from the simplicity of the algebra that these representations are equivalent: there exists an inertwining linear isomorphism $C : S \to \bar{S}$ such that

$$\bar{\gamma}_\mu = C\gamma_\mu C^{-1}, \tag{12}$$

where the Dirac matrices $\gamma_\mu = \gamma(e_\mu)$ correspond to an orthonormal basis in V such that $g_{\mu\nu} = 0$ for $\mu \neq \nu$, $g_{\mu\mu} = 1$ for $\mu = 1, \ldots, 2p+\varepsilon$ and -1 for $\mu = 2p+1+\varepsilon, \ldots, 2m$; the volume element $\eta = e_1 \ldots e_{2m}$ satisfies $\eta^2 = (-1)^{m+\varepsilon}$. The helicity automorphism is now defined by $\Gamma = \mathrm{i}^{m+\varepsilon}\gamma(\eta)$ so that $\Gamma^2 = I$ still holds and

$$\bar{\Gamma} = (-1)^{m+\varepsilon} C\Gamma C^{-1}.$$

The defining properties of the maps B and C determine them only up to non-zero complex factors which can be chosen so that

$$\bar{C}C = (-1)^{\frac{1}{2}(q-p)(q-p+1)} I \qquad \text{and} \qquad C^t \bar{B} C = B. \tag{13}$$

The 'charge conjugate' of $\varphi \in S$ is the spinor

$$\varphi_c = C^{-1}\bar{\varphi}. \tag{14}$$

If $q - p \equiv 0$ or 3 mod 4, then the representation is real: the space S decomposes into the direct sum of two real spaces of 'Majorana spinors' characterized by $\varphi_c = \varphi$ or $-\varphi$. If φ is a Weyl spinor, then so is φ_c and

$$\lambda(\varphi) + \lambda(\varphi_c) \equiv m + \varepsilon \bmod 2. \tag{15}$$

When $p \equiv q \bmod 4$ — and only in this case — there are non-zero spinors which are simultaneously Weyl and Majorana.

3.2. Multivectors Associated with Pairs of Spinors

Consider a pair (φ, ψ) of spinors. One associates with that pair the endomorphism $\varphi \otimes B\,\psi$ of S such that, for every $\chi \in S$, one has

$$(\varphi \otimes B\,\psi)(\chi) = \langle B\,\psi, \chi\rangle\,\varphi \quad \text{so that} \quad \mathrm{Tr}\,(\varphi \otimes B\psi) = \langle B\psi, \varphi\rangle.$$

Let γ be the isomorphism (8); there is a multivector (see e.g. (Pais, 1962) or (Budinich and Trautman, 1988b))

$$B(\psi, \varphi) = \sum_{k=0}^{2m} B_k(\psi, \varphi), \quad \text{where} \quad B_k(\psi, \varphi) \in \bigwedge^k \mathbf{C}^{2m}, \tag{16}$$

such that

$$\varphi \otimes B\,\psi = \gamma\,(B(\psi, \varphi)) \tag{17}$$

and

$$B_k(\psi, \varphi) = 2^{-m} \sum\nolimits_{\mu_1 < \ldots < \mu_k} \langle B\,\psi, \gamma^{\mu_k} \ldots \gamma^{\mu_1} \varphi\rangle e_{\mu_1} \ldots e_{\mu_k}. \tag{18}$$

The following theorem summarizes well-known results, which are obtained from the preceding definitions and the properties of the Dirac representation.

THEOREM 1. *Let γ be the representation (8), let $B(\psi, \varphi)$ be the multivector defined by (16–18) and $v \in \mathbf{C}^{2m}$. Then*

(i) $B(\psi, \gamma(v)\varphi) = (c(v) + e(v))B(\psi, \varphi)$;

(ii) $B_k(\psi, \varphi) = (-1)^{\frac{1}{2}(k-m)(k-m-1)} B_k(\varphi, \psi)$;

(iii) $*B(\psi, \varphi) = B(\psi, \Gamma\varphi)$;

(iv) $B_k(\psi, \Gamma\varphi) = (-1)^{k-m} B_k(\Gamma\psi, \varphi)$;

(v) *if φ and ψ are Weyl spinors, then*

$$\lambda(\varphi) + \lambda(\psi) + k + m \equiv 1 \bmod 2 \quad \textit{implies} \quad B_k(\psi, \varphi) = 0;$$

(vi) *in particular, if φ is a Weyl spinor, then*

$$k - m \not\equiv 0 \bmod 4 \quad \textit{implies} \quad B_k(\varphi, \varphi) = 0.$$

The representation (11) of the real algebra is faithful (injective), but γ is not surjective: the multivector defined by (17) is not real, in general. There holds, by virtue of (13), the equality

$$\overline{B(\psi, \varphi)} = B(\psi_c, \varphi_c). \tag{19}$$

Therefore, if $q-p\equiv 0$ or $3 \bmod 4$, and $\varphi_c=\varphi,\ \psi_c=\psi$, then the multivector $B(\psi,\varphi)$ is real. In general, the multivector $B_k(\varphi_c,\varphi)$ is real or pure imaginary, depending on whether $\frac{1}{2}(q-p)(q-p+1)+\frac{1}{2}(k-m)(k-m-1)$ is even or odd, as may be seen by referring to part (ii) of Theorem 1 and (13).

3.3. Pure spinors associated with $\mathbf{C}^{2m}$

Let again γ be the Dirac representation of $\mathbf{Cl}\,(2m)$ in S and $W=\mathbf{C}^{2m}$; if $\varphi\in S$ and $\varphi\neq 0$, then the vector space

$$N(\varphi)=\{v\in W:\gamma(v)\varphi=0\}$$

depends only on the *direction* of the spinor φ and is *totally null*, i.e. $v^2=0$ for every $v\in N(\varphi)$. If $N(\varphi)$ is maximal (in this case: m-dimensional), then φ is said to be *pure*. Let $W=N\oplus P$ be a decomposition of W into a direct sum of maximal, totally null (*mtn*) subspaces and let $(n_1,\dots,n_m,p_1,\dots,p_m)$ be the corresponding *Witt basis*,

$$n_\mu n_\nu+n_\nu n_\mu=p_\mu p_\nu+p_\nu p_\mu=0,\ \ n_\mu p_\nu+p_\nu n_\mu=\delta_{\mu\nu},$$

for $\mu,\nu=1,\dots,m$. Since the representation γ is faithful, there exists $\omega\in S$ such that $\varphi=\gamma(n_1\dots n_m)\omega\neq 0$ and then $N(\varphi)=N$ so that φ is pure. Conversely, given a pure spinor φ, one can find an *mtn* subspace P complementary to $N=N(\varphi)$. In terms of the Witt basis, $\eta=[n_1,p_1]\dots[n_m,p_m]$ and one sees that a pure spinor is Weyl. The collection of 2^m pure spinors $\gamma(p_{\mu_1}\dots p_{\mu_k})\varphi$, where $1\leq\mu_1<\dots<\mu_k\leq 2m$, is a 'Fock basis' in S. There is a bijective correspondence between the set of all *mtn* subspaces of W and the set of all directions of pure spinors. If v is a non-null vector, $v^2\neq 0$, and φ is a pure spinor, then $\gamma(v)\varphi$ is a pure spinor of opposite helicity and $N(\gamma(v)\varphi)=vN(\varphi)v^{-1}$. The group $\mathbf{O}(2m)$ acts transitively on the space of directions of pure spinors and $\mathbf{SO}(2m)$ acts transitively on the manifold of directions of pure spinors of one helicity: this is the complex, $\frac{1}{2}m(m-1)$–dimensional symmetric space $\mathcal{Q}_m=\mathbf{SO}\,(2m)/\mathrm{U}\,(m)$ (Ehresmann, Porteous).

Using Theorem 1 one proves

THEOREM 2. I. *Let φ be a non-zero Weyl spinor associated with $W=\mathbf{C}^{2m}$. The following conditions are equivalent:*

(i) *φ is pure and $N=N(\varphi)=\mathrm{span}\,\{n_1,\dots,n_m\}$;*

(ii) $B_m(\varphi,\varphi)=n_1\wedge n_2\wedge\dots\wedge n_m$ *and* $B_k(\varphi,\varphi)=0$ *for* $k\neq m$;

(iii) *the vector space* $N=\{B_1(\psi,\varphi)\in W:\psi\in S\}$ *is* mtn.

II. *If* φ *and* ψ *are pure spinors, then the dimension of* $N(\varphi) \cap N(\psi)$ *is the least integer* r *such that* $B_r(\psi, \varphi) \neq 0$*; moreover,* $B_r(\psi, \varphi) = n_1 \wedge \ldots \wedge n_r$ *and* $N(\varphi) \cap N(\psi) = \mathrm{span}\,\{n_1, \ldots, n_r\}$*, so that*

(iv) $\lambda(\varphi) + \lambda(\psi) + r + m \equiv 0 \bmod 2.$

If the pure spinors φ *and* ψ *are linearly independent, then*

(v) $\varphi + \psi$ *is pure iff* $r = m - 2$ *and then* $N(\varphi + \psi) \cap N(\psi) = N(\varphi) \cap N(\psi)$.

III. *Consider now a real vector space* V *with a scalar product of signature* $(2p + \varepsilon, 2q + \varepsilon)$ *and let* $W = \mathbf{C} \otimes V$*. For every spinor* φ *one has*

(vi) $N(\varphi_c) = \overline{N(\varphi)}.$

If φ *is pure, as a spinor associated with* W*, then so is* φ_c*; its* real index r *is the dimension of the intersection* $N(\varphi) \cap N(\varphi_c)$*. There is a decomposition of the space* $\mathcal{Q}_m$ *of directions of pure spinors of one helicity into submanifolds* $\mathcal{Q}_{p,q,\varepsilon,r}$ *of directions of pure spinors having the same helicity and real index,*

(vii) $\mathcal{Q}_m = \bigcup_r \mathcal{Q}_{p,q,\varepsilon,r},$ *where* $r = \varepsilon, \varepsilon{+}2, \ldots, \varepsilon{+}2\min\{p, q\}.$

Pure spinors with the least value ε *of the real index are* generic *in the sense that* $\mathcal{Q}_{p,q,\varepsilon,\varepsilon}$ *is open and dense in* $\mathcal{Q}_m$*. The action of the group* $\mathbf{SO}\,(2p + \varepsilon, 2q + \varepsilon)$ *is transitive on every* $\mathcal{Q}_{p,q,\varepsilon,r}$*, a manifold of real dimension* $m(m-1) - \frac{1}{2}r(r-1)$.

According to part (ii) of Theorem 2 *a pure spinor associated with* $\mathbf{C}^{2m}$ *is the square root of a self- (or anti-self-)dual and decomposable m-vector*. A proof of the equivalence of parts (i) and (iii) is based on the equation (Budinich and Trautman, 1988b)

$$c(v)B_1(\psi, \varphi) = B_0(\psi, \gamma(v)\varphi).$$

Another proof was given by Hughston and Mason (1988). Most of the statements in Parts I and II are due to Cartan and Chevalley, see also Benn and Tucker. The real case (Part III) is taken from Kopczyński and Trautman.

4. Analogies between Complex and Optical Geometries

4.1. The Algebraic Aspect

Consider a real, n-dimensional vector space V. A complex vector subspace N of $W = \mathbf{C} \otimes V$ defines a *complex flag* (K, L, J) in V, i.e. a pair (K, L) of subspaces of V such that $K \subset L$ and J is a complex structure in L/K, i.e. a linear map such that $J^2 = -\,\mathrm{id}$. Namely, given N, one puts $K = \mathrm{Re}\,(N \cap \bar{N})$, $L = \mathrm{Re}\,(N + \bar{N})$ and $J(n + \bar{n} \bmod K) = i(n - \bar{n}) \bmod K$, where $n \in N$. One easily sees that there is a bijective and natural correspondence between the set of all complex flags in V and the set of all complex subspaces of W. Denote by r the dimension of K. If, in particular, V and N are $2m$- and m-dimensional, respectively, and $r = 0$, then $L = V$ and J is a complex structure in V. When V is given a scalar product g, then a natural question, in relation to a complex flag, is to ask whether g *descends* to L/K and makes J orthogonal. This is answered by the following (Nurowski and Trautman, 1993)

THEOREM 3. *Let (K, L, J) be a complex flag in an even-dimensional real vector space V with a scalar product g and let N be the corresponding subspace of the complexification W of V. The following two conditions are equivalent:*
(i) *N is maximal among totally null subspaces of W;*
(ii) *g descends to a scalar product in L/K, the complex structure J is orthogonal and $K = L^{\perp}$.*

The case of $m = 2$, g of signature $(3, 1)$ and $r = 1$ leads to an 'optical structure': the line K is interpreted as corresponding to a ray of light and L/K is a 2-dimensional 'screen space'; see (Trautman, 1984 and 1985; Robinson and Trautman, 1988) and the references given there.

4.2. Complex, Cauchy–Riemann and Optical Geometries

4.2.1. *Geometries Defined by Subbundles of the Complexified Tangent Bundle*

Consider an n-dimensional, paracompact, connected, smooth manifold $\mathcal{M}$ and let $\mathcal{W}$ denote the complexification of its tangent bundle $\mathcal{V} = T\mathcal{M}$. The preceding algebraic constructions can be applied, 'pointwise', to the fibres of the bundles. In particular, a *complex flag geometry* $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ on $\mathcal{M}$ consists of two smooth subbundles $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{V}$, such that $\mathcal{K} \subset \mathcal{L} \subset \mathcal{V}$, and an automorphism $\mathcal{J}$ of the vector bundle $\mathcal{L}/\mathcal{K}$ such that $\mathcal{J}^2 = -\mathrm{id}$. By Theorem 3, such a geometry is equivalent to giving a smooth, complex vector subbundle $\mathcal{N}$ of the complex vector bundle $\mathcal{W}$. Denoting by $\mathcal{K}_x$ the fibre of $\mathcal{K}$ over $x \in \mathcal{M}$, and similarly for other bundles, one has $\mathcal{K}_x = \mathrm{Re}\,(\mathcal{N}_x \cap \overline{\mathcal{N}_x})$, etc.

Let $\mathcal{S}(\mathcal{N})$ be the module of sections of the vector bundle $\mathcal{N} \to \mathcal{M}$; a similar notation is used for other vector bundles over $\mathcal{M}$ and, in particular, $\mathcal{S}(\mathcal{M})$ is the algebra of smooth functions on $\mathcal{M}$. One says that the complex flag geometry defined by $\mathcal{N}$ is *integrable* if

$$[\mathcal{S}(\mathcal{N}), \mathcal{S}(\mathcal{N})] \subset \mathcal{S}(\mathcal{N}). \tag{20}$$

In the general case, when $0 < r < m < \frac{1}{2}(n+r)$, condition (20) implies

$$[\mathcal{S}(\mathcal{K}), \mathcal{S}(\mathcal{K})] \subset \mathcal{S}(\mathcal{K}), \tag{21}$$

and

$$[\mathcal{S}(\mathcal{K}), \mathcal{S}(\mathcal{L})] \subset \mathcal{S}(\mathcal{L}). \tag{22}$$

Therefore, the bundle $\mathcal{K}$ defines a *foliation* on $\mathcal{M}$, which is assumed to be *regular* in the sense that the set $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ of all its leaves has the structure of a smooth,
$(n-r)$-dimensional manifold such that the canonical map $\mathcal{M} \to \mathcal{M}'$ is a submersion. The following theorem rephrases well-known facts concerning complex and Cauchy–Riemann (CR) geometries (Wells, 1983; Yano and Kon, 1984).

THEOREM 4. *Let $\mathcal{N} \to \mathcal{M}$ be a complex vector subbundle of the complexified tangent bundle of an n-dimensional manifold $\mathcal{M}$; denote by $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ the associated complex flag geometry, and let m and r be the dimensions, complex in the first and real in the second case, respectively, of the fibres of the bundles $\mathcal{N}$ and $\mathcal{K}$. Then*

(i) *If $r = 0$ and $n = 2m$, then $\mathcal{J}$ defines an almost complex structure on $\mathcal{M}$ and (20) is its classical integrability condition, equivalent to the vanishing of the Nijenhuis tensor of $\mathcal{J}$;*

(ii) *If $r = 0$, but $n > 2m > 0$, then (20) is the integrability condition of a (non-trivial) CR geometry defined by $\mathcal{J}$ on the fibres of $\mathcal{L} \subset \mathcal{V}$;*

(iii) *In the general case, when $0 < r < m \leq \frac{1}{2}(n+r)$, condition (20) implies the Frobenius integrability condition (21) of $\mathcal{K}$; by virtue of (22) the quotient manifold $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ inherits, by projection, a subbundle $\mathcal{L}'$ of its tangent bundle and $\mathcal{J}$ descends to $\mathcal{M}'$, defining there a CR geometry. The latter geometry is simply a complex geometry when $n = 2m - r$.*

4.2.2. *Hermitian and Optical Geometries*
Let $\mathcal{M}$ be now a $2m$-dimensional manifold satisfying, in addition to what is stated at the beginning of §4.2.1, the following assumptions: $\mathcal{M}$ has a metric tensor g of signature $(2p+\varepsilon, 2q+\varepsilon)$, is orientable and has a *spin structure*, i.e. a principal bundle $\mathcal{P}$ over $\mathcal{M}$, with structure group $G = \mathbf{Spin}\,(2p+\varepsilon, 2q+\varepsilon)$, doubly-covering the bundle of orthonormal frames on $\mathcal{M}$ of coherent orientation; see Lawson and Michelsohn for details. The last assumption is, in fact, not essential because all that is needed here are bundles of 'projectivized' spinors which are associated with the bundles of orthonormal frames. It is often, however, convenient to work with spinor fields and to represent them by the corresponding maps from $\mathcal{P}$ to the typical fibre S equivariant with respect to the action of the group G: a spinor field is a map $\varphi : \mathcal{P} \to S$ such that $\varphi(\xi a) = \gamma(a^{-1})\varphi(\xi)$ for every $\xi \in \mathcal{P}$ and $a \in G$; similar representations are applied to vector fields and differential forms. For the sake of clarity, the same symbols will be used to represent such fields, irrespectively of whether they are considered as equivariant maps from $\mathcal{P}$ to a typical fibre or as sections of the corresponding associated bundle. The metric tensor determines a Levi-Civita connection and ∇ denotes the corresponding covariant derivative. With the pair (φ, ψ) of spinor fields one associates the multivector-valued field $\mathcal{B}(\psi, \varphi)$, given by

$$\mathcal{B}(\psi, \varphi)(\xi) = B(\psi(\xi), \varphi(\xi)).$$

The manifold $\mathcal{M}$ being orientable, it is meaningful to consider spinors of the same heliciy over $\mathcal{M}$. Generalizing the constructions due to Penrose, Atiyah *et al.* and O'Brian–Rawnsley, one defines the total space of the *twistor bundle* $\mathcal{T}_r$ of *real index* r as consisting of directions of all pure spinors on $\mathcal{M}$ of one, say positive, helicity and of real index r. According to Theorem 2, if a section of the twistor bundle exists, then it can be represented (in 'homogeneous coordinates') by a nowhere vanishing field $\varphi : \mathcal{P} \to S$ of pure spinors; such a section defines a complex vector bundle $\mathcal{N} \to \mathcal{M}$; its fibres are *mtn*s, $\mathcal{N}_x = N(\varphi(\xi))$, where $\xi \in \mathcal{P}_x$. With the field φ one associates the self-dual and decomposable complex m-form

$$\Phi = g(\mathcal{B}_m(\varphi, \varphi)), \tag{23}$$

so that

$$u \in \mathcal{S}(\mathcal{N}) \quad \text{iff} \quad c(u)\Phi = 0 \quad \text{iff} \quad g(u) \wedge \Phi = 0. \tag{24}$$

THEOREM 5. *Let φ be a nowhere vanishing field of pure spinors on $\mathcal{M}$, let $\mathcal{N}$ be the corresponding bundle of* mtn *subspaces of the complexified tangent bundle of $\mathcal{M}$ and let Φ be the m-form defined by (23). The integrability condition (20) is equivalent to each of the following:*

(i) *the* Penrose–Sommers equation $\varphi \wedge \nabla_u \varphi = 0$ *holds for every* $u \in \mathcal{S}(\mathcal{N})$*;*

(ii) *there exists a 1-form* μ *such that* $\mathrm{d}\Phi = \mu \wedge \Phi$*.*

A spinorial proof of this theorem is given by Hughston and Mason who provide references to Penrose and Sommers; they also point out that, locally, by rescaling Φ, one can reduce the 1-form μ to 0 ; this is a 'generalized Robinson theorem'.

If (20) holds and the dimension r of the fibres of the bundle $\mathcal{K}$ is > 0, then the leaves of the corresponding foliation are r-dimensional *null geodetic manifolds*; a similar result was obtained by Plebański and Hacyan in the context of complex, four-dimensional Riemannian geometry.

From now on it is assumed that the field φ of pure spinors is generic, i.e. $r = \varepsilon$. There are two cases to consider: either (i) g is *pseudo-Euclidean*, $\varepsilon = 0$, or (ii) g is *pseudo-Lorentzian*, $\varepsilon = 1$. The prefix *pseudo* is dropped whenever p or $q = 0$.

4.2.2.1. (i) *Hermitian Geometry.* In the pseudo-Euclidean case, a section of the twistor bundle $\mathcal{T}_0$ defines an orthogonal almost complex structure $\mathcal{J}$ on $\mathcal{M}$; the orthogonality property is equivalent to the statement that the bilinear map $j : \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \to \mathcal{S}(\mathcal{M})$ given by

$$j(u, v) = \langle g(u), \mathcal{J}(v) \rangle$$

defines a *two-form* on $\mathcal{M}$; using $\mathcal{J}$ to give a complex structure in the fibres of $\mathcal{V}$, one introduces in this bundle, with fibres of complex dimension m, the pseudo-Hermitian form $g + \mathrm{i}j$ of signature (p, q). There is a convenient, spinorial expression of j in terms of the field φ. Since $r = 0$ and φ is nowhere zero, so is the function $\mathcal{B}_0(\varphi_c, \varphi)$ and $j = \mathrm{i}\, g(\underline{\mathcal{B}_2}(\varphi_c, \varphi))/\mathcal{B}_0(\varphi_c, \varphi)$. Moreover, the form Φ can be scaled so that $\Phi \wedge \overline{\Phi} = \wedge^m j$. A manifold $\mathcal{M}$ with g of signature $(2p, 2q)$ and an orthogonal almost complex structure $\mathcal{J}$ is called an *almost pseudo-Hermitian* space; if $\mathcal{J}$ is integrable, then $\mathcal{M}$ is *pseudo-Hermitian*; it is a *pseudo-Kähler* space if, in addition, one of the following equivalent conditions is satisfied: (a) $\mathrm{d}j = 0$, (b) $\nabla\mathcal{J} = 0$, (c) there exists a field of 1-forms μ such that $\nabla\varphi = \mu \otimes \varphi$.

4.2.2.2. (ii) *Optical Geometry.* In the pseudo-Lorentzian case, a section of the twistor bundle $\mathcal{T}_1$ defines an *almost optical geometry*, i.e. a complex flag geometry $(\mathcal{K}, \mathcal{L}, \mathcal{J})$ on $\mathcal{M}$ with a metric tensor g of signature $(2p + \varepsilon, 2q + \varepsilon)$ such that $\mathcal{K} = \mathcal{L}^\perp$ is a real line bundle and $\mathcal{J}$ is orthogonal with respect to the metric induced by g in the fibres of $\mathcal{L}/\mathcal{K}$. An almost optical geometry satisfying the integrability condition (20) is called *optical*: the trajectories of $\mathcal{K}$ are then null geodesics and the quotient $\mathcal{M}' = \mathcal{M}/\mathcal{K}$ is a $(2m-1)$-dimensional CR manifold. In the proper Lorentzian case $(p = 1, q = 0)$ the integrability condition is equivalent to the statement that the trajectories of $\mathcal{K}$ form a congruence of *null geodesics without shear*, cf. p. 193 in vol. 2 of Penrose and Rindler. Such congruences play a fundamental role in the study of *algebraically special* gravitational fields; see e.g. Kramer *et al.*, Robinson and Trautman (1986 and 1988) and the numerous references given there.

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