The conformal geometry of complex quadrics and the fractional-linear form of Möbius transformations

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A new derivation is given of the Vahlen (1902) form of the local conformal transformations of $\mathbf{C}^n$, $v \mapsto (av+b)(cv+d)^{-1}$, where $v \in \mathbf{C}^n$ and a, b, c, d are suitable elements of the complex Clifford algebra $\mathrm{Cl}(n)$. The derivation is based on the homomorphism of groups $\mathrm{Spin}(n+2) \to \mathrm{SO}(n+2)$, the isomorphism of algebras $\mathrm{Cl}(n+2) \cong \mathbf{C}(2) \otimes \mathrm{Cl}(n)$, and the action of the Möbius group $\mathrm{SO}(n+2)$ on the quadric $\mathbf{Q}_n$, the conformal compactification of $\mathbf{C}^n$. It is shown how the conformal geometry of $\mathbf{Q}_n$ lifts, for every $n=1,2,\ldots$, to a unique conformal spin structure. The Hermite–Sylvester interpolation method is used to represent the map exp: $\mathrm{spin}(n) \to \mathrm{Spin}(n)$ in such a manner that $\exp a$ becomes a Clifford polynomial in $a \in \wedge^2 \mathbf{C}^n$.

I. INTRODUCTION

In 1910 Cunningham and Bateman[1] observed that Maxwell's equations without sources are invariant with respect to conformal transformations of Minkowski space. Somewhat later, Bessel–Hagen[2] derived the corresponding conservation laws. Ever since that time, invariance with respect to the conformal group—exact for particles of zero mass and approximate, in the limit of high energies,[3] otherwise—has attracted the attention of physicists. It has given rise to a wealth of new ideas and developments, especially in connection with twistors,[4] strings,[5] and two-dimensional conformal field theory.[6]

In geometry, the conformal group appeared already around 1850 in the work of Liouville[7] and Möbius.[8] In modern terminology and notation, their results can be summarized as follows: every global conformal transformation of the n-sphere $\mathbf{S}_n$ is induced by the action of an element of the group $\mathrm{O}(n+1,1)$ on null lines in the vector space $\mathbf{R}^{n+2}$ endowed with the quadratic form $x_1^2+\cdots+x_{n+1}^2-x_{n+2}^2$. Global conformal transformations of $\mathbf{S}_n$ form a Lie group; its connected component is the *Möbius group*. For $n>2$ every local conformal transformation of $\mathbf{S}_n$ extends to a global one, whereas $\mathbf{S}_2 \cong \mathbf{CP}_1 \cong \mathbf{C} \cup \{\infty\}$ admits local conformal transformations given by holomorphic functions. Among the latter, only the fractional-linear functions, i.e., those given by

$$z'=(az+b)/(cz+d), \quad \text{where} \quad a,b,c,d \in \mathbf{C} \quad \text{and} \quad ad-bc \neq 0 \tag{1.1}$$

extend to all $\mathbf{S}_2$. Since the coefficients a, b, c, d can be made to satisfy $ad-bc=1$ without changing the map (1.1), one has the exact sequence of group homomorphisms

$$1 \to \mathbf{Z}_2 \to \mathrm{SL}(2,\mathbf{C}) \to \mathrm{SO}_0(3,1) \to 1, \tag{1.2}$$

which exhibits $\mathrm{Spin}_0(3,1) \cong \mathrm{SL}(2,\mathbf{C})$ as the simply connected double cover of the Möbius group of $\mathbf{S}_2$. The action of the group $\mathrm{SL}(2,\mathbf{C})$ on $\mathbf{C} \cup \{\infty\}$ provides a realization of the Möbius

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group easier to handle than that of $SO_0(3,1)$ on $S_2 \subset R^3$. Explicit formulas for the action can be obtained by means of the stereographic projection $S_2 \to C \cup \{\infty\}$ given, in terms of the spherical coordinates (θ,φ), by $z = 2e^{i\varphi} \cot \frac{1}{2}\theta$, so that $d\theta^2 + \sin^2\theta\, d\varphi^2 = |dz|^2/(1+\frac{1}{4}|z|^2)^2$. The observation that the Lorentz group $SO_0(3,1)$ induces conformal transformations of the "celestial sphere" S_2 contributed to correcting[9] some early misconceptions about the visibility of the relativistic length contraction.[10]

Vahlen,[11] Ahlfors,[12] and, under their influence, several other authors,[13] noticed that the fractional-linear expression of the Möbius transformations can be generalized to higher dimensions by replacing the complex numbers a, b, c, d, and z by elements of a suitable Clifford algebra. Those authors did not, however, emphasize the role of the null lines acted upon by the conformal group.

In this article, we give a complete derivation of the fractional-linear form of the Möbius transformations by making use of the notions of Clifford algebras, Spin groups, and (projective) quadrics. Recall (see also Sec. III) that a complex, n-dimensional quadric Q_n is a manifold consisting of all null lines in the complex vector space C^{n+2}. The quadric Q_n has a natural conformal geometry, but no complex-bilinear Riemannian structure.[14] The complex orthogonal group $SO(n+2,C)$ acts transitively on Q_n and preserves its conformal geometry. The fractional-linear formula for this action is obtained from the homomorphism $\mathrm{Spin}(n+2,C) \to SO(n+2,C)$, by using the embedding $\mathrm{Spin}(n+2,C) \to \mathrm{Cl}(n+2) \cong C(2) \otimes \mathrm{Cl}(n)$, where $\mathrm{Cl}(n)$ is the Clifford algebra of C^n and $C(2)$ is the algebra of complex 2 by 2 matrices.

Complex quadrics play a fundamental role in (complex) conformal geometry: they are the holomorphic analogs of spheres.[15] In particular, the quadric Q_4 is the conformally compactified, complexified Minkowski space of twistor theory.[4,16]

The article is organized as follows: in Sec. II we give the necessary prerequisites on Clifford algebras[17] supplemented by a formulation of Hodge duality in the spirit of Kähler.[18] Section III contains an exposition of the elements of the conformal geometry of complex quadrics; it complements the results of Ref. 14. In Sec. IV we derive the exact sequences of homomorphisms connecting the conformal spin and orthogonal groups and extend the classical notion of spin structure on a manifold to the conformal case. Section V contains our new derivation of the fractional-linear form of the Möbius transformation. An easy by-product of our research, a description of the "conformal spin structure" on the complex quadrics, is given in Sec. VI. It complements the work[14] on proper spin structures on the real projective quadrics $(S_p \times S_q)/Z_2$. A practical method for computing the exponential map from the Lie algebra of $\mathrm{Spin}(n,C)$ to the group, based on Hermite–Sylvester interpolation, is given in Sec. VI.

II. CLIFFORD ALGEBRAS AND HODGE DUALITY

(1) Let V be an n-dimensional vector space over the field K of real or complex numbers. Assume V to be given a scalar product, i.e., a K bilinear, symmetric, and nondegenerate map $g: V \times V \to K$. The *Clifford algebra* $\mathrm{Cl}(g)$ is an associative algebra over K with unit element 1; it is generated by the elements of $K \oplus V \subset \mathrm{Cl}(g)$ subject to all the relations resulting from

$$v \cdot v = g(v,v), \quad v \in V. \tag{2.1}$$

The product of elements of $\mathrm{Cl}(g)$, the *Clifford product*, is denoted by a dot, as in Eq. (2.1). Most of the time, the dot is omitted; e.g., $v \cdot v$ is usually written as v^2. One shows that the Clifford algebra $\mathrm{Cl}(g)$ exists and is unique up to isomorphisms of the algebras. There holds the following *universal property*: if $\mathscr{A}$ is an algebra over K with unit element $1_{\mathscr{A}}$ and $f: V \to \mathscr{A}$ is a linear map with the Clifford property

$$f(v)^2 = g(v,v) 1_{\mathscr{A}}, \quad v \in V \tag{2.2}$$

then there exists a homomorphism of algebras with units, $F:\mathrm{Cl}(g)\to\mathscr{A}$, extending f. The linear map $V\to\mathrm{Cl}(g)$, $v\mapsto -v$, has the Clifford property and extends to the involutive *main automorphism* α_g of $\mathrm{Cl}(g)$. It defines the $\mathbf{Z}_2$-grading

$$\mathrm{Cl}(g)=\mathrm{Cl}_0(g)\oplus\mathrm{Cl}_1(g), \tag{2.3}$$

where

$$\mathrm{Cl}_\epsilon(g)=\{a\in\mathrm{Cl}(g):\alpha_g(a)=(-1)^\epsilon a\},\quad \epsilon=0,1. \tag{2.4}$$

Elements of $\mathrm{Cl}_0(g)$ [resp., $\mathrm{Cl}_1(g)$] are called even (resp., odd). The *main autiautomorphism* of $\mathrm{Cl}(g)$ is the linear isomorphism $\beta_g:\mathrm{Cl}(g)\to\mathrm{Cl}(g)$ characterized by

$$\beta_g(1)=1,\quad \beta_g(v)=v,\quad \text{and } \beta_g(ab)=\beta_g(b)\beta_g(a) \tag{2.5}$$

for every $v\in V$ and a, $b\in\mathrm{Cl}(g)$. We write α and β instead of α_g and β_g, respectively, whenever this cannot lead to ambiguities.

(2) Let $W=V\oplus K^2$ be given the scalar product h such that, if $w=(v,\lambda,\mu)\in W$, where $v\in V$ and $\lambda,\mu\in K$, then

$$h(w,w)=g(v,v)+\lambda\mu. \tag{2.6}$$

Denoting by $K(2)$ the algebra of 2 by 2 matrices with entries in K, one recognizes the tensor product $\mathscr{A}=K(2)\otimes\mathrm{Cl}(g)$ to be an algebra over K, consisting of all 2 by 2 matrices with entries in $\mathrm{Cl}(g)$. The linear map

$$W\to\mathscr{A}\quad\text{given by}\quad w=(v,\lambda,\mu)\mapsto\begin{pmatrix} v & \lambda\\ \mu & -v\end{pmatrix} \tag{2.7}$$

has the Clifford property

$$\begin{pmatrix} v & \lambda\\ \mu & -v\end{pmatrix}^2=h(w,w)\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \tag{2.8}$$

and extends to an isomorphism of algebras $\mathrm{Cl}(h)\to K(2)\otimes\mathrm{Cl}(g)$. Moreover, if a, b, c, $d\in\mathrm{Cl}(g)$ then

$$\alpha_h\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} \alpha_g(a) & -\alpha_g(b)\\ -\alpha_g(c) & \alpha_g(d)\end{pmatrix} \tag{2.9}$$

and

$$\beta_h\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} \bar d & \bar b\\ \bar c & \bar a\end{pmatrix}, \tag{2.10}$$

where $\bar a=\alpha_g\circ\beta_g(a)$. The latter notation is motivated by the following observation: if $K=\mathbf{R}$ and g is a negative-definite scalar product in $V=\mathbf{R}$ (resp., $V=\mathbf{R}^2$), then $\mathrm{Cl}(g)=\mathbf{C}$ [resp., $\mathrm{Cl}(g)=\mathbf{H}$, the algebra of quaternions] and $\bar a$ is the complex (resp., quaternion) conjugate of a.

(3) The Grassmann (exterior) algebra of multivectors

$$\wedge V=\bigoplus_{k=0}^{n}\wedge^k V,\quad \wedge^0 V=K \tag{2.11}$$

is $\mathbf{Z}$ graded: if $a\in\wedge^k V$ and $b\in\wedge^l V$, then $a\wedge b\in\wedge^{k+l}V$. The *interior product* of $a\in\wedge V$ by $v\in V$ is the multivector $v\lrcorner a$ characterized by the following properties: the map $a\mapsto v\lrcorner a$ is linear, if $a\in\wedge^k V$ and $b\in\wedge V$, then

$$v\lrcorner(a\wedge b)=(v\lrcorner a)\wedge b+(-1)^k a\wedge(v\lrcorner b),$$

$$\text{if } u\in V, \quad \text{then} \quad v\lrcorner u=g(u,v). \tag{2.12}$$

Consider the algebra $\mathscr{A}=\operatorname{End}\wedge V$ of all K-linear endomorphisms of the 2^n-dimensional vector space $\wedge V$. Its unit element is the identity endomorphism $\mathrm{id}_{\wedge V}$. The map

$$f:V\to\mathscr{A} \quad \text{given by} \quad f(v)a=v\wedge a+v\lrcorner a, \tag{2.13}$$

where $v\in V$ and $a\in\wedge V$, is linear and has the Clifford property, $f(v)^2=g(v,v)\mathrm{id}_{\wedge V}$. It extends to an injective homomorphism $F:\mathrm{Cl}(g)\to\mathscr{A}$ of algebras with units and defines the *isomorphism of vector spaces*

$$\iota:\mathrm{Cl}(g)\to\wedge V, \quad \text{where} \quad \iota(a)=F(a)1 \tag{2.14}$$

is the result of the evaluation of the endomorphism $F(a)$ on the unit 1 of the Grassmann algebra. As ι is natural, one can identify the vector spaces $\mathrm{Cl}(g)$ and $\wedge V$ and abuse the notation by omitting to write ι altogether. Since F is a homomorphism extending f, one has $F(va)=f(v)F(a)$ and Eq. (2.13) gives

$$v\cdot a=v\wedge a+v\lrcorner a \tag{2.15}$$

for $v\in V$ and $a\in\mathrm{Cl}(g)\cong\wedge V$.

Let a_k denote the component of $a\in\wedge V$ belonging to $\wedge^k V$, so that $a=a_k$ is equivalent to $a\in\wedge^k V$. By a repeated application of Eq. (2.15) one obtains

$$(a_k\cdot b_l)_m=0 \quad \text{for } m\begin{cases}<|k-l|, & \text{or}\\ \equiv k-l+1 \bmod 2, & \text{or}\\ >n-|n-k-l|.\end{cases} \tag{2.16}$$

Since $\beta(a_k)=(-1)^{k(k-1)/2}a_k$ one has

$$(a_k\, b_l)_m=(-1)^{kl+(1/2)(k+l-m)}(b_l\, a_k)_m. \tag{2.17}$$

It is worth noting that

$$(a_k\, b_l)_{k+l}=a_k\wedge b_l \tag{2.18}$$

and

$$(a\beta(b))_0=(a|b) \tag{2.19}$$

is the scalar product of a and $b\in\wedge V$. If $a=a_k$, $b=b_l$, and $k\neq l$, then $(a|b)=0$ follows from Eq. (2.16). If $K=\mathbf{R}$ and g is positive definite, then Eq. (2.19) defines a positive-definite extension of g to $\wedge V$ and, therefore, to $\mathrm{Cl}(g)$. If g is not definite, then this extension is neutral, i.e., $\wedge V$ contains totally null subspaces of the maximal dimension 2^{n-1}.

(4) Let (e_μ), $\mu=1,\dots,n$ be a frame (linear basis) in V, orthonormal with respect to g. We choose the vectors of the frame as follows. Let $g_{\mu\nu}=g(e_\mu,e_\nu)$, $\mu,\nu=1,\dots,n$. For $K=\mathbf{C}$ we take $g_{\mu\nu}=\delta_{\mu\nu}$ and for $K=\mathbf{R}$ and signature (p,q), we have $g_{\mu\nu}=0$ for $\mu\neq\nu$ and $g_{\mu\mu}=1$ for $\mu=1,\dots,p$ and $g_{\mu\mu}=-1$ for $\mu=p+1,\dots,p+q=n$. The volume element

$$\eta=e_1\cdots e_n\in\wedge^n V \tag{2.20}$$

satisfies

$$\eta^2=\begin{cases}(-1)^{n(n-1)/2}, & \text{for } K=\mathbf{C}\\ (-1)^{(p-q)(p-q-1)/2}, & \text{for signature } (p,q).\end{cases} \tag{2.21}$$

According to Kähler,[18] the *Hodge dual* of $a\in\wedge V$ is the multivector

$$*a=a\eta. \tag{2.22}$$

If $a=a_k$, then $*a=(*a)_{n-k}$ and $**a=\eta^2 a$ is given by Eq. (2.21), irrespective of the value of k.

Let $v\in V$ and $a\in\wedge V$. The Clifford multiplication being associative, $(va)\eta=v(a\eta)$, formula (2.15) yields

$$*(v\wedge a)=v\lrcorner *a. \tag{2.23}$$

Similarly

$$(a\cdot *b)_k=*(a\cdot b)_{n-k} \tag{2.24}$$

and, in particular, if a and b are of the same degree, then

$$(a|b)\eta=a\wedge *\beta(b). \tag{2.25}$$

For every a, b, $\in\wedge V$ one puts

$$2ab=[a,b]+\{a,b\},\quad\text{where}\quad [a,b]=ab-ba. \tag{2.26}$$

The bracket [,] (resp., {,}) makes $\mathbf{Cl}(g)$ into a Lie (resp., Jordan) algebra.

If k is *even*, then the product in $\wedge^k V$ given by

$$(a,b)\mapsto(ab)_k,\quad\text{where}\quad a,b\in\wedge^k V \tag{2.27}$$

makes $\wedge^k V$ into an algebra which is either commutative (when $k\equiv 0 \bmod 4$) or anticommutative (when $k\equiv 2 \bmod 4$). In particular, for $k=2$, one has

$$ab=-(a|b)+\tfrac{1}{2}[a,b]+a\wedge b,\quad\text{where}\quad a,b\in\wedge^2 V \tag{2.28}$$

and the commutator $[a,b]$ is a bivector, $[a,b]=2(ab)_2$. The bracket [,] makes $\wedge^2 V$ into the Lie algebra of the group $\mathrm{Spin}(g)$. If $n=2k\equiv 0 \bmod 4$ and $\eta^2=1$, then there is the decomposition

$$\wedge^k V=\wedge^k_+ V\oplus\wedge^k_- V \tag{2.29}$$

of k-vectors into self-dual and antiself-dual parts. Since, in this case, Eq. (2.24) gives $(a\cdot *b)_k=*(ab)_k$ for a, $b\in\wedge^k V$, the product (2.27) induces in each of the subspaces $\wedge^k_+ V$ and $\wedge^k_- V$ the structure of an algebra. In particular, for $n=4$ and $k=2$ one has

$$*[a,b]=[a,*b] \tag{2.30}$$

and Eq. (2.29) reduces to the well-known decomposition $\mathrm{spin}(4)=\mathrm{spin}(3)\oplus\mathrm{spin}(3)$. The next lowest dimension, where there is a nontrivial product (2.27), is $n=8$. The commutative algebras $\wedge^4_\pm K^8$ are each 35-dimensional.

Incidentally, Hodge duality is useful—though not essential—in proving the least obvious (i.e., the third) implication among those appearing in Eq. (2.16). Clearly, if $m>k+l$, then $(a_k\cdot b_l)_m=0$. The elements $a_k\cdot b_l$ and $*a_k\cdot *b_l$ differ at most by a sign. Since $*a_k$ and $*b_l$ are multivectors of degree $n-k$ and $n-l$, respectively, the mth component of their Clifford product vanishes for $m>2n-k-l$; if $k+l>n$, then this inequality coincides with the last one in Eq. (2.16).

III. THE CONFORMAL GEOMETRY OF COMPLEX QUADRICS

(1) Let W be an $(n+2)$-dimensional complex vector space with a scalar product h. If $w\in W'=W\setminus\{0\}$, then

$$[w]=\{\lambda w\in W:\lambda\in\mathbf{C}\} \tag{3.1}$$

is the *line* through w. The line is said to be *null*, if w is a null vector, $h(w,w)=0$. The set of all null lines, the *quadric*

$$Q=\{[w]:w\in W' \text{ and } h(w,w)=0\}, \tag{3.2}$$

is a compact, complex, n-dimensional manifold.

By considering a null curve $u:\mathbf{R}\to W'$ and differentiating both sides of the equation $h(u(t),u(t))=0$ with respect to t, one obtains the following description of the tangent bundle of Q: for $(w,u)\in W'\times W$ let $[(w,u)]$ be the equivalence class characterized by

$$[(w,u)]=[(w',u')], \quad \text{iff there are } \lambda,\mu\in\mathbf{C},\ \lambda\neq 0, \text{ such that } w'=\lambda w \text{ and } u'=\lambda u+\mu w. \tag{3.3}$$

The total space TQ of the tangent bundle is then

$$TQ=\{[(w,u)]:[w]\in Q \text{ and } h(w,u)=0\}. \tag{3.4}$$

The scalar product h induces in Q a *conformal geometry*: the *null cone* at $[w]\in Q$ is the set

$$N_{[w]}=\{[(w,u)]\in T_{[w]}Q:h(u,u)=0\}. \tag{3.5}$$

A simple topological argument[14] shows that Q does not admit any complex-bilinear Riemannian metric. It admits, however, a Hermitean (even Kähler) metric induced by a Hermitean positive form on W.

The special orthogonal group $\mathrm{SO}(h)=\{A\in\mathrm{GL}(W):\det A=1 \text{ and } h(Aw,Aw)=h(w,w) \text{ for every } w\in W\}$, considered as a group of transformations of Q, is called the *Möbius group*. If $w\neq 0$ is a null vector and $A\in\mathrm{SO}(h)$, then $A[w]=[Aw]$. The action of the Möbius group on Q is transitive and preserves the distribution of null cones: if $[(w,u)]\in N_{[w]}$, then $A[(w,u)]=[(Aw,Au)]\in N_{[Aw]}$. In other words, the Möbius group consists of conformal transformations of Q and, in fact, it is the connected component of the group of all conformal automorphisms of Q.[19]

(2) Assume now W and h to be as in Sec. II(2) with $K=\mathbf{C}$ and $\dim V=n>1$. If

$$i{:}V\to W' \quad \text{is given by} \quad i(v)=(v,-g(v,v),1) \tag{3.6}$$

then the map

$$j{:}V\to Q, \quad \text{where} \quad j(v)=[i(v)] \tag{3.7}$$

is injective, the image $j(V)$ is open and dense in Q, and the complement of the image is the compact $(n-1)$-dimensional "null cone at infinity" consisting of all elements of Q of the form $[(v, \lambda, 0)]$ where v is null, $g(v,v)=0$.

For every $v\in V$ the linear map

$$j_v{:}V\to i(v)^\perp, \quad j_v(u)=(u,-2g(u,v),0) \tag{3.8}$$

of V into the subspace $i(v)^\perp$ of W, orthogonal to the vector $i(v)$, is an (injective but not surjective) isometry

$$h(j_v(u),j_v(u))=g(u,u), u\in V. \tag{3.9}$$

Using the identification $TV\cong V\times V$, one finds the map $j_*{:}TV\to TQ$, tangent to j, given by Eq. (3.7), to be such that

$$j_*(v,u)=[(i(v), j_v(u))]. \tag{3.10}$$

By virtue of Eq. (3.9) the map j is conformal: if $u\in V$ is null, then $j_*(v,u)\in N_{[i(v)]}$. Since V is conformally flat, so is Q.

(3) The action of the Möbius group on Q induces local conformal transformations of V, defined as follows. For every $v\in V$ and $A\in \mathrm{SO}(h)$ one has

$$Aj(v)=[Ai(v)], \quad \text{where} \quad j(v)=[i(v)]. \tag{3.11}$$

In general, $Aj(v)$ does not belong to $j(V)$. Let

$$V(A)=\{v\in V{:}Aj(v)\in j(V)\}. \tag{3.12}$$

The set $V(A)$ is open and dense in V. The map

$$\widetilde{A}{:}V(A)\to V(A^{-1}), \quad \text{defined by} \quad j(\widetilde{A}v)=Aj(v), \quad v\in V(A) \tag{3.13}$$

is a composition of conformal transformations; therefore, it is a conformal diffeomorphism of $V(A)$ onto $V(A^{-1})$. We refer to $\widetilde{A}$ as the *local Möbius transformation* of V, defined by the global Möbius transformation A of Q.

(4) From now on, throughout the article, we assume $K=\mathbf{C}$ and use a *notation* emphasizing the dimension of the underlying complex vector space V. Thus $\mathrm{Cl}(n)$ is the Clifford algebra of $V=\mathbf{C}^n$ with the standard scalar product and $\mathrm{GL}(n)$ [resp. $\mathrm{O}(n)$] denotes the *complex* general linear (resp., orthogonal) group. A similar notation will apply to the quadrics and spin groups; it is at variance with the traditional notation, used in the Introduction, where the letter $\mathbf{C}$ appears explicitly in the context of complex orthogonal and spin groups.

IV. CPIN GROUPS AND STRUCTURES

(1) Consider the space $V=\mathbf{C}^n$ with its standard scalar product g. The complex conformal group $\mathrm{CO}(n)$ is the set of all $A\in \mathrm{GL}(n)$ for which there is $\lambda(A)\in \mathrm{GL}(1)=\mathbf{C}'$ such that

$$g(Av,Av)=\lambda(A)g(v,v), \quad \text{for every} \quad v\in V. \tag{4.1}$$

If $A \in \mathrm{O}(n)$ and $\mu \in \mathbf{C}'$, then $\mu A \in \mathrm{CO}(n)$ and there is the exact sequence of group homomorphisms

$$1 \to \mathbf{Z}_2 \to \mathbf{C}' \times \mathrm{O}(n) \to \mathrm{CO}(n) \to 1. \tag{4.2}$$

Noting that $I = \mathrm{id}_V$ can always be connected to $-I$ by a curve in $\mathrm{CO}(n)$ and that $-I \in \mathrm{SO}(n)$ if, and only if, n is even, one sees that

(i) for $n = 2m+1$ odd, the group $\mathrm{CO}(2m+1)$ is connected and isomorphic to $\mathbf{C}' \times \mathrm{SO}(2m+1)$, and

(ii) for $n = 2m$ even, there is the exact sequence

$$1 \to \mathbf{Z}_2 \to \mathbf{C}' \times \mathrm{SO}(2m) \to \mathrm{CO}_0(2m) \to 1, \tag{4.3}$$

where $\mathrm{CO}_0(2m)$ is the connected component of $\mathrm{CO}(2m)$.

(2) Since, for every $v \in V \subset \mathrm{Cl}(n)$, one has $g(v,v) = v^2$ (the Clifford square), from now on we always write v^2 instead of $g(v,v)$. The *Clifford group,*[20] or the *conformal spin group,* associated with V, is the set $\mathrm{Cpin}(n)$ of products of all finite sequences of non-null vectors, $\mathrm{Cpin}(n) = \{u_1 \cdots u_k \in \mathrm{Cl}(n) : u_i \in V,\ u_i^2 \neq 0 \text{ for } i = 1,...,k \text{ and } k = 1,2,...,\}$, with a multiplication induced by the Clifford product. If $s \in \mathrm{Cpin}(n)$, then $s\bar{s} = \bar{s}s \in \mathbf{C}'$; recall [Sec. II(2)] that the map $s \to \bar{s} = \alpha \circ \beta(s)$ is $\mathbf{C}$ *linear* and $u_1 \cdots u_k u_1 \cdots u_k = (-1)^k u_1^2 \cdots u_k^2$. If $u \in V$ is non-null, then the map $v \mapsto -uvu^{-1} (v \in V)$ is a reflection in the hyperplane orthogonal to u. For every $s \in \mathrm{Cpin}(n)$, define

$$\rho(s) : V \to V \quad \text{by} \quad \rho(s)v = sv\bar{s} \tag{4.4}$$

so that $\rho(s)$ is linear and $(\rho(s)v)^2 = (\bar{s}s)^2 v^2$, i.e., $\rho(s) \in \mathrm{CO}(n)$ with $\lambda(\rho(s)) = (\bar{s}s)^2$.

By the Cartan–Dieudonné Theorem,[21] the homomorphism $\rho : \mathrm{Cpin}(n) \to \mathrm{CO}(n)$ is surjective. To compute its kernel, note that $\rho(s)v = v$ for every $v \in V$ implies $\bar{s}s = \pm 1$ so that either $sv = vs$ or $sv = -vs$. The volume element η commutes (resp., anticommutes) with all vectors for n odd (resp., even). Since the volume element (2.20) is normalized so that $\beta(\eta)\eta = 1$, one has $\rho(\eta) = -I$ and obtains $\ker \rho = \{\pm 1, \pm \sqrt{-1}\eta\}$. Therefore, the kernel is isomorphic to $\mathbf{Z}_4$ for $n \equiv 0$ or 1 mod 4 and $\mathbf{Z}_2 \times \mathbf{Z}_2$ for $n \equiv 2$ or 3 mod 4.

For every n, the group $\mathrm{Cpin}(n)$ is the disjoint union of two nonempty subsets

$$\mathrm{Cpin}_\epsilon(n) = \mathrm{Cpin}(n) \cap \mathrm{Cl}_\epsilon(n), \quad \epsilon = 0 \quad \text{and} \quad 1, \tag{4.5}$$

which are both open and closed in $\mathrm{Cpin}(n)$: the group is not connected and $\mathrm{Cpin}_0(n)$ is the connected component containing 1. The latter group gives rise to the exact sequences

$$1 \to \mathbf{Z}_2 \to \mathrm{Cpin}_0(2m+1) \xrightarrow{\rho} \mathrm{CO}_0(2m+1) = \mathrm{CO}(2m+1) \to 1, \tag{4.6}$$

$$1 \to \mathbf{Z}_4 \to \mathrm{Cpin}_0(4m) \xrightarrow{\rho} \mathrm{CO}_0(4m) \to 1, \tag{4.7}$$

$$1 \to \mathbf{Z}_2 \times \mathbf{Z}_2 \to \mathrm{Cpin}(4m+2) \xrightarrow{\rho} \mathrm{CO}_0(4m+2) \to 1. \tag{4.8}$$

(3) One defines

$$\mathrm{Pin}(n) = \{s \in \mathrm{Cpin}(n) : \beta(s)s = 1\} \tag{4.9}$$

and, by restricting ρ to $\mathrm{Pin}(n)$, obtains the exact sequence

$$1 \to \mathbf{Z}_2 \to \mathrm{Pin}(n) \xrightarrow{\rho} \mathrm{O}(n) \to 1, \tag{4.10}$$

where now $\rho(s)v=sv\bar{s}=sv\alpha(s^{-1})$ and $\ker \rho|_{\mathrm{Pin}(n)} = \{\pm 1\} = \mathbf{Z}_2$. Similarly, the connected component of Pin(n)

$$\mathrm{Spin}(n)=\mathrm{Pin}(n)\cap \mathbf{Cl}_0(n) \tag{4.11}$$

gives rise to the exact sequence

$$1\to \mathbf{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \to 1. \tag{4.12}$$

It is well-known that Spin(2)=GL(1), Spin(3)=SL(2), Spin(4)=SL$(2)\times$SL(2), Spin(5) =Sp(4), and Spin(6)=SL(4).

There are also the obvious exact sequences

$$1\to \mathbf{Z}_2 \to \mathbf{C}'\times \mathrm{Pin}(n) \to \mathrm{Cpin}(n) \to 1 \tag{4.13}$$

and

$$1\to \mathbf{Z}_2 \to \mathbf{C}'\times \mathrm{Spin}(n) \to \mathrm{Cpin}_0(n) \to 1. \tag{4.14}$$

(4) Let Γ and $\tilde{\Gamma}$ be one of the groups

$$\mathrm{CO}(n), \mathrm{CO}_0(n), \mathrm{O}(n), \mathrm{SO}(n)$$

and

$$\mathrm{Cpin}(n), \mathrm{Cpin}_0(n), \mathrm{Pin}(n), \mathrm{Spin}(n),$$

respectively, so that, in every case, there is the epimorphism

$$\rho:\tilde{\Gamma}\to\Gamma, \quad \rho(s)v=sv\bar{s}, \quad \text{where} \quad s\in\tilde{\Gamma} \quad \text{and} \quad v\in V. \tag{4.15}$$

Let $\pi:P\to X$ be a Γ-structure on a complex, n-dimensional manifold X. In other words, P is a restriction of the bundle of complex linear frames of X to the subgroup Γ of GL(n). For example, a conformal geometry on X is equivalent to a CO(n)-structure on X. Recall that a frame $p\in P$ at $x\in X$ is a linear isomorphism $p:V\to T_xX$; if $A\in\Gamma$, then the composition $p\circ A\in P$ is another frame at x. A *$\tilde{\Gamma}$-structure* on X is a prolongation of the Γ-bundle $P\to X$ to $\tilde{\Gamma}$, i.e., a principal $\tilde{\Gamma}$-bundle $\tilde{\pi}:\tilde{P}\to X$ such that there is a surjective map $\sigma:\tilde{P}\to P$, $\tilde{\pi}=\pi\circ\sigma$ and $\sigma(\tilde{p}a)=\sigma(\tilde{p})\rho(a)$ for every $\tilde{p}\in\tilde{P}$ and $a\in\tilde{\Gamma}$. This notion generalizes the classical one of the spin structure. If $\tilde{\Gamma}$=Cpin(n) or Cpin$_0(n)$, then one uses the expression "conformal spin structure."

(5) Let X be a manifold with a Γ-structure $\pi:P\to X$ and consider a connected Lie group G, acting transitively on X, and preserving this structure. Let $H\subset G$ be the isotropy (stability) group of a point $x\in X$, so that $X\cong G/H$, and let

$$\tau:H\to\Sigma \tag{4.16}$$

be the *linear isotropy representation* of H, determined by the tangent action of H in T_xX: given a fixed frame p at x, one puts

$$\tau(t)=p^{-1}\circ t_*\circ p, \tag{4.17}$$

where t_* is the map tangent to the map $t:X\to X$, $t\in H$. The set $G\times_\tau\Gamma$ of equivalence classes of the form $[(s,A)]_\tau$, where $(s,A)\in G\times\Gamma$, and

$$[(s,A)]_\tau=[(s',A')]_\tau \quad \text{iff there is} \quad t\in H \quad \text{such that} \quad s'=st \quad \text{and} \quad A=\tau(t)A', \tag{4.18}$$

can be identified with P: one sends $[(s,A)]_\tau$ to the frame $s_*\circ p\circ A$ and checks that this is a well-defined map, equivariant with respect to the action of Γ.

A homomorphism

$$\tilde{\tau}:H\to\tilde{\Gamma} \tag{4.19}$$

is said to be a *lift* of τ to $\tilde{\Gamma}$ if $\rho\circ\tilde{\tau}=\tau$. Given such a lift, one constructs the set

$$\tilde{P}=G\times_{\tilde{\tau}}\tilde{\Gamma}, \tag{4.20}$$

makes it into a manifold, defines $\sigma:\tilde{P}\to P$ by $\sigma([(s,a)]_{\tilde{\tau}})=[(s,\rho(a))]_\tau$, and the action of $\tilde{\Gamma}$ in $\tilde{P}$ by $[(s,a)]_{\tilde{\tau}}b=[(s,ab)]_{\tilde{\tau}}$, where $s\in G$ and $a,\ b\in\Gamma$. Clearly, the principal $\tilde{\Gamma}$-bundle $\tilde{\pi}=\pi\circ\sigma:\tilde{P}\to X$ is a $\tilde{\Gamma}$-structure on X. Conversely, an argument given in Ref. 14 shows that, if G is *simply connected*, then every $\tilde{\Gamma}$-structure on X can be obtained in this manner.

V. THE FRACTIONAL-LINEAR FORM OF LOCAL MÖBIUS TRANSFORMATIONS

(1) Consider the complex quadric $\mathbf{Q}_n$ consisting of all null lines in the $(n+2)$-dimensional vector space $W=V\oplus\mathbf{C}^2$, with the scalar product h, cf. Secs. II(2) and III(1). The homomorphism

$$\rho:\mathrm{Spin}(n+2)\to\mathrm{SO}(n+2),\quad \rho(s)w=sws^{-1}, \tag{5.1}$$

where $s\in\mathrm{Spin}(n+2)$ and $w=(v,\lambda,\mu)\in W$, defines a transitive action of $\mathrm{Spin}(n+2)$ on Q_n: the map

$$\mathbf{Q}_n\ni[w]\mapsto\rho(s)[w]=[sws^{-1}]\in\mathbf{Q}_n \tag{5.2}$$

is the Möbius transformation and, since ρ is surjective, all Möbius transformations of $\mathbf{Q}_n$ can be so represented.

(2) According to Sec. II(2), there is the isomorphism of algebras with units, $\mathrm{Cl}(n+2)\cong\mathbf{C}(2)\oplus\mathrm{Cl}(n)$, resulting from the map (2.7). If $w=i(v)$ is the null vector given by Eq. (3.6), then its image in $\mathbf{C}(2)\oplus\mathrm{Cl}(n)$ is

$$\begin{pmatrix} v & v\bar{v} \\ 1 & \bar{v}\end{pmatrix}=\begin{pmatrix} v \\ 1\end{pmatrix}(1,\bar{v}). \tag{5.3}$$

An element

$$s=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \tag{5.4}$$

of $\mathrm{Cl}(n+2)$ belongs to $\mathrm{Spin}(n+2)$ if, and only if, the following conditions are satisfied:

(i) s is even; according to Eq. (2.9) this is equivalent to

$$\text{as elements of } \mathrm{Cl}(n),\ a \text{ and } d \text{ are even and } b \text{ and } c \text{ are odd}, \tag{5.5}$$

(ii) $\beta_h(s)s=1$; according to Eq. (2.10) this is equivalent to

$$\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}; \tag{5.6}$$

(iii) for every $w \in W$ there is $w' \in W$ such that

$$sws^{-1} = w'. \tag{5.7}$$

The normalization condition (5.6) can be written as

$$\bar{d}a + \bar{b}c = 1 \tag{5.8}$$

and

$$\bar{d}b + \bar{b}d = \bar{c}a + \bar{a}c = 0, \tag{5.9}$$

whereas Eq. (5.7) is equivalent to

$$a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbf{C}, \quad b\bar{d}, a\bar{c} \in V, \tag{5.10}$$

$$av\bar{b} - bv\bar{a}, cv\bar{d} - dv\bar{c} \in \mathbf{C} \quad \text{and} \quad av\bar{d} - bv\bar{c} \in V \tag{5.11}$$

for every $v \in V = \mathbf{C}^n$. It is clear that the relations (5.8)–(5.11) are also satisfied when

$$a,\ b,\ c, \text{ and } \ d \text{ are replaced by } \bar{d},\ \bar{b},\ \bar{c}, \ \text{ and } \ \bar{a}\ , \tag{5.12}$$

respectively.

(3) Referring to Sec. III(3), we can now determine the local Möbius transformation $\widetilde{\rho(s)}$, $s \in \mathrm{Spin}(n+2)$. We take $v \in V(\rho(s))$, so that $[si(v)s^{-1}] = [i(v')] \in j(V)$, replace w and w' in Eq. (5.7) by $i(v)$ and $\mu i(v')$, respectively, and, by virtue of Eq. (5.3), obtain

$$\begin{pmatrix} av+b \\ cv+d \end{pmatrix} (\overline{cv+d}, \overline{av+b}) = \mu \begin{pmatrix} v' \\ 1 \end{pmatrix} (1, \bar{v}'). \tag{5.13}$$

Since $i(v) \neq 0$ implies $\mu \neq 0$, we see that $(\overline{cv+d})/\mu$ is the inverse of $cv+d$ and the *fractional-linear formula for the local Möbius transformation* $\widetilde{\rho(s)}$

$$v' = \widetilde{\rho(s)} v = (av+b)(cv+d)^{-1} \tag{5.14}$$

holds for every

$$v \in V(\rho(s)) = \{v \in V : cv+d \text{ is invertible}\}. \tag{5.15}$$

To describe explicitly the domain $V(\rho(s))$ of the definition of the local Möbius transformation (5.14) we note that, by virtue of Eqs. (5.10) and (5.12), the element $\bar{d}c$ is a vector and

$$(\overline{cv+d})(cv+d) = \bar{d}d + 2g(\bar{d}c, v) - \bar{c}cv^2 \in \mathbf{C} \tag{5.16}$$

holds for every $v \in V$. There are three cases to consider:

(i) In the generic case $\bar{c}c \neq 0$ and $V(\rho(s))$ equals V with the *null cone* of vertex at $\bar{d}c/\bar{c}c$ removed.

(ii) If $\bar{c}c = 0$ and $\bar{d}c \neq 0$, then the vector $\bar{d}c$ is null and $V(\rho(s))$ consists of V minus the *null hyperplane* of the equation $2g(\bar{d}c, v) + \bar{d}d = 0$.

(iii) If $\bar{c}c = 0$ and $\bar{d}c = 0$, then $c = 0$, $\bar{d}d \neq 0$ and $V(\rho(s)) = V$. In this case s equals

$$\begin{pmatrix} a & au \\ 0 & \bar{a}^{-1} \end{pmatrix}, \quad \text{where} \quad a \in \mathrm{Cpin}_0(n), \quad u \in V \tag{5.17}$$

and $\widetilde{\rho(s)}$ is a global affine transformation of V, consisting of a translation by u, followed by a composition of a rotation and a dilation.

VI. THE CONFORMAL SPIN STRUCTURE ON A COMPLEX QUADRIC

The group $G=\mathrm{Spin}(n+2)$ is simply connected for $n\geqslant 1$ and acts transitively on the complex quadric $X=\mathbf{Q}_n$. We can, therefore, use the method outlined in Sec. IV(5) to determine the conformal spin structure on the quadric.

(1) The isotropy group H of the "point at infinity" $[w_\infty]$, where $w_\infty=(0,1,0)\in W$ is

$$H=\{t\in G{:}tw_\infty t^{-1}=\lambda w_\infty,\ \lambda\in\mathbf{C}'\}. \tag{6.1}$$

Using Eqs. (5.4) in (6.1) and taking Eqs. (5.8)–(5.11) into account, one obtains

$$a=\bar{d}^{-1}\in\mathrm{Cpin}_0(n)\quad\text{and}\quad\lambda=a\bar{a}, \tag{6.2}$$

$$b=au,\quad\text{where}\quad u\in V=\mathbf{C}^n,\quad\text{and}\quad c=0 \tag{6.3}$$

so that every element of H is of the form (5.17), i.e.,

$$t=\begin{pmatrix} a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix} \tag{6.4}$$

and H is isomorphic to the semidirect product of groups, $H\cong\mathbf{C}^n\times\mathrm{Cpin}_0(n)$.

According to the Cartan–Dieudonné Theorem, the first factor in Eq. (6.4) is the product of an even sequence of no more than n non-null elements of V. If u is non-null, then the second factor can be written as a product of two vectors

$$\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}=\begin{pmatrix} u & 0\\ 0 & -u\end{pmatrix}\begin{pmatrix} u/u^2 & 1\\ 0 & -u/u^2\end{pmatrix}.$$

If $u\neq 0$ is null, then four vectors are necessary.

$$\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}=\begin{pmatrix} u & \frac{1}{4}\\ 4 & -u\end{pmatrix}\begin{pmatrix} u & -\frac{1}{4}\\ 4 & -u\end{pmatrix}\begin{pmatrix} u & -\frac{1}{4}\\ -4 & -u\end{pmatrix}\begin{pmatrix} u & \frac{1}{4}\\ -4 & -u\end{pmatrix}.$$

(2) The liner isotropy representation (4.16) is obtained as follows. Consider the frame (linear isomorphism)

$$p{:}V\to T_{[w_\infty]}Q\quad\text{given by}\quad p(v)=[(w_\infty,(v,0,0))]. \tag{6.5}$$

According to the definition (4.17)

$$\tau(t)v=p^{-1}\circ t_*\circ p(v)=p^{-1}[(tw_\infty t^{-1},\,t(v,0,0)t^{-1})].$$

If t is given by Eq. (6.4), then $tw_\infty t^{-1}=\lambda w_\infty$, where $\lambda=a\bar{a}$, and

$$t(v,0,0)t^{-1}=(ava^{-1},-2\lambda g(u,v),0)=(ava^{-1},0,0)\bmod w_\infty.$$

Therefore

$$\tau(t)v=\lambda^{-1}ava^{-1}\quad\text{and}\quad\Sigma=\mathrm{CO}_0(n). \tag{6.6}$$

The lift of τ to $\tilde{\Sigma}=\mathrm{Cpin}_0(n)$ given by

$$\tilde{\tau}(t)=\lambda^{-1}a, \quad \text{where} \quad t\in H \quad \text{and} \quad \lambda=\bar{a}a \tag{6.7}$$

is unique because H is connected. Therefore, for every $n>1$, *the quadric* $\mathbf{Q}_n$ *has a unique conformal spin structure.* This is in contrast with the result[22] on the proper spin structure associated with the Kähler metric on $\mathbf{Q}_n$: the latter structure exists only for $n=1$ or n even.

(3) In Riemannian geometry, if G is a group of isometries acting on a manifold transitively and *effectively*, then the linear isotropy representation $\tau:H\to \mathrm{O}(n)$ is injective and G is the total space of a restriction of the bundle of linear frames to H.[14] In our case the situation is rather different: there is a nontrivial kernel $N=\ker\tau\subset H$; its connected component is $N_0=\ker\tilde{\tau}\cong\mathbf{C}^n$ and $H/N_0\cong\mathrm{Cpin}_0(n)$. The bundle $G/N_0=\mathrm{Spin}(n+2)/\mathbf{C}^n\to\mathbf{Q}_n$ is isomorphic to the $\mathrm{Cpin}_0(n)$-bundle $\tilde{P}\to\mathbf{Q}_n$ defining the conformal spin structure and $P\cong G/N$.

VII. HERMITE INTERPOLATION AND THE EXPONENTIAL MAP

The well-known formula

$$\exp \mathbf{v}\sigma=\cosh\lambda+\lambda^{-1}\mathbf{v}\sigma\sinh\lambda, \tag{7.1}$$

where λ is a square root of $\mathbf{v}^2$, $\mathbf{v}\in\mathbf{C}^3$, and the sigmas are Pauli matrices, provides a convenient parametrization of the group $\mathrm{Spin}(3)=\mathrm{SL}(2)$. In this section, we present a method for generalizing (7.1) to arbitrary Spin groups.

(1) Let $\mathscr{A}$ be a complex, N-dimensional, associative algebra with unit element. As such, it has a natural topology and a faithful and continuous representation γ in the algebra $\mathbf{C}(n)$ of complex, n by n matrices, where $n\leqslant N$. An entire analytic function $f:\mathbf{C}\to\mathbf{C}$ extends, in an obvious manner, to a function $f:\mathbf{C}(n)\to\mathbf{C}(n)$. Since $\gamma(\mathscr{A})$ is a closed subspace of $\mathbf{C}(n)$, for every $a\in\mathscr{A}$, the element $f(\gamma(a))\in\mathbf{C}(n)$ belongs to $\gamma(\mathscr{A})$ and defines an element of $\mathscr{A}$. To put it shortly, every entire analytic function $f:\mathbf{C}\to\mathbf{C}$, such as a polynomial or exp, extends to a map $f:\mathscr{A}\to\mathscr{A}$. For every $a\in\mathscr{A}$ the set $\{1,a,...,a^N\}$ is linearly dependent: there thus exists a polynomial p of degree M, where $1\leqslant M\leqslant N$, such that $p(a)=0$. Therefore, every positive power of a can be represented as a linear combination of the elements $1,a,...,a^{M-1}$. If f is entire, then there are complex numbers φ_0, $\varphi_1,...,\varphi_{M-1}$ such that

$$f(a)=\varphi_0+\varphi_1 a+\cdots+\varphi_{M-1}a^{M-1}. \tag{7.2}$$

(2) A method for computing the coefficients appearing in Eq. (7.2) is based on *Hermite* (or Lagrange–Sylvester) *interpolation.*[23] Recall that, given a sequence $(z_1,...,z_m)$ of m distinct complex numbers, one associates with a complex function f the Lagrange interpolation polynomial

$$\sum_{i=1}^{m} f(z_i)p_i(z)/p_i(z_i),$$

where $p_i(z)=p(z)/(z-z_i)$ and $p(z)=(z-z_1)\cdots(z-z_m)$.

The Taylor polynomial of degree l at 0

$$f_l(z)=\sum_{k=0}^{l} f^{(k)}(0)z^k/k! \tag{7.3}$$

gives an approximation of differential order l to a function f smooth in a neighborhood of 0. Hermite and Sylvester found a generalization of these two approximation methods.

(3) Let $(z_1,...,z_m)$ and $(l_1,...,l_m)$ be sequences of distinct complex numbers and of non-negative integers, respectively. Such a pair of sequences defines, and is defined by, the normalized polynomial

$$p(z)=(z-z_1)^{l_1+1}\cdots(z-z_m)^{l_m+1} \tag{7.4}$$

of degree $M=l_1+\cdots+l_m+m$. Put

$$p_i(z)=p(z)/(z-z_i)^{l_i+1} \tag{7.5}$$

so that $p_i(z_i)\neq 0$ and, if $i\neq j$, then

$$p_i^{(l)}(z_j)=0 \quad \text{for } l=1,...,l_j \quad \text{and } i,j=1,...,m. \tag{7.6}$$

The function q_{ik} $(k=1,...,l_i, i=l,...,m)$, defined by

$$p_i(z)q_{ik}(z)=(z-z_i)^k \tag{7.7}$$

is rational and has a zero of order k at z_i, but is not a polynomial unless $m=1$. If this function is replaced on the left side of Eq. (7.7) by its Taylor polynomial s_{ik} of degree l_i at z_i

$$s_{ik}(z)=\sum_{r=k}^{l_i} q_{ik}^{(r)}(z_i)(z-z_i)^r/r! \tag{7.8}$$

then the resulting function

$$h_{ik}(z)=p_i(z)s_{ik}(z) \tag{7.9}$$

is a polynomial of degree $<M$. Note that the polynomials h_{il_i} $(i=1,...,m)$ are of degree $M-1$. Using Eq. (7.6) and the formula

$$h_{ik}(z)=(z-z_i)^k+p_i(z)(s_{ik}(z)-q_{ik}(z)) \tag{7.10}$$

and noting that $s_{ik}-q_{ik}$ has a zero of order l_i+1 at z_i, one obtains

$$h_{ik}^{(l)}(z_j)=k!\delta_{ij}\delta_{kl} \quad \text{for } k=1,...,l_i, \quad l=1,...,l_j, \quad \text{and } i,j=1,...,m. \tag{7.11}$$

Therefore, if f is smooth, then its *Hermite interpolation polynomial associated with p*

$$\hat{f}(z)=\sum_{i=1}^{m}\sum_{k=0}^{l_i} f^{(k)}(z_i)h_{ik}(z)/k! \tag{7.12}$$

is of degree $<M$ and

$$\hat{f}^{(k)}(z_i)=f^{(k)}(z_i), \quad \text{for } k=1,...,l_i, \quad \text{and } i=1,...,m. \tag{7.13}$$

The hat map is idempotent. If f is a polynomial, then $\hat{f}=0$ is equivalent to f being divisible by p. If f is an entire function, then $f(z)=\lim_{l\to\infty} f_l(z)$ for every $z\in\mathbf{C}$, where f_l is the polynomial (7.3). The difference $f_l-\hat{f}_l$ is a polynomial divisible by p. Therefore, if $a\in\mathscr{A}$ and p are as in Sec. VI(1), then

$$f_l(a)=\hat{f}_l(a) \quad \text{for } l=0,1,..., \quad \text{and } f(a)=\lim_{l\to\infty} f_l(a)=\hat{f}(a).$$

These observations can be summarized in

Proposition: Let f be an entire analytic function and let $\mathscr{A}$ be a complex, finite-dimensional, associative algebra with unit element. If $a\in\mathscr{A}$ and $p(a)=0$, where p is a (normalized) polynomial of positive degree, then

$$f(a)=\hat{f}(a), \tag{7.14}$$

where $\hat{f}$ is the Hermite interpolation polynomial of f, associated with p.

It is convenient, but not necessary, to take for p the minimal polynomial of a, i.e., the normalized polynomial of lowest positive degree such that $p(a)=0$.

(4) Let $\mathscr{A}$ be the even Clifford algebra $\mathrm{Cl}_0(n)$ associated with $V=\mathbb{C}^n$. The vector space of bivectors, $\wedge^2 V\subset\mathscr{A}$, is the underlying space of the Lie algebra of the group $\mathrm{Spin}(n)\subset\mathscr{A}$. The map $\exp:\mathscr{A}\to\mathscr{A}$, restricted to $\wedge^2 V$, has values in the group $\mathrm{Spin}(n)$ and coincides with the exponential map in the sense of the theory of Lie groups. The above Proposition can be used to compute it explicitly.

According to Eq. (2.28), if a is a bivector, then its Clifford square is $a^2=-(a|a)+a\wedge a$. Therefore, there exists a polynomial P of degree not larger than

$$\dim \bigoplus_k \wedge^{4k} V=2^{n-2}+\tfrac{1}{2}\,\mathrm{Re}(1+\sqrt{-1})^n \tag{7.15}$$

and such that $p(a)=P(a^2)=0$. Putting $\lambda^2=-(a|a)$ and $\mu^4=(a\wedge a)_0^2$, one obtains

$$p(z)=z^2-\lambda^2,\quad \text{for } n\leqslant 3$$

and

$$p(z)=(z^2-\lambda^2)^2-\mu^4,\quad \text{for } n=4 \text{ and } 5.$$

Formula (7.1) is obtained from Eqs. (7.12) and (7.14) by putting $a=\mathbf{v}\sigma$ and $f=\exp$.

If the bivector a is nilpotent, i.e., if $a^M=0$ for a positive integer M, then $\exp a=1+a/1!+\cdots+a^{M-1}/(M-1)!$. If $a^2=0$, then a is a *null* bivector: it is decomposable, its scalar square vanishes, and $s=\exp a=1+a$. The "null rotation" (Ref. 24) $\rho(s)$ of a vector v is easily obtained from Eq. (2.15)

$$\rho(s)v=(1+a)v(1-a)=v-2va+2(va)a. \tag{7.16}$$

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