80

SPIN STRUCTURES ON HYPERSURFACES AND THE SPECTRUM OF THE DIRAC OPERATOR ON SPHERES*

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Abstract. Recent results on pin structures on hypersurfaces in spin manifolds are reviewed. A new form of the Dirac operator is used to compute its spectrum on n-dimensional spheres. This constrbution is based on two papers by the author, where details and proofs can be found (Ref.4 and 5).

1. This research has been motivated by, and can be summarized in, the following observations:
(i) In odd dimensions, it is appropriate to use the *twisted* adjoint representation $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ to find a cover of the full orthogonal group $\mathrm{O}(n)$ which extends the standard homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Here ρ is given by $\rho(a)v = \alpha(a)va^{-1}$, where $v \in \mathbf{R}^n$, $a \in \mathrm{Pin}(n) \subset \mathrm{Cl}(n)$ and α is the grading (main) automorphism of the Clifford algebra $\mathrm{Cl}(n)$ [1]. Using the twisted representation leads to modifying the Dirac operator [2].
(ii) The bundles of "Dirac spinors" over even-dimensional spheres are trivial [3]; this observation generalizes to hypersurfaces in $\mathbf{R}^{n+1}$: every such hypersurface, even if it is non-orientable, admits a pin structure with a trivial bundle of Dirac (n even) or Pauli (n odd) spinors [4]
(iii) The spectrum and the eigenfunctions of the Laplace operator Δ on the n-dimensional unit sphere $\mathbf{S}_n$ are easily obtained from the formula

$$\sum_{i=1}^{n+1} \partial^2/\partial x_i^2 = r^{-2}\Delta + r^{-n}\partial/\partial r(r^n\partial/\partial r) \tag{1}$$

This formula generalizes to a foliation of $\mathbf{R}^{n+1}$ by hypersurfaces and extends to the Dirac operator, allowing a simple computation of the Dirac spectrum of n-spheres [5].

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2. Consider the vector space $\mathbf{R}^n$ with the standard scalar product $(u \mid v)$ and the associated positive-definite quadratic form $(u \mid u)$, where $u = (u^\mu) \in \mathbf{R}^n$, $\mu = 1, \dots, n$. The corresponding Clifford algebra $\mathrm{Cl}(n)$ contains $\mathbf{R} \bigoplus \mathbf{R}^n$, one has

$$uv + vu = -2(u \mid v), \quad \text{where} \quad u, v \in \mathbf{R}^n, \tag{2}$$

and uv is the Clifford product of u and v. Let (e_μ) be the canonical frame in $\mathbf{R}^n$ so that $u = u^\mu e_\mu$ for every $u \in \mathbf{R}^n$; similarly, $(e_i), i = 1, \dots, n+1$, is the canonical frame in $\mathbf{R}^{n+1}$. The group $\mathrm{Pin}(n)$ is defined as the subset of $\mathrm{Cl}(n)$ consisting of products of all finite sequences of unit vectors.

Let $\mathrm{Cl}(n) = \mathrm{Cl}_0(n) \bigoplus \mathrm{Cl}_1(n)$ be the decomposition of $\mathrm{Cl}(n)$ defining its $\mathbf{Z}_2$ grading so that $\mathrm{Spin}(n) = \mathrm{Pin}(n) \bigcap \mathrm{Cl}_0(n)$. Let $a = a_0 + a_1$ be the corresponding decomposition of $a \in \mathrm{Cl}(n)$. The map $h : \mathrm{Cl}(n) \to \mathrm{Cl}_0(n+1)$ given by $a \to a_0 + a_1 e_{n+1}$ is an isomorphism of algebras with units. By restriction, it defines the commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathrm{Pin}(n) & \xrightarrow{h} & \mathrm{Spin}(n+1) \\ \rho \downarrow & & \downarrow \rho \\ \mathrm{O}(n) & \xrightarrow{H} & \mathrm{SO}(n+1) \end{array} \tag{3}$$

where the horizontal (resp., vertical) arrows are injective (resp., surjective).

For every n, there is a representation γ of $\mathrm{Cl}(n)$ and a representation γ' of $\mathrm{Cl}(n+1)$ in the same complex vector space S. The representation γ' extends γ in the sense that $\gamma = \gamma' \circ h$. One puts

$$\gamma_i = \gamma'(e_i) \quad i = 1, \dots, n+1. \tag{4}$$

and defines the helicity automorphism $\Gamma = (-1)^{n(n-1)/4} \gamma_1 \gamma_2 \dots \gamma_n$ so that $\Gamma^2 = 1$. Note that $\gamma(e_\mu) = \gamma_\mu \gamma_{n+1}$ and $\gamma(e_\mu e_\nu) = \gamma_\mu \gamma_\nu$. For $n = 2m$, γ is the Dirac representation in a complex vector space of dimension 2^m and γ' is one of two Pauli representations, characterized, say, by $\gamma_{n+1} = \sqrt{-1}\Gamma$. For $n = 2m - 1$, γ' is the Dirac representation, whereas γ is a faithful representation that decomposes into two irreducible Pauli representations. This terminology generalizes the one used by physicists in dimensions 3 and 4.

3. Consider now an n-dimensional pin manifold M, i.e. a Riemannian manifold with a pin structure

$$Q \xrightarrow{\sigma} P \xrightarrow{\pi} M \tag{5}$$

where P is the $\mathrm{O}(n)$-bundle of all orthonormal frames on M so that $\sigma(q) = (\sigma_\mu(q))$, $q \in Q$, is an orthonormal frame at $\tilde{\pi}(q) = \pi \circ \sigma(q) \in M$ and $\tilde{\pi} : Q \to M$ is a Pin(n)-bundle such that $\sigma \circ \delta(a) = \delta(\rho(a)) \circ \sigma$, where $\delta(a)$ is the (right) translation by $a \in \mathrm{Pin}(n)$ of elements of Q. The Levi-Civita connection on M defines a "spin" connection on the pin-bundle $Q \to M$ which can be described by giving on Q a collection of n horizontal vector fields ∇_μ $(\mu = 1, \ldots, n)$ such that, for every $q \in Q$, one has $T_q\tilde{\pi}(\nabla_\mu(q)) = \sigma_\mu(q)$.

By restriction, one has the representation γ: $\mathrm{Pin}(n) \to \mathrm{GL}(S)$ and one defines a spinor field on M, with its pin structure (5), as a map $\psi : Q \to S$, equivariant with respect to the action on $\mathrm{Pin}(n)$, $\psi \circ \delta(a) = \gamma(a^{-1}) \circ \psi$. Alternatively, and equivalently, a spinor field can be described as a section of the bundle $E \to M$, associated with $Q \to M$ by the representation γ.

The Dirac operator $\nabla = \gamma^\mu \nabla_\mu$ transforms spinor fields into spinor fields.

4. A hypersurface M in an $(n+1)$-dimensional connected Riemannian manifold M' is an n-manifold M with an immersion $f : M \to M'$. The metric tensor on M' induces a Riemannian metric on M. If M' is orientable and P' is its bundle of orthonormal frames of coherent orientation, then the bundle P of all orthonormal frames on M can be identified with the set

$$\{(x,p) \in M \times P' : p = (p_i), i = 1, \ldots, n+1 \text{ where } p \text{ is a frame at } f(x) \text{ such that } p_{n+1} \text{ is orthogonal to } T_x f(T_x M) \subset T_{f(x)} M' \Big\}.$$

The group $\mathrm{O}(n)$ acts in P via H. Assume now that M' has a spin structure $Q' \xrightarrow{\sigma'} P' \to M'$; a spin-structure on M is (5), where $Q \to P$ is the Z_2-bundle induced [6] from $Q' \to P'$ by the map $F : P \to P', F(x,p) = p$, i.e.

$$Q = \{(p,q) \in P \times Q' : F(p) = \sigma'(q)\}.$$

As an example illustrating this construction, one can mention the embedding of real projective spaces, $\mathbf{RP}_n \to \mathbf{RP}_{n+1}$. Since $\mathbf{RP}_{4m+3}$ is a spin manifold, there is a pin structure on $\mathbf{RP}_{4m+2}$ [7].

Immersions of M, which are differentiably homotopic one to another, give rise to equivalent pin structures on M, but otherwise not, in general. For example, the "identity" and the "square" immersions of S_1 in $\mathbf{R}^2$ give rise to the non-trivial and the trivial spin structures on the circle, respectively.

Assume now that the spin structure on M' is trivial, i.e. there exists a map $g : Q' \to \mathrm{Spin}(n+1)$ such that $g(qa) = g(q)a$ for every $q \in Q'$ and $a \in \mathrm{Spin}(n+1)$. *The pin structure on the hypersurface M need not be trivial, but the bundle $E \to M$ of spinors, associated by γ with $Q \to M$, is isomorphic to the direct product $M \times S$.*

Indeed, the bundle E can be identified with the set of equivalence classes of the form $[(p,q,\phi)]$, where $(p,q,\phi) \in P \times Q' \times S$, $F(p) = \sigma'(q)$ and $[(p,q,\phi)] = [(p',q',\phi')]$ iff there is $a \in \mathrm{Pin}(n)$ such that $p' = p\rho(a)$, $q' = qh(a)$ and $\phi = \gamma(a)\phi'$. The map $[(p,q,\phi)] \to (\pi(p), \gamma'(g(q))\phi)$ trivializes E. For example, if M is a hypersurface in $\mathbf{R}^{n+1}$, then its bundle of Dirac or Pauli spinors is trivial. Since $\mathbf{RP}_3 = \mathrm{SO}(3)$ has a trivial spin bundle, the bundle of two-component "Dirac" spinors on $\mathbf{RP}_2$ is also trivial. In general, the bundles of Weyl (half) spinors on even-dimensional hypersurfaces in $\mathbf{R}^{n+1}$ are not trivial (example: even-dimensional spheres).

5. Let $f : M \to M'$ be an embedding (i.e. injective immersion) of the hypersurface M in the manifold M' with a *trivial* spin structure $Q' \to P' \to M'$. The maps $P \to P'$ and $Q \to Q'$ are then also injective and the extension $Q"$ of the Pin(n)-bundle Q to the group Spin(n) is also trivial. A spinor fiels $\psi : Q \to S$ extends to a map $\psi" : Q" \to S$ such that $\psi"(qa) = \gamma'(a^{-1})\psi"(q)$ for every $q \in Q"$ and $a \in$Spin$(n+1)$. Instead of working with ψ, one can now take a global section s of the trivial bundle $Q" \to M$ and the composition $\Psi = \psi" \circ s : M \to S$ as an equivalent way of describing the spinor field. One defines the Dirac operator D acting on Ψ by the formula

$$D\Psi = (\nabla\psi)" \circ s. \tag{6}$$

6. The above considerations are particularly useful and simple when M is an orientable hypersurface embedded in $\mathbf{R}^{n+1}$. This being so, let (X^i) be the unit normal vector field on M and let (x^i) be the Cartesian coordinates in $\mathbf{R}^{n+1}$. Each of the $n(n+1)/2$ vector fields

$$X_{ij} = X_j\partial_i - X_i\partial_j, \quad \text{where} \quad \partial_i = \partial/\partial x_i, \quad 1 \le i < j \le n,$$

is tangent to M. Introducing the notation

$$\sigma_{ij} = (\gamma_i\gamma_j - \gamma_j\gamma_i)/2, \quad \mathbf{X} = X^I\gamma_i, \quad \text{div } X = \partial_i X^i,$$

so that

$$\sigma_{ij} = \delta_{ij} + \gamma_i\gamma_j,$$

one can write (6) as

$$D\Psi = \frac{1}{2}\mathbf{X}(\sigma^{ij}X_{ij} - \text{div}X)\Psi. \tag{7}$$

The right side of (7) is invariant with respect to the replacement of X by $-X$ and one can show that the assumption of orientability of M is irrelevant.

Assume now that $\mathbf{R}^{n+1}$ is foliated by a family of hypersurfaces so that the field X of unit normals is defined over an open subset of $\mathbf{R}^{n+1}$. The identity

$$\gamma^i\partial_i = \mathbf{X}(X^i\partial_i + \tfrac{1}{2}\sigma^{ij}X_{ij}) \tag{8}$$

leads to a decomposition of the Dirac operator $\gamma^i\partial_i$ on $\mathbf{R}^{n+1}$ into parts tangential and transverse to the foliation,

$$\gamma^i\partial_i = D + \mathbf{X}(\partial/\partial r + \tfrac{1}{2}\text{div}X), \tag{9}$$

where $\partial/\partial r = X^i\partial_i$ is the derivative in the "radial" direction, transverse to the foliation. There is an analogous formula for the Laplace operator

[4]. Since the operator D anticommutes with $\mathbf{X}$ and $\mathbf{X}^2 = -1$, if Ψ is an eigenfunction of D, then $(1+\mathbf{X})\Psi$ is an eigenfunction of $\mathbf{X}D$ with the same eigenvalue. Therefore, for M orientable, it is enough to consider the spectrum of the latter operator.

7. As a simple application, consider the spectrum of the Dirac operator on the unit sphere $\mathbf{S}_n$. The space $\mathbf{R}^{n+1}$ with its origin removed is foliated by the spheres $r = \sqrt{(x_1^2 + \ldots + x_{n+1}^2)} =$ const. so that $X^i = x^i/r$,the vector fields X_{ij} are generators of rotations, $\mathbf{div}X = n/r$ and equation (9) gives

$$\mathbf{X}\gamma^i\partial_i = \mathbf{XD} - (\partial/\partial r + n/2r). \tag{10}$$

Let $\Phi{:}\mathbf{R}^{n+1} \to S$ be a spinor-valued harmonic polynomial of degree $l+1$, where $l = 0, 1, \ldots$ The polynomial $\Psi = (\gamma^i\partial_i)\Phi$ is of degree l and is annihilated by the Dirac operator $\gamma^i\partial_i$. Therefore, on the unit sphere $r = 1$, one has

$$\mathbf{XD}\Psi = (l + n/2)\Psi \text{ and } \mathbf{XD}\ \mathbf{X}\Psi = -(l + n/2)\mathbf{X}\Psi. \tag{11}$$

and the spectrum of the Dirac operator on $\mathbf{S}_n$, for $n > 1$, is the set of all numbers of the form $\pm(l + n/2)$, where $l = 0, 1, 2, \ldots$. There is a gap of length n and 0 is never an eigenvalue, this being a simple consequence of the celebrated Lichnerowicz theorem [8]. For $n = 1$, there are two spin structures. The previous formula applies to the non-trivial structure; for the trivial one, the spectrum is $\mathbf{Z}$.

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