Spinors and the Dirac operator on hypersurfaces. I. General theory

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It is shown that a hypersurface immersed isometrically into the Euclidean space $\mathbb{R}^{n+1}$, where $n=2\nu$ or $2\nu+1$, has a pin structure such that the associated bundle of 2^ν-component spinors is trivial. This is used to derive a new formula for the Dirac operator on hypersurfaces. The Dirac operator is slightly modified to be compatible with the twisted adjoint representation of the pin group. When $\mathbb{R}^{n+1}$ is foliated by hypersurfaces, then the Dirac operator in $\mathbb{R}^{n+1}$ splits into a radial and a tangential part with respect to the foliation. There is a corresponding new formula for the Laplacian.

I. INTRODUCTION

The Dirac equation, introduced in 1928 in the context of special relativity theory,[1] was soon afterwards generalized to curved spaces,[2] but it started attracting the attention of pure mathematicians only about 35 years later.[3–5] Lichnerowicz[3] obtained the first global result on the properties of the Dirac operator $\nabla\!\!\!\!/$: on a compact Riemannian space with a positive Ricci scalar R, the operator $\nabla\!\!\!\!/$ has no eigenfunctions with eigenvalue 0. This "vanishing theorem" and its generalizations[4] have been used by Gromov and Lawson[6] to determine the existence of Riemannian metrics with positive R on compact, non-simply-connected manifolds. Much work has been done to give a geometrical estimate of the eigenvalues of $\nabla\!\!\!\!/$, on Killing spinors and the index theorem[5] for $\nabla\!\!\!\!/$; there is a review and a bibliography of this material in Ref. 7.

Under the influence of the Kaluza–Klein theory and supergravity, mathematicians and physicists have determined spin structures on symmetric spaces[8,9] and found, in many cases, the spectra and eigenfunctions of the Dirac operator on those spaces.[10–12]

In this paper I consider pin structures and the Dirac operator on hypersurfaces in the flat, Euclidean space $\mathbb{R}^{n+1}$. The hypersurfaces need not be orientable nor admit any symmetries; they may have a complicated topology and be merely immersed (self-intersections are allowed). Thus, for example, the Klein bottle immersed in $\mathbb{R}^3$ is covered by these considerations. It is shown here that the hypersurfaces always have a pin structure and the associated bundle of Dirac or Pauli spinors is trivial. This observation is used to derive a convenient, "global" form of the Dirac operator on hypersurfaces in $\mathbb{R}^{n+1}$.

The original motivation was to derive for $\nabla\!\!\!\!/$ an equation analogous to the formula

$$\Delta(\mathbb{R}^{n+1})=r^{-2}\Delta(\mathbb{S}_n)+r^{-n}\left(\frac{\partial}{\partial r}\right)r^n\frac{\partial}{\partial r} \tag{1.1}$$

expressing the Laplacian in $\mathbb{R}^{n+1}$ in terms of the Laplacian on the unit n-sphere $\mathbb{S}_n$ and an operator involving differentiation only in the direction of the coordinate $r=(x_1{}^2+\cdots+x_{n+1}{}^2)^{1/2}$. It turned out that there are general formulas for both Δ and $\nabla\!\!\!\!/$, expressing these operators as sums of parts tangential and transverse with respect to a foliation of $\mathbb{R}^{n+1}$ by hypersurfaces. These formulas are suitable for finding the eigenfunctions of Δ and $\nabla\!\!\!\!/$ by the method of separation of variables and for approximate computations. The triviality of the associated spinor bundles makes easier global considerations: there is no need to consider local sections of the principal bundle of spin frames—which, in general, is nontrivial—and one thereby avoids the necessity of using "double-valued spinor wave functions".[13]

The paper is organized as follows: in Secs. II and III there is a review of Clifford algebras, spin groups, and their representations. Section IV is devoted to pin structures and to a formulation of the Dirac operator, suitable also for odd-dimensional spaces. Section V contains a general lemma on trivial associated bundles; the lemma is applied in Sec. VI to spin structures on hypersurfaces. The new formulas for the Dirac and Laplace operators are presented in Sec. VII and applied to spheres in Part II of the paper. In agreement with the custom prevailing in physics, tensor, but not spinor, indices often appear and the Einstein summation convention is used throughout the paper.

II. CLIFFORD ALGEBRAS AND THEIR REPRESENTATIONS

Standard notions and results on Clifford algebras and their representations are summarized here in a form adapted to the needs of the paper; proofs and further details can be found in Refs. 14–17.

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Let g be a scalar product of signature (k,l), $k+l=n$, in the real vector space $V=\mathbb{R}^n$. The scalar product is chosen so that the standard linear frame (e_μ) in V is orthonormal with respect to g, i.e., if $g_{\mu\nu}=g(e_\mu,e_\nu)$, then $g_{\mu\nu}=0$ for $\mu\neq\nu$, $g_{\mu\mu}=1$ for $\mu=1,\ldots,k$, and $g_{\mu\mu}=-1$ for $\mu=k+1,\ldots,n$. The Clifford algebra associated with the pair (V,g) is denoted by $\mathrm{Cl}(k,l)$. The space V is a subspace of the algebra and generates it. If $u,v\in V$, then

$$uv+vu=2g(u,v). \tag{2.1}$$

The scalar product g defines an isomorphism from V to the dual space V^*, also denoted by g: if $u,v\in V$, then $g(v)$ is a one-form on V such that $\langle u,g(v)\rangle=g(u,v)$.

The main automorphism α of $\mathrm{Cl}(k,l)$ is defined by $\alpha(1)=1$ and $\alpha(v)=-v$ for $v\in V$. It defines the $\mathbb{Z}_2$-grading of the Clifford algebra,

$$\mathrm{Cl}(k,l)=\mathrm{Cl}_0(k,l)\oplus\mathrm{Cl}_1(k,l), \tag{2.2}$$

where

$$\mathrm{Cl}_\varepsilon(k,l)=\{a\in\mathrm{Cl}(k,l)\,|\,\alpha(a)=(-1)^\varepsilon a\},\quad \varepsilon=0,1.$$

Every element $a\in\mathrm{Cl}(k,l)$ decomposes according to (2.2), $a=a_0+a_1$, where $a_\varepsilon\in\mathrm{Cl}_\varepsilon(k,l)$. The components a_0 and a_1 are called the even and the odd part of a, respectively.

Recall that the vector space $\Lambda V=\oplus\Lambda^k V$ is an algebra with respect to the exterior product of multivectors. If $\lambda\in V^*$ and $w\in\Lambda V$, then the inner product of w by λ, $\lambda\,\lrcorner\,w\in\Lambda V$, is characterized by (i) linearity with respect to w, (ii) $\lambda\,\lrcorner\,1=0$ and $\lambda\,\lrcorner\,v=\langle v,\lambda\rangle$ for every $v\in V$, and (iii) if $u\in\Lambda^k V$, then

$$\lambda\,\lrcorner\,(u\wedge w)=(\lambda\,\lrcorner\,u)\wedge w+(-1)^k u\wedge(\lambda\,\lrcorner\,w).$$

Dually, if $v\in V$ and $\omega\in\Lambda V^*$, then there is the inner product $v\,\lrcorner\,\omega\in\Lambda V^*$. In particular, if ω is a k-form and $v_2,\ldots,v_k\in V$, then $(v\,\lrcorner\,\omega)(v_2,\ldots,v_k)=\omega(v,v_2,\ldots v_k)$.

The underlying vector space of $\mathrm{Cl}(k,l)$ can be identified, in a natural manner, with that of the exterior algebra ΛV. The canonical isomorphism of vector spaces, $\tau:\Lambda V\to\mathrm{Cl}(k,l)$, is characterized by $\tau(1)=1$, $\tau(v)=v$ for every $v\in V$, and

$$v\cdot\tau(w)=\tau(v\wedge w+g(v)\,\lrcorner\,w) \tag{2.3}$$

for every $v\in V$ and $w\in\Lambda V$. The product on the left of (2.3) is the Clifford multiplication. If $1\leqslant\mu_1<\mu_2<\cdots<\mu_p\leqslant n$, then (2.3) gives $\tau(e_{\mu_1}\wedge\cdots\wedge e_{\mu_p})=e_{\mu_1}\cdots e_{\mu_p}$.

The space $\Lambda^2 V$ of bivectors has a natural structure of a Lie algebra that can be identified with $\mathrm{so}(k,l)$, the Lie algebra of the group $\mathrm{SO}(k,l)$. If $A\in\Lambda^2 V$, then $A\circ g:V\to V$ is defined by $(A\circ g)(v)=-g(v)\,\lrcorner\,A$ and the Lie bracket is given by

$$[A,B]\circ g=[A\circ g,B\circ g]. \tag{2.4}$$

The image of $\Lambda^2 V$ by τ is also a Lie algebra with respect to the Clifford bracket and

$$\tau([A,B])=\tfrac{1}{2}(\tau(A)\tau(B)-\tau(B)\tau(A)) \tag{2.5}$$

for every $A,B\in\Lambda^2 V$. The map $A\to\tau(A)/2$ is an isomorphism of Lie algebras. The factor $\frac{1}{2}$ reappears in the formulas linking spin and linear connections (see Sec. IV).

In this paper, an important role is played by the isomorphism of algebras with units,

$$h:\mathrm{Cl}(k,l)\to\mathrm{Cl}_0(k,l+1) \tag{2.6}$$

defined by $h(a_0+a_1)=a_0+a_1e_{n+1}$ and holding for every pair of non-negative integers k and l. For every such pair, there is a representation γ of $\mathrm{Cl}(k,l)$ and a representation γ' of $\mathrm{Cl}(k,l+1)$ in the *same* complex vector space S. The representation γ' *extends* γ in the sense that

$$\gamma=\gamma'\circ h. \tag{2.7}$$

Putting

$$\gamma_i=\gamma'(e_i)\quad\text{for}\quad i=1,\ldots,n+1=k+l+1, \tag{2.8}$$

and defining the *helicity automorphism*

$$\Gamma=(-1)^{(k-l)(k-l-1)/4}\gamma_1\gamma_2\cdots\gamma_n,$$

so that $\Gamma^2=I$, one obtains

$$\Gamma\gamma_\mu=(-1)^{n+1}\gamma_\mu\Gamma\quad\text{for}\quad\mu=1,\ldots,n, \tag{2.9}$$

and

$$\gamma(e_\mu)=\gamma_\mu\gamma_{n+1},\quad\gamma(e_\mu e_\nu)=\gamma_\mu\gamma_\nu, \tag{2.10}$$

for μ, $\nu=1,\ldots,n$. Up to complex equivalence, the representations are uniquely defined by the following conditions.

(i) If $n=k+l$ is *even*, $n=2\nu$, then γ is the faithful and irreducible "Dirac representation" of the simple algebra $\mathrm{Cl}(k,l)$ in a space S of complex dimension 2^ν and γ' is one of two irreducible "Pauli representations," say the one characterized by (2.7) and $\gamma_{n+1}=\sqrt{-1}\Gamma$.

(ii) If $n=k+l$ is *odd*, $n=2\nu-1$, then γ' is the Dirac representation of the simple algebra $\mathrm{Cl}(k,l+1)$ in a space S of complex dimension 2^ν and γ is the faithful "Cartan representation" which decomposes into the direct sum of two irreducible Pauli representations in spaces of complex dimension $2^{\nu-1}$.

Except for the name "Cartan representation," which seldom appears, this terminology is a natural generalization of the one used by physicists for dimensions 3 and 4. (Physicists, however, attach the names of Dirac, Pauli,

TABLE I. Dimensions and names of spinor spaces.

dim	spinors	
vectors	2^ν	$2^{\nu-1}$
$2\nu-1$	Cartan	Pauli
2ν	Dirac	Weyl

and Weyl to spinors rather than to representations.) To make our conventions visible at a glance, we summarize them in Table I, which contains information about the dimensions of vector and spinor spaces and the names of spinors. The name "helicity" automorphism for Γ usually appears only when n is even: its eigenvectors are Weyl spinors. For n odd, the eigenvectors of Γ are Pauli spinors.

From the point of view presented in this paper, there are two reasons for considering the Cartan representation: first, it is a faithful representation of the full Clifford algebra of an odd-dimensional vector space and, second, if M is a hypersurface in $\mathbb{R}^{2\nu}$, then a Dirac spinor field in $\mathbf{R}^{2\nu}$ restricts to a Cartan spinor field on M. The latter observation is crucial in Sec. VII.

III. SPINOR GROUPS

With every pair (k,l) of non-negative integers there are associated spinor groups $\mathrm{Pin}(k,l)$ and $\mathrm{Spin}(k,l)$ defined as follows. Let $u\in V=\mathbf{R}^{k+l}$ be a unit vector, $u^2=1$ or -1, and $v\in V$; the map $v\to -uvu^{-1}$ is a reflection in the hyperplane orthogonal to u. The vector $-u$ gives the same reflection and this ambiguity is the root of the "double-valuedness of the spinor representations," cf. (3.2) below. The group $\mathrm{Pin}(k,l)$ is the subset of $\mathrm{Cl}(k,l)$ consisting of products of all finite sequences of unit vectors with group multiplication induced from that of the algebra. There are two important homomorphisms from $\mathrm{Pin}(k,l)$ to the orthogonal group $\mathrm{O}(k,l)$:

(i) the adjoint representation of the group Pin in the vector space V, ad: $\mathrm{Pin}(k,l)\to\mathrm{O}(k,l)$, defined by $\mathrm{ad}(a)v=ava^{-1}$, where $v\in V$;

(ii) the twisted adjoint representation ρ in the same vector space, given by

$$\rho(a)v=\alpha(a)va^{-1}. \tag{3.1}$$

For every pair (k,l) there is the exact sequence

$$1\to\mathbf{Z}_2\to\mathrm{Pin}(k,l)\to\mathrm{O}(k,l)\to 1. \tag{3.2}$$

If $n=k+l$ is *even*, then $-uvu^{-1}=(u\,\mathrm{vol})v(u\,\mathrm{vol})^{-1}$, where

$$\mathrm{vol}=e_1e_2\cdots e_n \tag{3.3}$$

is the *volume element* and there is an exact sequence like (3.2) with ρ replaced by ad. In the adjoint representation, the reflection in the hyperplane orthogonal to u is covered by the elements $u\,\mathrm{vol}$ and $-u\,\mathrm{vol}$. This is familiar to physicists who favor the adjoint representation: in Minkowski space $\mathbb{R}^4$, the time reflection is represented on Dirac spinors by the matrices $\pm\gamma_1\gamma_2\gamma_3$.

The spin group is the subgroup of $\mathrm{Pin}(k,l)$ consisting of all its even elements,

$$\mathrm{Spin}(k,l)=\mathrm{Pin}(k,l)\cap\mathrm{Cl}_0(k,l). \tag{3.4}$$

The groups $\mathrm{Spin}(k,l)$ and $\mathrm{Spin}(l,k)$ are isomorphic to each other, but the groups $\mathrm{Pin}(k,l)$ and $\mathrm{Pin}(l,k)$, in general, are not. This observation has interesting geometrical[8] and physical[18] consequences. The representations ad and ρ coincide when restricted to the spin group and, for every pair (k,l), there is the exact sequence

$$1\to\mathbf{Z}_2\to\mathrm{Spin}(k,l)\to\mathrm{SO}(k,l)\to 1. \tag{3.5}$$

If $n=(k,l)$ is *even*, then the Clifford–Hodge duality map

$$\mathrm{Cl}(k,l)\to\mathrm{Cl}(k,l),\quad \text{given by } a\to a\,\mathrm{vol}, \tag{3.6}$$

restricted to V has the Clifford property, $(v\,\mathrm{vol})^2=-v^2(\mathrm{vol})^2$. By universality of Clifford algebras, the map $v\to v\,\mathrm{vol}$ extends to an isomorphism of algebras

$$j{:}\mathrm{Cl}(k,l)\to\begin{cases}\mathrm{Cl}(k,l), & \text{if } (\mathrm{vol})^2=-1,\\ \mathrm{Cl}(l,k), & \text{if } (\mathrm{vol})^2=1.\end{cases} \tag{3.7}$$

By restriction to the pin groups this yields an isomorphism, also denoted by j, such that

$$\mathrm{ad}=\rho\circ j. \tag{3.8}$$

Therefore, up to the isomorphism j [and signature swap for $(\mathrm{vol})^2=1$], the covering homomorphisms ad and ρ are equivalent. For this reason, in even-dimensional spaces, there is no essential difference between the twisted adjoint representation ρ and the adjoint representation ad, traditionally used by physicists.

The situation is entirely different for $n=k+l$ *odd*: in this case, the volume element is in the center of the full algebra and the automorphism α is not inner. To represent reflections faithfully, it is necessary to use the twisted adjoint representation. For the sake of uniformity, it is convenient to *use the twisted adjoint representation in all cases*, and this is done in this paper. There is, however, the possibility of embedding $\mathrm{Cl}(k,l)$ into the larger algebra $\mathrm{Cl}(k,l+1)$: for every $a\in\mathrm{Cl}(k,l)$ one has

$$h(\alpha(a))=e_{n+1}h(a)e_{n+1}^{-1}$$

and (3.9)

$$\gamma(\alpha(a)) = \gamma_{n+1}\gamma(a)\gamma_{n+1}^{-1},$$

so that α appears as an inner automorphism in the larger algebra.

By restriction of the homomorphisms h, γ, and γ' of the Clifford algebras, one obtains homomorphisms of the corresponding spinor groups. The restrictions are denoted by the same letters. For every k and l with $k+l=2\nu$ or $2\nu-1$ ($\nu=1,2,\ldots$) one has the commutative diagram of group homomorphisms

$$\begin{array}{ccccccc} \mathrm{Pin}(k,l) & \xrightarrow{h} & \mathrm{Spin}(k,l+1) & \xrightarrow{\mathrm{inj}} & \mathrm{Pin}(k,l+1) & \xrightarrow{\gamma'} & \mathrm{GL}(S) \\ \rho\downarrow & & \rho\downarrow & & \rho\downarrow & & \\ \mathrm{O}(k,l) & \xrightarrow{H} & \mathrm{SO}(k,l+1) & \xrightarrow{\mathrm{inj}} & \mathrm{O}(k,l+1) & & \end{array}, \qquad (3.10)$$

where

$$H(A)e_\mu = Ae_\mu, \quad \mu=1,\ldots,n$$

and (3.11)

$$H(A)e_{n+1} = (\det A)e_{n+1},$$

$A\in\mathrm{O}(k,l)$, and S is a complex vector space of dimension 2^ν.

The Lie algebra $\mathrm{spin}(k,l)$ of the group $\mathrm{Spin}(k,l)$ can be identified with the vector subspace $\tau(\Lambda^2 V)$ of $\mathrm{Cl}_0(k,l)$; it is spanned by the elements $e_\mu e_\nu$, where $1\leqslant\mu<\nu\leqslant n$; the bracket coincides with the commutator obtained from the associative Clifford product. The Lie algebra $\mathrm{so}(k,l)$ of the group $\mathrm{SO}(k,l)$ is $\Lambda^2 V$ with the bracket defined in (2.4). If $\Sigma\in\mathrm{Spin}(k,l)$, then $\exp t\Sigma$ is a one-parameter subgroup of $\mathrm{Spin}(k,l)$. The homomorphism $\rho:\mathrm{Spin}(k,l)\to\mathrm{SO}(k,l)$ induces the derived isomorphism of Lie algebras:

$$\rho_*:\mathrm{spin}(k,l)\to\mathrm{so}(k,l) \qquad (3.12)$$

given by

$$\rho_*(\Sigma) = \frac{d}{dt}\rho(\exp t\Sigma)\big|_{t=0}, \quad \text{i.e., } \rho_*(\Sigma)u = [\Sigma,u],$$

where $u\in V$. Since $[\tau(A),u] = 2A\circ g(u)$, one obtains

$$\rho_*^{-1} = \tfrac{1}{2}\tau\,|\,\mathrm{so}(k,l), \quad \text{i.e., } \rho_*^{-1}(e_\mu\wedge e_\nu) = \tfrac{1}{2}e_\mu e_\nu, \quad \mu\neq\nu. \qquad (3.13)$$

IV. PIN STRUCTURES AND THE DIRAC OPERATOR

Consider an n-dimensional smooth manifold M with a metric tensor g of signature (k,l), $k+l=n$. The principal bundle $\pi:P\to M$ of all linear frames on M, orthonormal with respect to g, has $\mathrm{O}(k,l)$ as its structure group and is endowed with the Levi–Civita connection determined by g. One says that M has a *pin structure* [sometimes: $\mathrm{Pin}(k,l)$-structure][13–16] if there is a manifold Q and a map ρ: $Q\to P$ such that $\pi\circ\rho$: $Q\to M$ is a principal bundle with structure group $\mathrm{Pin}(k,l)$ and $\rho(qa)=\rho(q)\rho(a)$, where $qa=\delta(a)q$ is the result of the action of $a\in\mathrm{Pin}(k,l)$ on the "pin frame" $q\in Q$; similarly $\rho(q)\rho(a)$ is the result of the action of $\rho(a)\in\mathrm{O}(k,l)$ on the linear frame $\rho(q)\in P$. Explicitly, applying (3.1) to the μth vector e_μ of the standard frame in $V=\mathbb{R}^n$, one has

$$e_\nu\rho^\nu{}_\mu(a) = \alpha(a)e_\mu a^{-1}, \quad a\in\mathrm{Pin}(k,l), \qquad (4.1)$$

and

$$\rho_\mu(qa) = \rho_\nu(q)\rho^\nu{}_\mu(a), \quad q\in Q, \qquad (4.2)$$

where $\rho_\mu(q)$ is the μth vector of the orthonormal frame $\rho(q)$.

Since the groups $\mathrm{O}(k,l)$ and $\mathrm{O}(l,k)$ are isomorphic, but the groups $\mathrm{Pin}(k,l)$ and $\mathrm{Pin}(l,k)$, in general, are not, one should distinguish from each other the pin structures on nonorientable manifolds corresponding to the covering homomorphisms $\mathrm{Pin}(k,l)\to\mathrm{O}(k,l)$ and $\mathrm{Pin}(l,k)\to\mathrm{O}(k,l)$. For example, it is known[8] that the real projective spaces of dimension $\equiv 0 \bmod 4$ (resp., $2 \bmod 4$) have two inequivalent $\mathrm{Pin}(4k,0)$ structures (resp., $\mathrm{Pin}(0,4k+2)$ structures).

If the manifold M is orientable, then its bundle of frames can be restricted to $\mathrm{SO}(k,l)$. If, moreover, it admits a pin structure, then the bundle Q of "pin frames" can be restricted to the group $\mathrm{Spin}(k,l)=\mathrm{Spin}(l,k)$ and one says that M has a *spin structure*. Conversely, the injections $\mathrm{Spin}(k,l)\to\mathrm{Pin}(k,l)$ and $\mathrm{Spin}(l,k)\to\mathrm{Pin}(l,k)$ can be used to extend a spin structure to two pin structures corresponding to the two coverings of $\mathrm{O}(k,l)$.

In theoretical physics, one often considers the behavior of spinor fields ("wave functions of fermions") under reflections. For this reason, it is appropriate to use pin structures even on orientable manifolds, such as the space-time manifolds of relativistic physics.

The standard, differential geometric notions relative to principal bundles with connections (Ref. 19, Ch. II) can be applied to the bundle $Q\to M$. The *soldering form* on Q is a one-form $\theta=e_\mu\theta^\mu$ with values in $V\subset \mathrm{Cl}(k,l)$ defined as follows: if $u\in T_qQ$, then $\langle u,\theta^\mu(q)\rangle$ is the μth component of the projection of u to M with respect to the linear frame $\rho(q)$. The soldering form is of type ρ,

$$\delta(a)^*\theta^\mu=\rho^\mu{}_\nu(a^{-1})\theta^\nu \quad \text{for} \quad a\in \mathrm{Pin}(k,l), \tag{4.3}$$

where $\delta(a)^*$ denotes the pullback by $\delta(a)$.

The one-parameter subgroup $\exp(t\tau(A)/2)$ of $\mathrm{Spin}(k,l)$, where $A\in\Lambda^2V$, induces the vertical vector field $W(A)$ on Q,

$$W(A)f=\frac{d}{dt}f\circ\delta\left(\exp\frac{1}{2}t\tau(A)\right)\Big|_{t=0},\ f:Q\to\mathbb{R}. \tag{4.4}$$

One puts $W_{\mu\nu}=W(e_\mu\wedge e_\nu)$ and notes $\langle W_{\mu\nu},\theta^\rho\rangle=0$.

Given a representation γ: $\mathrm{Pin}(k,l)\to \mathrm{GL}(S)$, as described in Sec. III, one defines a *spinor field* (Harvey[16] would say: a pinor field) on M as a zero-form of type γ, i.e., a smooth map $\psi:Q\to S$, equivariant with respect to the action of $\mathrm{Pin}(k,l)$,

$$\delta(a)^*\psi=\gamma(a^{-1})\psi \quad \text{for} \quad a\in\mathrm{Pin}(k,l). \tag{4.5}$$

Alternatively and equivalently, a spinor field can be defined as a section of the bundle $E\to M$ associated with $Q\to M$ by the representation γ.

Substituting in (4.5) the function $\exp(te_\mu e_\nu/2)$ for a, differentiating with respect to t, evaluating at $t=0$, and defining

$$\sigma_{\mu\nu}=\tfrac{1}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu) \tag{4.6}$$

one obtains

$$W_{\mu\nu}\psi=-\sigma_{\mu\nu}\psi,\quad 1\leqslant\mu<\nu\leqslant n, \tag{4.7}$$

as the infinitesimal form of the transformation law (4.5).

The Levi–Civita connection on P lifts to a "spinor connection" on Q described by a one-form ω of type ad, with values in the Lie algebra $\mathrm{spin}(k,l)$. Referring to (3.13) one obtains the explicit formula[14,17]

$$\omega=\tfrac{1}{4}\omega^{\mu\nu}e_\mu e_\nu, \tag{4.8}$$

where $(\omega^{\mu\nu})$ is the pullback by $\rho:Q\to P$ of the Levi–Civita, $\mathrm{so}(k,l)$-valued connection one-form on P. Note that $\omega^{\mu\nu}+\omega^{\nu\mu}=0$. Under the action of $\mathrm{Pin}(k,l)$ the connection form transforms according to

$$\delta(a)^*\omega=a^{-1}\omega a \tag{4.9}$$

and it satisfies

$$W(A)\,\lrcorner\,\omega=\tfrac{1}{2}\tau(A). \tag{4.10}$$

The last property is equivalent to

$$W_{\mu\nu}\,\lrcorner\,\omega^{\rho\sigma}=\delta^\rho_\mu\delta^\sigma_\nu-\delta^\sigma_\mu\delta^\rho_\nu. \tag{4.11}$$

According to general theory, the covariant exterior derivative $D\psi$ of a spinor field ψ is the horizontal part of $d\psi$. By virtue of (4.7) and (4.11), the vertical part of $d\psi$ is $-\frac{1}{2}\sigma_{\mu\nu}\omega^{\mu\nu}\psi$ so that

$$D\psi=d\psi+\tfrac{1}{2}\sigma_{\mu\nu}\omega^{\mu\nu}\psi=\theta^\mu\nabla_\mu\psi \tag{4.12}$$

where $\nabla_\mu(\mu=1,...,n)$ is a horizontal vector field on Q, such that $\langle\nabla_\mu,\theta^\nu\rangle=\delta_\mu{}^\nu$. In other words, for every $q\in Q$, the collection of $\frac{1}{2}n(n+1)$ one-forms $(\theta^\mu(q),\omega^{\nu\rho}(q))$ constitutes a linear frame in $T_q{}^*Q$, dual with respect to the linear frame $(\nabla_\mu(q),W_{\nu\rho}(q))$ in T_qQ.

By virtue of (4.3) and (4.5), the covariant derivative of ψ transforms according to

$$\delta(a)^*\nabla_\mu\psi=\gamma(a^{-1})\nabla_\nu\psi\rho^\nu{}_\mu(a). \tag{4.13}$$

Applying the homomorphism $\gamma:\mathrm{Cl}(k,l)\to\mathrm{End}\,S$ to both sides of (4.1) and using (2.10) and (3.9), one obtains

$$\gamma_\nu\rho^\nu{}_\mu(a)=\gamma(a)\gamma_\mu\gamma(a^{-1}). \tag{4.14}$$

Therefore, the *Dirac operator*

$$\nabla\!\!\!\!/\;=\gamma^\mu\nabla_\mu \tag{4.15}$$

transforms in such a way that $\nabla\!\!\!\!/\;\psi$ is a spinor field,

$$\delta(a)^*\nabla\!\!\!\!/\;\psi=\gamma(a^{-1})\nabla\!\!\!\!/\;\psi \quad \text{for} \quad a\in\mathrm{Pin}(k,l). \tag{4.16}$$

It should be noted that the operator (4.15) looks like the standard Dirac operator, but, in fact, involves a modification due to the definition (2.8) of the matrices γ_μ. In the standard approach, one uses the matrices $\gamma(e_\mu)$. The standard Dirac operator behaves well under (is equivariant with respect to) the adjoint representation, whereas the modified operator is adapted to the twisted adjoint representation ρ.

If ψ is a spinor field satisfying (4.5), then $\gamma_{n+1}\psi$ transforms with respect to the twisted representation $\gamma\circ\alpha$. If M has a spin structure, then one can restrict ψ to the reduction of Q to $\mathrm{Spin}(k,l)$ and consider there also the spinor field $\gamma_{n+1}\psi$. In this case, if ψ is an eigenfunction of $\nabla\!\!\!\!/$ with eigenvalue λ, then $\gamma_{n+1}\psi$ is an eigenfunction of $\nabla\!\!\!\!/$ with eigenvalue $-\lambda$: the *spectrum of* $\nabla\!\!\!\!/$ *on orientable manifolds is symmetric* with respect to the origin.

If n is *even*, then γ_{n+1} coincides, up to a factor of $\sqrt{-1}$, with the helicity endomorphisms Γ of S; its eigenvectors are Weyl spinors. If M has a spin structure, then it admits fields of Weyl spinors; if such a field is an eigenfunction of $\nabla\!\!\!\!/$, then the eigenvalue is 0.

If n is *odd*, then the eigenvectors of Γ are Pauli spinors and γ_{n+1} anticommutes with Γ. Since now Γ commutes with γ_μ $(\mu=1,\ldots,n)$, there are Pauli spinor fields on an odd-dimensional manifold with pin structure, even if the manifold is nonorientable. The Dirac operator $\nabla\!\!\!\!/$ commutes with Γ and, therefore, can be restricted to fields of Pauli spinors: this is the *Pauli operator*, generalizing the operator $\sigma\cdot$grad of nonrelativistic quantum mechanics. The spectrum of the Pauli operator need not be symmetric, even if the manifold is orientable.[20]

It is convenient to express the Dirac operator in terms of differential forms on Q. The horizontal volume n-form at $q\in Q$ is

$$\eta(q)=\theta^1(q)\wedge\cdots\wedge\theta^n(q). \tag{4.17}$$

The field η is well defined even if M is nonorientable. It transforms according to

$$\delta(a)^*\eta=\eta\,\det\rho(a)\quad\text{for}\quad a\in\mathrm{Pin}(k,l). \tag{4.18}$$

Putting

$$\eta_\mu=\nabla_\mu\lrcorner\,\eta,\quad \eta_{\mu\nu}=\nabla_\nu\lrcorner\,\eta_\mu,\quad\text{etc.}, \tag{4.19}$$

and noting the formulas

$$\theta^\mu\wedge\eta_\nu=\delta_\nu{}^\mu\eta,\quad \theta^\sigma\wedge\eta_{\mu\nu}=\delta_\nu{}^\sigma\eta_\mu-\delta_\mu{}^\sigma\eta_\nu,\ \text{etc.} \tag{4.20}$$

one checks the validity of

$$\eta\nabla\!\!\!\!/\,\psi=\gamma^\mu D\psi\wedge\eta_\mu\,. \tag{4.21}$$

V. A CONDITION EQUIVALENT TO THE TRIVIALITY OF AN ASSOCIATED BUNDLE

Consider a principal bundle $\pi:P\to M$ with structure group G and a representation γ of G in a vector space S. The homomorphism $\gamma:G\to\mathrm{GL}(S)$ defines an associated vector bundle

$$E=P\times_G S\to M. \tag{5.1}$$

The canonical map $P\times S\to E$ is characterized by $(p,\phi)\to[(p,\phi)]=[(pa,\gamma(a^{-1})\phi)]$, where $p\in P$, $\phi\in S$, $a\in G$, and square brackets denote an equivalence class with respect to the action of G.

Lemma: The bundle E is trivial if and only if there exists a group G', a homomorphism $h:G\to G'$, and an extension $\gamma':G'\to\mathrm{GL}(S)$ of the representation γ such that the associated principal G'-bundle $P\times_G G'\to M$ is trivial.

Indeed, assume first that E is trivial, as a vector bundle, i.e., that there is a trivializing map

$$E\ni[(p,\phi)]\to(\pi(p),f(p)\phi)\in M\times S \tag{5.2}$$

such that

$$f:P\to\mathrm{GL}(S)\quad\text{and}\quad f(pa)=f(p)\circ\gamma(a) \tag{5.3}$$

for every $p\in P$ and $a\in G$. Take now $G'=\mathrm{GL}(S)$, $h=\gamma$, and $\gamma'=\mathrm{id}$ so that γ' extends γ. The bundle $P\times_G G'\to M$ is trivial because it has a global section corresponding to the equivariant map $e:P\to G'$, where $e(p)=f(p)^{-1}$ for every $p\in P$. Conversely, given an extension γ' of γ and a homomorphism $h:G\to G'$ such that $P\times_G G'\to M$ is trivial, there is a map $e:P\to G'$ such that $e(pa)=h(a^{-1})e(p)$ for every $p\in P$ and $a\in G$, If $f:P\to\mathrm{GL}(S)$ is defined by $f(p)=\gamma'(e(p)^{-1})$, then the map $[(p,\phi)]\to(\pi(p),f(p)\phi)$ trivializes the vector bundle $E\to M$.

In Sec. VI the Lemma is applied to prove the triviality of the spinor bundle associated with the pin structure on a hypersurface immersed in flat space.

I give here another example of application of the Lemma. It played a role in motivating the research presented in this paper: during a conference held in Trieste in 1986, I learned from Simone Gutt that her work on Killing spinors implied the triviality of the bundles of Dirac spinors on even-dimensional spheres.

Example: If G is a closed subgroup of G' and $P=G'\to G'/G=M$, then the principal G'-bundle $P\times_G G'\to M$ is trivial because it admits the global section $aG\to[(a,a^{-1})]$, where $a\in G'$. In particular, if $G'=\mathrm{Spin}(n+1)$ and $G=\mathrm{Spin}(n)$, then $M=\mathbb{S}_n$ is the n-dimensional sphere. Since the spinor representation γ of $\mathrm{Spin}(n)$ in S extends to a representation of γ' of $\mathrm{Spin}(n+1)$, as described in Sec. III, our Lemma proves the triviality of the bundle of spinors

$$\mathrm{Spin}(n+1)\times_{\mathrm{Spin}(n)}S$$

associated with the spin structure $\mathrm{Spin}(n+1)\to\mathrm{SO}(n+1)\to\mathbb{S}_n$ of the n-sphere.[21]

VI. PIN STRUCTURES ON HYPERSURFACES

A simple topological argument[22] shows that a hypersurface in a Euclidean or Lorentzian space has a pin structure. To be more precise, consider an n-dimensional hypersurface (Ref. 18, Ch. VIII), i.e., a connected n-dimensional manifold M with an immersion $f:M\to\mathbb{R}^{n+1}$. Assume that M has a proper Riemannian metric induced from $\mathbb{R}^{n+1}$ by f. Denoting by $T^\perp M$ the line bundle normal to TM, we have that $TM\oplus T^\perp M$ is trivial and its total Stiefel–Whitney class w is equal to 1. From the Whitney product theorem[23] one obtains

$$1=w(TM\oplus T^{\perp}M)$$
$$=1+w_1(TM)+w_1(T^{\perp}M)+w_1(TM)w_1(T^{\perp}M)$$
$$+w_2(TM). \tag{6.1}$$

Therefore,

$$w_1(TM)=w_1(T^{\perp}M) \quad \text{and} \quad w_1(TM)^2+w_2(TM)=0. \tag{6.2}$$

There are two cases to consider, depending on the signature of the metric in the ambient space:

(a) $\mathbb{R}^{n+1}$ is the flat Euclidean space with the metric defined by the standard, positive-definite scalar product. The hypersurface M need not be orientable and the second equality in (6.2) is the condition for the existence of a $\mathrm{Pin}(0,n)$-structure on M.[24]

(b) $\mathbb{R}^{n+1}$ is the flat Lorentzian space with the metric defined by a scalar product of signature $(n,1)$. Assume that the normal bundle is timelike, i.e., its fibers are spanned by vectors with negative squares. The metric induced on M by f is then proper Riemannian and the bundle $T^{\perp}M$ is trivial. Therefore, M is orientable, $w_1(TM)=0$, and Eq. (6.2) shows that it has a spin structure, $w_2(TM)=0$.

Only case (a) is considered from now on. The notation

$$\mathrm{Pin}(n) \quad \text{instead of} \quad \mathrm{Pin}(0,n) \tag{6.3}$$

is used and "pin structure" means "$\mathrm{Pin}(0,n)$-structure."

Let $\pi:P\to M$ be the principal $\mathrm{O}(n)$-bundle of all orthonormal frames on the hypersurface M isometrically immersed in the flat Euclidean space $\mathbb{R}^{n+1}$. One extends it to the trivial bundle $P'=M\times\mathrm{SO}(n+1)$ by defining the embedding $k:P\to P'$ as follows. Let (e_i), $i=1,\dots,n+1$, be the standard frame in the vector space $V'=\mathbb{R}^{n+1}$. The dual frame (e^i) is characterized by $\langle e_j,e^i\rangle=\delta_j^i$. Denote by f_*u the image of $u\in T_mM$ by the map tangent to f at $m\in M$ and consider the vector f_*u as an element of V'. If $p=(p_\mu)$, $\mu=1,\dots,n$, is an element of P, then one puts $F_\mu(p)=f_*p_\mu$ and defines another unit vector $F_{n+1}(p)$ by requiring it to be orthogonal to $f_*(T_mM)$ in V' and such that the collection $(F_i(p))$, $i=1,\dots,n+1$, is a frame in V' of the same orientation as that of (e_i). This gives a map

$$F=(F^i_j):P\to\mathrm{SO}(n+1), \quad F^i_j(p)=\langle F_j(p),e^i\rangle, \tag{6.4}$$

such that $F(pA)=F(p)H(A)$, where $A\in\mathrm{O}(n)$ and H is as in (3.11). Note that

$$F_{n+1}(pA)=(\det A)F_{n+1}(p) \quad \text{for} \quad A\in\mathrm{O}(n), \tag{6.5}$$

i.e., the normal vector $F_{n+1}(p)$ changes sign when the orientation of the frame p is reversed.

The map $k:P\to P'$ defined by $k(p)=(\pi(p),F(p))$ is an injective homomorphism of principal bundles over M. The manifold P' is doubly covered by the manifold $Q'=M\times\mathrm{Spin}(n+1)$. Consider the induced bundle

$$Q=\{(m,a)\in Q'\,|\,(m,\rho(a))\in k(P)\} \tag{6.6}$$

and denote by l the canonical injection of Q into Q'. Since k is injective, there is a map $\rho:Q\to P$ such that $k\circ\rho(m,a)=(m,\rho(a))$. This makes Q into a double cover of P. Defining the action of $\mathrm{Pin}(n)$ in Q by $(m,a)b=(m,ah(b))$, where $b\in\mathrm{Pin}(n)$ and h is as in (3.10), one establishes the commutativity of the diagram

$$\begin{tikzcd} Q\times\mathrm{Pin}(n) \arrow[r] \arrow[d] & Q \arrow[d] \arrow[dr] & \\ & & M \\ P\times\mathrm{O}(n) \arrow[r] & P \arrow[ur] & \end{tikzcd} \tag{6.7}$$

and thus completes the construction of a pin structure on the hypersurface M. The relations between the bundles Q, Q', $\mathbb{R}^{n+1}\times\mathrm{Spin}(n+1)$, etc., are summarized in the diagram

$$\begin{tikzcd} Q \arrow[r, "l"] \arrow[d, "\rho"'] & Q' \arrow[r] \arrow[d] & \mathbb{R}^{n+1}\times\mathrm{Spin}(n+1) \arrow[d] \\ P \arrow[r, "k"] \arrow[dr] & P' \arrow[r] \arrow[d] & \mathbb{R}^{n+1}\times\mathrm{SO}(n+1) \arrow[d] \\ & M \arrow[r, "f"] & \mathbb{R}^{n+1} \end{tikzcd} \tag{6.8}$$

Consider now the bundle of spinors $E\to M$ associated with the principal $\mathrm{Pin}(n)$-bundle $Q\to M$ by the representation $\gamma:\mathrm{Pin}(n)\to\mathrm{GL}(S)$. The elements of E are equivalence classes $[(m,a,\phi)]$ of elements of $Q\times S\subset M\times\mathrm{Spin}(n+1)\times S$, the equivalence being given by

$$(m,a,\phi)\equiv(m,ah(b),\gamma(b^{-1})\phi), \tag{6.9}$$

where $(m,a)\in Q$, $\phi\in S$, and $b\in\mathrm{Pin}(n)$. Since, according to (2.7), the representation $\gamma':\mathrm{Spin}(n+1)\to\mathrm{GL}(S)$ extends γ, the situation is covered by the Lemma of Sec. V and *the bundle $E\to M$ is trivial.* A trivializing map is given by

$$[(m,a,\phi)]\to(m,\gamma'(a)\phi). \tag{6.10}$$

Let $\psi:Q\to S$ be a spinor field; by adapting (4.5) to the present situation one can write

$$\psi(m,ah(b))=\gamma(b^{-1})\psi(m,a), \tag{6.11}$$

where $(m,a)\in Q$ and $b\in\mathrm{Pin}(n)$. Therefore, one can extend ψ, in a unique manner, to a map ψ': $Q'\to S$ such that

$$\psi'\circ l=\psi \quad\text{and}\quad \psi'(m,ab)=\gamma'(b^{-1})\psi'(m,a) \tag{6.12}$$

for every $m\in M$ and $a,b\in\mathrm{Spin}(n+1)$. Consider the *standard section* s of the trivial bundle $Q'\to M$,

$$s:M\to Q'=M\times\mathrm{Spin}(n+1),\quad s(m)=(m,1), \tag{6.13}$$

where 1 is the unit element of $\mathrm{Spin}(n+1)$. The corresponding section of the trivial bundle $P'\to M$ has a simple geometrical meaning: it defines over M the constant field of standard frames (e_i). Defining

$$\Psi:M\to S \quad\text{by}\quad \Psi=\psi'\circ s, \tag{6.14}$$

one introduces a "spinor field" Ψ globally defined on the base M, with values in the typical fiber S of E, rather than in the fibers of E. The Dirac operator can be translated to an operator acting on Ψ that is denoted by ∇.

It is clear that the result on the existence of a pin structure on M is valid under the weaker assumption that M is a hypersurface in a spin manifold N. If the principal bundle of spin frames on N is trivial, then the associated vector bundle of Dirac or Pauli spinors on M is also trivial, but, in general, the bundle of Weyl spinors on an even-dimensional hypersurface is not.

VII. NEW FORMULAS FOR THE DIRAC AND LAPLACE OPERATORS ON HYPERSURFACES

If ψ is a spinor field on a hypersurface M in $\mathbb{R}^{n+1}$, then so is the field $\boldsymbol{\nabla}\psi$ and one can extend the latter field, in the manner indicated in (6.12), to $(\boldsymbol{\nabla}\psi)':Q'\to S$. One then puts

$$\nabla\Psi=(\boldsymbol{\nabla}\psi)'\circ s, \tag{7.1}$$

where s and Ψ are as in (6.13) and (6.14), respectively. It is clear that (7.1) defines a linear differential operator of the first order acting on S-valued functions on M. To compute it explicitly, one can proceed according to the following plan: first, one extends $D\psi$ to a one-form $(D\psi)'$ of type γ' on Q'. Second, one uses Eq. (4.21), and its extension to Q', to write the Dirac operator. Third, the extension is pulled back to M by means of the standard section $s:M\to Q'$.

According to the general theory of connections on principal bundles (Ref. 18, Ch. II), the $\mathrm{spin}(n)$-valued connection form ω on Q extends to the $\mathrm{spin}(n+1)$-valued connection form ω' on Q', $\omega'=\frac{1}{4}\omega'^{ij}e_ie_j$ $(i,j=1,\dots,n+1)$, where $\omega'^{ij}+\omega'^{ji}=0$ and

$$l^*\omega'^{in+1}=0,\quad l^*\omega'^{\mu\nu}=\omega^{\mu\nu}. \tag{7.2}$$

Therefore, the extension $(D\psi)'$ of the derivative $D\psi$ can be written as

$$(D\psi)'=d\psi'+\tfrac{1}{2}\sigma_{ij}\omega'^{ij}\psi', \tag{7.3}$$

where

$$\sigma^{ij}=\tfrac{1}{4}(\gamma^i\gamma^j-\gamma^j\gamma^i). \tag{7.4}$$

To alleviate the exposition, assume that M is orientable and let P_0 be a restriction of P to $\mathrm{SO}(n)$; then $Q_0=\rho^{-1}(P_0)$ is the corresponding restriction of Q to $\mathrm{Spin}(n)$. The hypersurface M has now a well-defined field X of unit normals given by

$$X(\pi(p))=F_{n+1}(p) \quad\text{for}\quad p\in P_0 \tag{7.5}$$

and[25]

$$s^*\omega'^{ij}=X^idX^j-X^jdX^i, \tag{7.6}$$

where X^j is the jth component of X with respect to (e_i). Denoting by l_0 the injection of Q_0 into Q', one can introduce a V'^*-valued n-form (η_i) on Q' in such a way that

$$l_0^*\eta_i=\delta_i^{n+1}\eta \quad\text{and}\quad \delta(a)^*\eta_i=\eta_j\rho_i^j(a) \tag{7.7}$$

for $a\in\mathrm{Spin}(n+1)$. Similarly, there is the $\Lambda^2\, V'^*$-valued $(n-1)$-form (η_{ij}) on Q' such that

$$l_0^*\eta_{ij}=(\delta_i^{n+1}\delta_j^\mu-\delta_j^{n+1}\delta_i^\mu)\eta_\mu,\ \text{etc.} \tag{7.8}$$

Multiplying both sides of Eq. (4.21) on the left by γ^{n+1} one finds that the resulting equation extends to Q' to yield

$$\gamma^i\eta_i(\nabla\psi)'=\sigma^{ij}(D\psi)'\wedge\eta_{ij}. \tag{7.9}$$

Pulling both sides of this equation back to M by the section s, using (7.3), (7.6), and properties of the sigma matrices resulting from

$$\gamma_i\gamma_j+\gamma_j\gamma_i=-2\delta_{ij}\quad (i,j=1,\dots,n+1), \tag{7.10}$$

one obtains the new form of the *Dirac operator on the hypersurface M*,

$$\nabla\Psi=(\gamma_kX^k)(\sigma^{ij}X_{ij}-\tfrac{1}{2}\operatorname{div}X)\Psi, \tag{7.11}$$

where $\operatorname{div}X=\partial_kX^k$,

$$X_{ij}=X_j\partial_i-X_i\partial_j, \tag{7.12}$$

and $\partial_i=\partial/\partial x^i$ denotes differentiation with respect to the Cartesian coordinate x^i in $\mathbb{R}^{n+1}$, i.e., $\partial_i\Psi=e_i\,\lrcorner\, d\Psi$. The vector indices $i,j,k=1,\dots,n+1$ relative to V' are raised and lowered by means of the Euclidean metric tensor with components δ_{ij}. For example, $(\gamma_k X^k)^2 = -X_k X^k = -1$. If M is the hyperplane $x^{n+1}=0$, then $X_i=\delta_i^{n+1}$ and $\nabla\Psi=\gamma^\mu\partial_\mu\Psi$.

The $\frac{1}{2}n(n+1)$ vector fields (differential operators) $X_{ij}(1<i<j<n+1)$ are tangent to M; they satisfy the identity $X_{[k}X_{ij]}=0$ and the integrability condition $X_{[ij}X_{k]}=0$. The latter condition is used in the derivation of (7.11) from (7.9).

The anticommutativity of the operator (4.15) with γ_{n+1} results in

$$(\gamma_k X^k)\nabla + \nabla(\gamma_k X^k) = 0. \tag{7.13}$$

Note that the right sides of Eqs. (7.6) and (7.11) are invariant with respect to the replacement of X by $-X$: this shows that the assumption of orientability is irrelevant and the formula for the Dirac operator holds good also for nonorientable M.

Assume now that there is (at least locally) a *foliation* of $\mathbb{R}^{n+1}$ by a family of hypersurfaces. The field of unit normals X is now defined on (an open subset of) $\mathbb{R}^{n+1}$. From $2\sigma_{ij}=\delta_{ij}+\gamma_i\gamma_j$ one obtains the identity

$$\gamma^i\partial_i = (\gamma_k X^k)(X^i\partial_i + \sigma^{ij}X_{ij}). \tag{7.14}$$

Introducing the derivative in the "radial" direction, $\partial/\partial r = X^i\partial_i$ one obtains a decomposition of the Dirac operator in $\mathbb{R}^{n+1}$ into parts tangential and transverse to the foliation,

$$\gamma^i\partial_i = \nabla + (\gamma_k X^k)\left(\frac{\partial}{\partial r} + \frac{1}{2}\,\mathrm{div}\,X\right). \tag{7.15}$$

There is an analogous formula for the Laplacian,

$$\sum_{i=1}^{n+1} \partial_i^2 = \frac{1}{2} X_{ij}X^{ji} + \frac{\partial^2}{\partial r^2} + (\mathrm{div}\,X)\,\frac{\partial}{\partial r}. \tag{7.16}$$

The first term on the right side of Eq. (7.16) is the Laplacian on the hypersurface; it generalizes the often-used formula connecting the Laplacian on the two-sphere with the square of the quantum-mechanical operator of orbital momentum.

Formula (7.15) bears a close relation to Eq. (4.1) of Ref. 20. Spinor fields on hypersurfaces have been recently considered by Baum[26] in the context of Killing spinors; the physical aspects of spectral asymmetry of the Dirac operator were investigated by Gibbons.[27]

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