Simple spinors and real structures

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A concept of a real index associated with any maximal totally null subspace in a complexified vector space endowed with a scalar product, and also with any complex simple (pure) spinor, is introduced and elaborated. It is shown that the real Pin group acts transitively on the projective space of simple spinors with any given real index. The connection between the real index and multivectors associated with a simple spinor and its charge conjugate is established. The simple spinors with extreme values of the real index deserve a special attention: The generic simple spinors for which the real index is minimal and simple-r spinors for which it is maximal.

I. INTRODUCTION

A corollary to the classical Witt theorem[1] asserts that the group of orthogonal automorphisms of a vector space W with a scalar product acts transitively on the set of all totally null (isotropic) subspaces of a given dimension. In particular, if W is complex $2m$-dimensional, then the set of all *maximal* (i.e., m-dimensional) *totally null* (MTN) subspaces has the structure of a complex, $\frac{1}{2}m(m-1)$-dimensional manifold Q_m, diffeomorphic to[2] $\mathrm{O}(2m)/\mathrm{U}(m)$. The space Q_m is homogeneous also under the action of the *real* orthogonal group $\mathrm{O}(2m)$. This observation goes beyond the corollary to the Witt theorem, which asserts transitivity only under the action of the larger, *complex* orthogonal group. The latter group has several "real forms" $\mathrm{O}(k,l)$, $k+l=2m$, besides the compact group $\mathrm{O}(2m)=\mathrm{O}(2m,0)$. It is clear that the "pseudo-orthogonal" groups, i.e., such that $kl\neq 0$, in general, do not act transitively on the set of all totally null (TN) subspaces of W of a given dimension. In this paper we solve the problem of decomposing Q_m into orbits of $\mathrm{O}(k,l)$: each such orbit is characterized by the dimension r of the intersection $N\cap\bar{N}$, where $\bar{N}$ is the complex conjugate of the MTN space $N\in Q_m$, considered as a subspace of the complexification W of a real, $2m$-dimensional vector space V with a scalar product of the signature (k,l). We call r the *real index* of N. In Sec. II we show that

$$r\equiv k \bmod 2\equiv l \bmod 2,\quad 0\leqslant r\leqslant \min(k,l),$$

and there is one orbit of the real dimension $m(m-1)-(r/2)(r-1)$ for every value of r, satisfying the above conditions.

For every subspace N of a complexified vector space, we denote by $\Re N$ the vector space consisting of real parts of all vectors belonging to N. We emphasize than $\Re N$ is usually larger than the vector space consisting of all real vectors belonging to N. If $N\in Q_m$ has real index r, then

$$\tilde{N}=\Re(N/(N\cap\bar{N}))$$

is a real vector space of dimension $2(m-r)$. It has a complex structure given by the map $\tilde{N}\ni n\mapsto n\,\lrcorner\,\tilde{j}\in\tilde{N}$, where $\tilde{j}\in\Lambda^2\tilde{N}$ is the *Kähler bivector* whose representatives $j\in\Lambda^2 W$ are described in Sec. III. This complex structure is compatible with the metric structure induced in $\tilde{N}$ by that in W. Generically, the real index r is either 0 or 1. In the first case, one has $\tilde{N}=V$ and $\tilde{j}$ determines a metric complex structure in V. In the second, $K=\Re(N\cap\bar{N})$ is a real null direction, a "ray," and $\tilde{j}$ determines a complex structure in the generalized "screen space" $\tilde{N}$ of the optical geometry.[3,4]

The problem we are considering is closely related to, and motivated by, the question of giving a useful interpretation of simple spinors in the real case. Recall that Cartan[5] introduced the notion of simple spinors (Chevalley[6] called them "pure") associated with complex vector spaces: a spinor $\varphi\neq 0$ is *simple* if the TN subspace $M(\varphi)=\{u\in W\colon u\varphi=0\}$ is maximal. Here, $u\varphi$ denotes the evaluation on φ of the endomorphism associated with u by the faithful irreducible representation of the complex Clifford algebra $\mathrm{Cl}(2m)$ in the complex 2^m-dimensional space S of Dirac spinors. In the traditional physicist's notation, one writes $u\varphi=u^\mu\gamma_\mu\varphi$, where γ_μ ($\mu=1,\dots,2m$) are the generalized Dirac matrices subject to $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$; see Refs. 7 and 8 for details. Cartan showed that, if φ is simple, then it is a Weyl spinor, i.e., it is an eigenvector of $\gamma_1\cdots\gamma_{2m}$, the eigenvalue called its helicity. There is a natural, bijective correspondence between Q_m and the set of directions of sim-

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ple spinors. One can assign a helicity to the elements of Q_m; this manifold is a disjoint sum of two connected manifolds containing MTN subspaces of definite helicity; each of them is diffeomorphic to $SO(2m)/U(m)$.

The embedding $V \to W$ of the real vector space V into its complexification induces an antilinear (semilinear) map $\varphi \mapsto \varphi_c$, where φ_c is the "charge conjugate" of the spinor φ; see Ref. 7 and Sec. III. We show that $M(\varphi_c) = \overline{M(\varphi)}$ and, therefore, the simplicity of φ is equivalent to that of φ_c. We say that φ is simple of real index r if $M(\varphi)$ is an MTN subspace of real index r. Defining a simple spinor to be generic if its real index is minimal (0 or 1), we obtain that the set of directions of generic simple spinors is open in Q_m; it coincides with Q_m, if, and only if, k or $l \leqslant 1$. Cartan's characterization of simple spinors by properties of multivectors, constructed bilinearly out of spinors, can be extended to the real case (Sec. III). Namely, let $B_p(\phi, \psi)$ be the p-vector associated with any $\phi, \psi \in S$, and suppose that $\varphi \in S$ is simple. Then φ has the real index equal to r if, and only if, $B_r(\varphi_c, \varphi) \neq 0$ and $B_{r-2}(\varphi_c, \varphi) = 0$. Moreover, $B_r(\varphi_c, \varphi)$ is proportional to exterior product of vectors spanning $N \cap \overline{N}$ and $B_{r+2p}(\varphi_c, \varphi)$ is proportional to $\wedge^p j \wedge B_r(\varphi_c, \varphi)$. In particular, if $r = 0$, then $B_2(\varphi_c, \varphi) = -iB(\varphi_c, \varphi) j \neq 0$.

Our results imply an interesting analogy between complex and optical geometries. Consider an even-dimensional pseudo-Riemannian manifold $\mathscr{M}$. Call the manifold pseudo-Euclidean if the signature of the metric tensor is $(2\kappa, 2\lambda)$ and pseudo-Lorentzian if it is $(2\kappa + 1, 2\lambda + 1)$, where κ and λ are non-negative integers. One drops the prefix "pseudo" if $\kappa\lambda = 0$. Giving a field of directions of generic simple spinors on a pseudo-Euclidean manifold is equivalent to endowing it with an almost complex structure. Similarly, such a field on a pseudo-Lorentzian manifold determines a (generalized) *optical geometry*,[3,4] i.e., a pair of real vector bundles $\mathscr{K}$ and $\mathscr{L}$ such that $\mathscr{K} \subset \mathscr{L} \subset T\mathscr{M}$, the bundles $\mathscr{K}$ and $\mathscr{L}$ have fibers of dimension 1 and codimension 1, respectively, and there is given a metric complex structure in the fibers of the bundle $\mathscr{L}/\mathscr{K}$. According to Hughston and Mason,[9] both the integrability of the almost complex structure and the shear-free geodetic property of the optical geometry are equivalent to the Penrose–Sommers condition,[10,11]

$$\varphi \wedge \nabla_u \varphi = 0.$$

Above, φ is any (local) field of simple spinors which determines the structure and $\nabla_u \varphi$ denotes its covariant derivative in the direction of an arbitrary section u of the bundle $\mathscr{N} \to \mathscr{M}$ of MTN spaces defined by φ, the fiber is $\mathscr{N}_x = M(\varphi(x))$ for every $x \in \mathscr{M}$. The analogy between the complex and optical geometries will be pursued further in subsequent publications.

II. MAXIMAL TOTALLY NULL SUBSPACES IN A COMPLEXIFIED SPACE

We consider a real vector space V of an even dimension $2m$, endowed with a scalar product, and its complexification $W = \mathbf{C} \otimes V$. The scalar product will be denoted by the central dot; its signature in V is (k,l), $k + l = 2m$. The Clifford product will be denoted by $\vee$.

Let N be a TN subspace of W. There are two interesting canonical subspaces of N. First, this is $N \cap \overline{N} = \{u \in N: \bar{u} \in N\} = \mathbf{C} \otimes K$, where K is the space of all real vectors contained in N; $K = N \cap V$. The complex dimension of $N \cap \overline{N}$ equal to the real dimension of K will be denoted r and called the *real index* of N.

The second canonical subspace of N is

$${}^{\perp}N = \{u \in N: \bar{u} \cdot v = 0 \forall v \in N\} = N \cap \overline{N}^{\perp}.$$

Notice that, since N is TN, $N \cap \overline{N} \subset {}^{\perp}N$. Let M be a subspace complementary to $N \cap \overline{N}$ in N, $N = \mathbf{C} \otimes K \oplus M$. Decompose M into the direct sum, $M = M_0 \oplus M_+ \oplus M_-$, where M_0, M_+, and M_- are subspaces of M, orthogonal each to the other, such that $M_0 \subset {}^{\perp}N$ and the Hermitian scalar product

$$(u,v) \mapsto \bar{u} \cdot v$$

is positive and negative definite for M_+ and M_-, respectively. In effect, we have

$$N = \mathbf{C} \otimes K \oplus M_0 \oplus M_+ \oplus M_- \tag{1}$$

and the dimensions r, n_0, n_+, and n_- of corresponding subspaces in the decomposition (1) do not depend on the choice of M.

The real parts of all vectors belonging to N form a subspace $\Re N$ of V that is the direct sum of subspaces

$$\Re N = K \oplus \Re M_0 \oplus \Re M_+ \oplus \Re M_- \tag{2}$$

of dimensions r, $2n_0$, $2n_+$, and $2n_-$, respectively. The signature of the scalar product in $\Re M_+$ is $(2n_+, 0)$ and in $\Re M_-$ it is $(0, 2n_-)$; immediate restrictions on the dimensions of the decomposition (1) are

$$2n_+ \leqslant k, \quad 2n_- \leqslant l. \tag{3}$$

In the orthogonal complement of $\Re M_+ \oplus \Re M_-$ in V, the signature of the scalar product is $(k - 2n_+, l - 2n_-)$. Since the sum of the first two components in (2) is a real TN subspace of this orthogonal complement, the inequalities

$$r + 2n_0 \leqslant k - 2n_+, \quad r + 2n_0 \leqslant l - 2n_- \tag{4}$$

must hold. The inequalities (3) and (4) are the only restrictions on the non-negative integers r, n_0, n_+, and

n_- associated with any TN subspace of W; if they are fulfilled, we can construct a corresponding TN subspace N.

From now on, let N be an MTN subspace of W. Then, we have

$$r+n_0+n_+ +n_- =m. \tag{5}$$

The equality (5), if compared with the inequalities (4), gives

$$n_0=0,\quad 2n_+ =k-r,\quad 2n_- =l-r. \tag{6}$$

The formulas (6) and (3) imply that the admissible values of the real index of an MTN subspace are

$$r=\begin{cases}0,2,\ldots,\min(k,l), & \text{for } k \text{ and } l \text{ even;}\\ 1,3,\ldots,\min(k,l), & \text{for } k \text{ and } l \text{ odd.}\end{cases}$$

Since the scalar product in the orthogonal complement of $\Re M$ in V has the neutral signature (r,r), there exists its MTN subspace L complementary to the MTN subspace K,

$$V=K\oplus L\oplus \Re M. \tag{7}$$

We shall use

(k_j)—a basis of K, $j=1,\ldots,r$,

(l_j)—a basis of L, $j=1,\ldots,r$,

(m_α)—a basis of M_+, $\alpha=1,\ldots,n_+$,

(n_β)—a basis of M_-, $\beta=1,\ldots,n_-$.

They can be chosen in such a way that their nonvanishing scalar products are

$$k_i\cdot l_j=\tfrac{1}{2}\delta_{ij},\quad \bar{m}_\alpha\cdot m_\beta=\tfrac{1}{2}\delta_{\alpha\beta},\quad \bar{n}_\alpha\cdot n_\beta=-\tfrac{1}{2}\delta_{\alpha\beta}.$$

Although for an MTN subspace N the equality ${}^\perp N=N\cap\bar{N}$ holds, for its proper subspace $J\subset N$, the inequality ${}^\perp J\neq J\cap\bar{J}$ can occur. Let I be a subspace of ${}^\perp J$ complementary to $J\cap\bar{J}$, ${}^\perp J=J\cap\bar{J}\oplus I$. It can be decomposed,

$$I=G\oplus H, \tag{8}$$

where $G=I\cap(\mathbf{C}\otimes K)$ and $H=I\cap M$. Let μ and ν denote the dimensions of the respective components in (8). Since G cannot contain a real vector, each basis of this space will have the form

$$(k_{2j-1}+ik_{2j}),\quad j=1,\ldots,\mu, \tag{9}$$

where (k_j), $j=1,\ldots,2\mu$, is a sub-basis of K. Consistently with the decomposition $M=M_+\oplus M_-$; each basis of H can be written as

$$(m_\alpha+n_\alpha),\quad \alpha=1,\ldots,\nu, \tag{10}$$

where (m_α) and (n_α), $\alpha=1,\ldots,\nu$, are sub-bases of M_+ and M_-, respectively. Since $(\bar{m}_\alpha+\bar{n}_\alpha)\cdot(m_\beta+n_\beta)=\bar{m}_\alpha\cdot m_\beta+\bar{n}_\alpha\cdot n_\beta=0$, they can be orthonormalized simultaneously,

$$\bar{m}_\alpha\cdot m_\beta=\tfrac{1}{2}\delta_{\alpha\beta}=-\bar{n}_\alpha\cdot n_\beta.$$

The only restrictions on the non-negative integers μ and ν are

$$2\mu\leqslant r,\quad 2\nu\leqslant 2\min(n_+,n_-)=\min(k-r,l-r).$$

If, moreover, $\dim J=\iota$ is given, the further quite obvious restriction on μ and ν is

$$\mu+\nu\leqslant\min(\iota,m-\iota). \tag{11}$$

Theorem 1: Let N and P be two MTN subspaces of $W=\mathbf{C}\otimes V$ such that $\dim(N\cap P)=m-1$. Then the following four conditions are equivalent:

(I) ${}^\perp(N\cap P)=N\cap P\cap\overline{N\cap P}$;

(II) N and P have the same real index;

(III) there exists a real vector $u\in N+P$ such that $u^2=\pm 1$;

(IV) there exists a real reflection mapping N onto P, $P=u\vee N\vee u^{-1}$.

Proof: Put $J=N\cap P$ and consider it as a subspace of N. The restrictions (11) imply

$$\mu+\nu\leqslant 1. \tag{12}$$

The first solution of (12) is $\mu=\nu=0$, which means that condition I, i.e., ${}^\perp J=J\cap\bar{J}$ is satisfied. Take a vector $q\in N-J$ such that $\bar{q}\cdot u=0$ for any $u\in J$; such a vector exists and is determined up to the transformations

$$q\to aq+n,\quad \text{where } 0\neq a\in\mathbf{C},\quad n\in J\cap\bar{J}. \tag{13}$$

The vector q must belong to one of the following three classes:

$$\begin{array}{llll}(1) & \bar{q}\cdot q=\tfrac{1}{2}, & q=m_1, & (r,n_+-1,n_-),\\ (2) & \bar{q}\cdot q=-\tfrac{1}{2}, & q=n_1, & (r,n_+,n_--1),\\ (3) & \bar{q}=q, & q=k_1, & (r-1,n_+,n_-).\end{array}$$

In the left column above we used the freedom left by the transformations (13). The middle column contains the vector q expressed in terms of a suitably chosen basis of

N. The right column contains signatures of the Hermitian scalar product in J, if its signature in N is (r,n_+,n_-).

Any vector $p\in P-N$ will satisfy $p\cdot q\neq 0$, since, otherwise, $N+P$ would be a TN subspace of the dimension $m+1$. On the other hand, any solution of the equations

$$p\cdot q=\tfrac{1}{2},\quad p\cdot J=\{0\},\quad p^2=0, \tag{14}$$

will be given up to the transformations

$$p\mapsto p+n,\quad \text{where } n\in J. \tag{15}$$

Each vector admitted by the transformations (15) will belong to $P-N$; it is therefore sufficient to find one solution of (14):

(1) $p=\bar{m}_1,\quad u=q+p,\quad u^2=1,$

(2) $p=\bar{n}_1,\quad u=q-p,\quad u^2=-1,$

(3) $p=l_1,\quad u=q\pm p,\quad u^2=\pm 1.$

In the middle and the right column above, the vector u, having all properties required by condition III is constructed. It is clear also that the signature of the Hermitian scalar product in P is (r,n_+,n_-) in any of the three classes, as it is required by condition II.

The second solution of (12) is $\mu=1$, $\nu=0$. Consistently with (8), we can choose $q=k_1-ik_2$ as an element of $N-J$. Then as a solution of (14), we can take

$$p=\tfrac{1}{2}(l_1+il_2). \tag{16}$$

One can easily check that any vector $u\in N+P$ such that $u^2\neq 0$ must be complex. Since the matrix

$$\begin{pmatrix} \bar{p}\cdot p & \bar{p}\cdot\bar{q} \\ q\cdot p & q\cdot\bar{q}\end{pmatrix}=\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}$$

has the eigenvalues $\pm\frac{1}{2}$ and they determine the signature of the Hermitian scalar product in span$\{\bar{q},p\}$, its signature in P is $(r-2,n_++1,n_-+1)$.

The third and last solution of (12) is $\mu=0$, $\nu=1$. Consistently with (10), we can choose $q=m_1-n_1$ as an element of $N-J$. The vector

$$p=\tfrac{1}{2}(\bar{m}_1+\bar{n}_1) \tag{17}$$

will be a solution of (14). Real, non-null elements of $N+P$ do not exist. The space span$\{p,\bar{p}\}$ is TN with respect to the Hermitian scalar product, therefore its signature in P is $(r+2,n_+-1,n_--1)$.

Thus the equivalence of conditions I, II, and III is proven. Condition IV implies condition II, since any real isomorphism will preserve r. To show that condition III implies condition IV, let us introduce two simple spinors ω and φ such that their corresponding MTN subspaces are

$$N=M(\omega)\equiv\{u\in W\colon u\omega=0\}\quad\text{and}\quad P=M(\varphi).$$

Then (cf. Proposition 4 in Ref. 8) $\varphi\sim u\omega$, and $u\in\mathrm{Pin}(k,l)$ generates the reflection $\rho_u\in\mathrm{O}(k,l)$ such that $P=\rho_u N=u\vee N\vee u^{-1}$. Q.E.D.

The proof of the following lemma, generalizing (14) and (15), will be left to the reader.

Lemma: Let (n_j), $j=1,...,m$, be a basis of a given MTN subspace N. There exist vectors (p_j), $j=1,...,m$, which span an MTN subspace and satisfy

$$n_j\cdot p_k=\tfrac{1}{2}\delta_{jk};$$

they are given up to the transformations

$$p_j\mapsto p_j+\sum_k Z_{jk}n_k,$$

where the coefficients $Z_{jk}=-Z_{kj}$ are otherwise arbitrary complex numbers.

Theorem 2: The group $\mathrm{O}(k,l)$ [resp., $\mathrm{SO}(k,l)$] acts transitively on each set of all MTN subspaces of $W=\mathbf{C}\otimes V$ with a given real index (resp., with a given real index and with a given helicity).

Proof: Let N and P be two MTN subspaces of W and consider $J=N\cap P$ as a subspace of N.

Suppose that

$$\mu+\nu<m-\iota \tag{18}$$

[cf. (11)]. Then there exists a vector $q\in N-J$ which belongs to one of the classes (1)–(3) discussed in the proof of Theorem 1. For an orthogonal complement J' of span$\{q\}$ in N, we have $\mu'=\nu'=0$. We can now take $p\in P$ such that the conditions (14) hold. According to Theorem 1, there exists a reflection ρ_{u_1} that maps N onto $N_1=J'\oplus\mathrm{span}\{p\}$. The subspace $J_1=N_1\cap P$ has the dimension $\iota_1=\iota+1$. We can repeat this procedure as long as the strong inequality (18) holds. After i steps the MTN subspace N_i is formed out of N by means of a real orthogonal transformation consisting of i real reflections.

Thus the proof is reduced to the case

$$\mu+\nu=m-\iota. \tag{19}$$

In the case (19), the subspace of N spanned by

$$k_{2j-1}-ik_{2j},\quad j=1,...,\mu,\quad\text{and}\quad m_\alpha-n_\alpha,\quad \alpha=1,...,\nu$$

[cf. (9) and (10)], is complementary to $J=N\cap P$. According to (16), (17), and the lemma, the corresponding Fock basis of P/J is

$$p_j=\frac{1}{2}(l_{2j-1}+il_{2j})+\sum_k Z_{jk}(k_{2k-1}-ik_{2k})$$

$$+\sum_\beta Z_{j\beta}(m_\beta-n_\beta),$$

$$p_\alpha=\frac{1}{2}(\bar{m}_\alpha+\bar{n}_\alpha)+\sum_k Z_{\alpha k}(k_{2k-1}-ik_{2k})$$

$$+\sum_\beta Z_{\alpha\beta}(m_\beta-n_\beta),$$

where $Z_{\alpha\beta}=-Z_{\beta\alpha}$, $Z_{\alpha k}=-Z_{k\alpha}$, and $Z_{jk}=-Z_{kj}$.

We are entitled to perform the following transformation of m_α and n_α:

$$m'_\alpha+n'_\alpha=m_\alpha+n_\alpha+2\sum_k \bar{Z}_{\alpha k}(k_{2k-1}+ik_{2k}),$$

$$m'_\alpha-n'_\alpha=m_\alpha-n_\alpha.$$

For such transformation $p'_\alpha=p_\alpha$, therefore $Z'_{\alpha k}=-Z'_{k\alpha}=0$. Writing $Z_{jk}=X_{jk}+iY_{jk}$, where X_{jk} and Y_{jk} are real, we can choose new l_j:

$$l'_{2j-1}=l_{2j-1}+2\sum_k (X_{jk}k_{2k-1}+Y_{jk}k_{2k}),$$

$$l'_{2j}=l_{2j}+2\sum_k (Y_{jk}k_{2k-1}-X_{jk}k_{2k}),$$

which gives $Z'_{jk}=0$. Dropping the primes, we get

$$p_j=\tfrac{1}{2}(l_{2j-1}+il_{2j}),$$

$$p_\alpha=\frac{1}{2}(\bar{m}_\alpha+\bar{n}_\alpha)+\sum_\beta Z_{\alpha\beta}(m_\beta-n_\beta).$$

There remains the freedom of transformations of the form

$$m'_\alpha+n'_\alpha=\sum_\beta \bar{A}_{\beta\alpha}(m_\beta+n_\beta),$$

$$m'_\alpha-n'_\alpha=\sum_\beta A^{-1}_{\alpha\beta}(m_\beta-n_\beta);$$

the corresponding transformations of $Z=(Z_{\alpha\beta})$ read

$$Z'=A^TZA. \tag{20}$$

In order to compare the real indices of N and P, it is sufficient to compare the real indices of

$$\operatorname{span}\{k_{2j-1}+ik_{2j},k_{2j-1}-ik_{2j},m_\alpha+n_\alpha,m_\alpha-n_\alpha\} \tag{21}$$

and

$$\operatorname{span}\{k_{2j-1}+ik_{2j},l_{2j-1}+il_{2j},m_\alpha+n_\alpha,p_\alpha\}, \tag{22}$$

where $j=1,\dots,\mu$, and $\alpha=1,\dots,\nu$; the remaining basis vectors of $N\cap P$ do not matter. The real index of the space (21) is 2μ. To find the real index of the space (22), we need to discuss the equation

$$\sum_\alpha\left(x_\alpha(m_\alpha+n_\alpha)+\frac{1}{2}y_\alpha(\bar{m}_\alpha+\bar{n}_\alpha)\right.$$

$$\left.+\sum_\beta y_\alpha Z_{\alpha\beta}(m_\beta-n_\beta)\right)$$

$$=\sum_\alpha\left(\bar{x}_\alpha(\bar{m}_\alpha+\bar{n}_\alpha)+\frac{1}{2}\bar{y}_\alpha(m_\alpha+n_\alpha)\right.$$

$$\left.+\sum_\beta \bar{y}_\alpha\bar{Z}_{\alpha\beta}(\bar{m}_\beta-\bar{n}_\beta)\right).$$

The Latin-indexed vectors are not introduced above, since it is obvious that their linear combinations can be real only if they vanish. The above equation leads to

$$x_\alpha=\frac{1}{2}\bar{y}_\alpha,\quad \sum_\alpha y_\alpha Z_{\alpha\beta}=0. \tag{23}$$

Suppose the rank of the skew-symmetric matrix Z is 2ρ. Then there are $\nu-2\rho$ complex linearly independent solutions of (23) and $2\nu-4\rho$ real linearly independent vectors in the space (22). Therefore, the real indices of N and P are equal if, and only if,

$$\mu=\nu-2\rho.$$

According to a classical theorem of algebra, the freedom (20) enables to reduce the matrix Z to its canonical form:

$$Z=\tfrac{1}{2}\operatorname{diag}(\underbrace{0,\dots,0}_{\mu},\underbrace{\epsilon,\dots,\epsilon}_{\rho}),$$

where

$$\epsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Thus the vectors p_α can be chosen as

$$p_\alpha=\tfrac{1}{2}(\bar{m}_\alpha+\bar{n}_\alpha),$$

where $\alpha = 1,...,\mu$, and

$$p_\alpha = \tfrac{1}{2}(\bar{m}_\alpha + \bar{n}_\alpha) + \tfrac{1}{2}(m_{\alpha+1} - n_{\alpha+1}),$$

$$p_{\alpha+1} = \tfrac{1}{2}(\bar{m}_{\alpha+1} + \bar{n}_{\alpha+1}) - \tfrac{1}{2}(m_\alpha - n_\alpha),$$

where $\alpha = \mu + 2\beta - 1$ and $\beta = 1,...,\rho$. The $2(\mu+\nu)$-dimensional space (21) decomposes into the orthogonal direct sum of four-dimensional subspaces,

$$\bigoplus_{j=1}^{\mu} N_j \oplus \bigoplus_{\beta=1}^{\rho} N'_\beta,$$

where

$$N_j = \mathrm{span}\{k_{2j-1}, k_{2j}, m_j, n_j\},$$

$$N'_\beta = \mathrm{span}\{m_\alpha, n_\alpha, m_{\alpha+1}, n_{\alpha+1}\}, \quad \alpha = \mu + 2\beta - 1.$$

Similarly the space (22) decomposes into the orthogonal direct sum of four-dimensional subspaces,

$$\bigoplus_{j=1}^{\mu} P_j \oplus \bigoplus_{\beta=1}^{\rho} P'_\beta,$$

where

$$P_j = \mathrm{span}\{k_{2j-1} + ik_{2j},\ m_j + n_j,\ l_{2j-1} - il_{2j}, \bar{m}_j + \bar{n}_j\},$$

$$P'_\beta = \mathrm{span}\{m_\alpha + n_\alpha,\ m_{\alpha+1} + n_{\alpha+1},$$

$$\bar{m}_\alpha + \bar{n}_\alpha + m_{\alpha+1} - n_{\alpha+1},$$

$$\bar{m}_{\alpha+1} + \bar{n}_{\alpha+1} - m_\alpha + n_\alpha\}, \quad \alpha = \mu + 2\beta - 1.$$

A pair of the above four-dimensional subspaces is nonorthogonal in the ordinary or in the Hermitian sense if, and only if, it is either (N_j, P_j) or (N'_β, P'_β). Therefore the whole problem is reduced to μ problems of the first kind and to ρ problems of the second kind; both types in the complexified eight-dimensional space with the signature of the scalar product equal to (4,4).

(1) Problem of the first kind ($r=2$):

$$N = \mathrm{span}\{k_1 + ik_2, m_1 + n_1, k_1 - ik_2, m_1 - n_1\},$$

$$P = \mathrm{span}\{k_1 + ik_2, m_1 + n_1, l_1 - il_2, \bar{m}_1 + \bar{n}_1\}.$$

(2) Problem of the second kind ($r=0$):

$$N = \mathrm{span}\{m_1 + n_1,\ m_2 + n_2,\ m_1 - n_1,\ m_2 - n_2\},$$

$$P = \mathrm{span}\{m_1 + n_1,\ m_2 + n_2,\ \bar{m}_1 + \bar{n}_1 + m_2 - n_2,$$

$$\bar{m}_2 + \bar{n}_2 - m_1 + n_1\}.$$

For the problem of the first kind, an explicit real orthogonal transformation Q such that $QN = P$ is

$$Q(k_1 + ik_2) = m_1 + n_1,$$

$$Q(m_1 + n_1) = k_1 + ik_2,$$

$$Q(k_1 - ik_2) = \bar{m}_1 + \bar{n}_1,$$

$$Q(m_1 - n_1) = l_1 + il_2.$$

For the problem of the second kind, it is

$$Q(m_1 + n_1) = \bar{m}_2 + \bar{n}_2 - m_1 + n_1,$$

$$Q(m_2 + n_2) = m_2 + n_2,$$

$$Q(m_1 - n_1) = -m_1 - n_1,$$

$$Q(m_2 - n_2) = \bar{m}_1 + \bar{n}_1 + m_2 - n_2.$$

Q.E.D.

Remark 1: If $\min(k,l) \leqslant 1$, then the action of the group $O(k,l)$ on the set of all MTN subspaces is transitive, because the real index in such a case is unique.

Remark 2: For complex orthogonal transformations, the minimal number of reflections mapping N onto P is

$$m - \dim(N \cap P).$$

The same occurs for real orthogonal transformations if $\min(k,l) \leqslant 3$. If $\min(k,l) \geqslant 4$ this is no longer true, because the number of reflections for the problems of the first and the second types appearing in the proof of Theorem 2 exceeds the expected 2.

Let β be the main antiautomorphism of the Clifford algebra and let

$$\mathrm{Pin}_\pm(k,l) = \{s \in \mathrm{Pin}(k,l):\ \beta(s) \vee s = \pm 1\}.$$

The notation $\mathrm{Spin}_\pm(k,l)$ $O_\pm(k,l)$, and $SO_\pm(k,l)$ are self-explanatory. For more information about the groups $\mathrm{Pin}(k,l)$ and $\mathrm{Spin}(k,l)$ consult Refs. 12 and 13.

Theorem 3: The group $O_+(k,l)$ [resp., $SO_+(k,l)$] acts transitively on each set of all MTN subspaces of $W = \mathbf{C} \otimes V$ with a given real index $r > 0$ (resp., with a given real index $r > 0$ and with a given helicity).

Proof: If the real index $r > 0$, then K and L contain at least one basis vector, k_1 and l_1, respectively. The element of the Clifford algebra,

$$s = (k_1 + l_1) \vee (k_1 - l_1)$$

belongs to $\mathrm{Spin}_-(k,l)$ and also to the stability group of the subspace $N, s \vee N \vee s^{-1} = N$.

Q.E.D.

The case $r = 0$ is different; its discussion is postponed to the next section.

III. CHARGE CONJUGATE OF A SIMPLE SPINOR

Representing the Clifford algebra of a complexified vector space as the endomorphisms algebra of the spinor space, $\mathrm{Cl}(W) \simeq \mathrm{End}\, S$, we have[7] two interesting linear isomorphisms, B: $S \to S^*$ and C: $S \to \bar{S}$ determined, up to arbitrary factors, by

$$\begin{aligned} \gamma_\mu{}^T &= (-1)^m B \gamma_\mu B^{-1}, \\ \bar{\gamma}_\mu &= C \gamma_\mu C^{-1}; \end{aligned} \tag{24}$$

where S^* is the dual of S and the vector space $\bar{S}$ has the set structure and the additive structure the same as S, however, the multiplication by complex numbers is complex conjugate to that in S. The *charge conjugate*[7] of a spinor $\varphi \in S$ is $\varphi_c = C\bar{\varphi} \in S$.

Theorem 4: There always holds $M(\varphi_c) = \overline{M(\varphi)}$.

Proof: Since for any $u,v \in V$ the formula (24) holds too, we have

$$\begin{aligned} (u+iv)\bar{C}\bar{\varphi} = 0 &\Leftrightarrow (\bar{u} - i\bar{v}) C\varphi = 0 \Leftrightarrow C(u-iv)\varphi \\ &= 0 \Leftrightarrow (u-iv)\varphi = 0. \end{aligned} \tag{25}$$

Q.E.D.

Remark: The complex conjugation $\mathrm{End}\, S \leftrightarrow \mathrm{End}\, \bar{S}$, used exclusively in (24) and in (25), should not be confused with that in W.

Corollary 1: The spinor φ is simple if, and only if, φ_c is simple.

Corollary 2: The spinor φ is *Majorana*, i.e., $\varphi_c \sim \varphi$, if, and only if, $\overline{M(\varphi)} = M(\varphi)$, i.e., its real index $r = \dim M(\varphi)$.

Corollary 3: The spinor φ is simple and Majorana if, and only if, $M(\varphi)$ is an MTN subspace such that $r = m$. In this case, the scalar product in V has a neutral signature.

Below we adopt the following definition of the scalar product of p-vectors:

$$(u_1 \wedge \cdots \wedge u_p) \cdot (v_1 \wedge \cdots \wedge v_p) = \det(u_i \cdot v_j), \tag{26}$$

and the following definition of the exterior product:

$$u \wedge v = A(u \otimes v) = A(u \vee v), \tag{27}$$

where u, v are arbitrary multivectors and A is the idempotent antisymmetrization operator.

When $M \subset N$ is chosen, we can define the *Kähler bivector* $j \in \Lambda^2 W$ as follows:

$$j \cdot (\bar{u} \wedge v) = \begin{cases} i\bar{u} \cdot v, & \text{for any } u,v \in M, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

If m_A is a basis of M, denote $g_{A\dot{B}} = m_A \cdot \bar{m}_{\dot{B}}$ and let $g^{\dot{A}B}$ be the inverse matrix, $g^{\dot{A}B} g_{B\dot{C}} = \delta^{\dot{A}}_{\dot{C}}$, $g_{A\dot{B}} g^{\dot{B}C} = \delta^C_A$. Then the component form of the Kähler bivector reads

$$j = i g^{\dot{B}A} m_A \wedge \bar{m}_{\dot{B}}.$$

Generically, the Kähler bivector depends on the choice of M; in the case $r=0$ only, this bivector is uniquely determined by $N = M(\varphi)$ (cf. Sec. I). Notice also that j is real. The symbol $\wedge^p j$ will denote its pth exterior power.

If $\phi,\psi \in S$, then let $B_p(\phi,\psi)$ be an element of $\Lambda^p W$ such that for any $v \in \Lambda^p W$:

$$B_p(\phi,\psi) \cdot v = \langle B\phi, v\psi \rangle =: B(\phi, v\psi).$$

Theorem 5: The algebraic constraints for a simple spinor φ to have the real index equal to r are

$$B_r(\varphi_c,\varphi) \neq 0, \quad B_{r-2}(\varphi_c,\varphi) = 0. \tag{29}$$

The nonvanishing multivectors $B_p(\varphi_c,\varphi)$ are

$$B_r(\varphi_c,\varphi) \sim k_1 \wedge \cdots \wedge k_r \tag{30}$$

and

$$B_{r+2p}(\varphi_c,\varphi) = (i^{-p}/p!) B_r(\varphi_c,\varphi) \wedge \wedge^p j, \tag{31}$$

for $p = 1,2,\ldots,m-r$.

Proof: The formulas (29) and (30) are direct consequences of Theorem 4, Proposition 9 in Ref. 7, and Eqs. (14) in Ref. 7. It remains to prove (31).

If u is a vector and v is a $(p-1)$-vector, then the convention (27) gives

$$v \wedge u = (-1)^p u \lrcorner v + v \vee u.$$

If $u_1,\ldots,u_q \in M(\varphi)$ and v is a $(p-q)$-vector, then

$$\begin{aligned} v \wedge u_1 \wedge \cdots \wedge u_q = (-1)^{q(p-(q-1)/2)} u_1 \\ \lrcorner (\cdots (u_q \lrcorner v) \cdots) \quad (\mathrm{mod}\, I_\varphi), \end{aligned} \tag{32}$$

where $I_\varphi = \{s \in \mathrm{Cl}(W)\colon s\varphi = 0\}$ is the left ideal in $\mathrm{Cl}(W)$ generated by $M(\varphi)$.

Consider

$$\begin{aligned} B_{r+2p}(\varphi_c,\varphi) \cdot (l_{i_1} \wedge \cdots \wedge l_{i_\lambda} \wedge \bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_\nu} \\ \wedge m_{B_1} \wedge \cdots \wedge m_{B_\mu} \wedge k_{j_1} \wedge \cdots \wedge k_{j_\kappa}). \end{aligned}$$

Equation (32) implies that the expression (33) equals

$$(-1)^{(\kappa+\mu)[r-(\kappa+\mu-1)/2]} B_{r+2p-2\kappa-2\mu}(\varphi_c,\varphi) \cdot$$

$$(m_{B_1} \lrcorner (\cdots (m_{B_\mu} \lrcorner (k_{j_1} \lrcorner (\cdots (k_{j_\kappa} \lrcorner (l_{i_1} \wedge \cdots$$

$$\wedge l_{i_\lambda} \wedge \bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_\nu}))\cdots)))\cdots)). \qquad (34)$$

If $\kappa + \mu > p$, then in (34) we get $B_{r+2p-2\kappa-2\mu}(\varphi_c,\varphi) = 0$; moreover the interior product in (34) vanishes if $\mu > \nu$ or $\kappa > \lambda$. Since $(\varphi_c)_c \sim \varphi$, we have $B_p((\varphi_c)_c,\varphi_c) \sim B_p(\varphi_c,\varphi)$, and since $M(\varphi_c) = \mathbf{C} \otimes K \oplus \bar{M}$, the remarks concerning vanishing of (34) = (33) hold too, if the roles of μ and ν are reversed. To have nonvanishing expression (33), we need $\mu = \nu$ and, since $r + 2p = \kappa + \lambda + \mu + \nu$, we obtain $\lambda \geqslant r + \kappa$ and this leads finally to $\kappa = 0$, $\lambda = r$, and $\mu = \nu = p$. In this only one nontrivial case, the expression (33) becomes

$$(-1)^{(p/2)(p-1)}(B_r(\varphi_c,\varphi)\cdot(l_1 \wedge \cdots \wedge l_r))$$

$$\times (m_{B_1} \lrcorner (\cdots (m_{B_p} \lrcorner (\bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_p}))\cdots)).$$

We have

$$m_{B_1} \lrcorner (\cdots (m_{B_p} \lrcorner (\bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_p}))\cdots)$$

$$= (-1)^{(p/2)(p-1)} \sum_\sigma \operatorname{sgn} \sigma (\bar{m}_{\dot{A}_1} \cdot m_{B_{\sigma(1)}}) \cdots (\bar{m}_{\dot{A}_p} \cdot m_{B_{\sigma(p)}}).$$

This should be compared to [cf. (26) and (28)]

$$(\underbrace{j \wedge \cdots \wedge j}_{p}) \cdot (\bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_p} \wedge m_{B_1} \wedge \cdots \wedge m_{B_p})$$

$$= p! \sum_\sigma \operatorname{sgn} \sigma (j \cdot (\bar{m}_{\dot{A}_1} \wedge m_{B_{\sigma(1)}})) \cdots (j \cdot (\bar{m}_{\dot{A}_p} \wedge m_{B_{\sigma(p)}}))$$

$$= i^p p! \sum_\sigma \operatorname{sgn} \sigma (\bar{m}_{\dot{A}_1} \cdot m_{B_{\sigma(1)}}) \cdots (\bar{m}_{\dot{A}_p} \cdot m_{B_{\sigma(p)}}).$$

That finally proves the formula (29). Q.E.D.

Remark: If $r < 2$, it is understood that the second condition in (29) is not present.

The case $r = 0$ has been exempted from Theorem 2. The maps B and C can be gauged so that $B(\varphi_c,\varphi)$ becomes real for any φ. The condition $r = 0$ is equivalent to $B(\varphi_c,\varphi) \neq 0$. For real vectors $u \in V$, we have $B((u\varphi)_c, u\varphi) = u^2 B(\varphi_c,\varphi)$, therefore $B(\varphi_c,\varphi)$ is invariant only with respect to elements of the group $\mathrm{Pin}_+(k,l)$. The inequalities

$$B(\varphi_c,\varphi) > 0 \quad \text{and} \quad B(\varphi_c,\varphi) < 0$$

determine, generically, two disjoint sets of simple spinor directions (and of corresponding MTN subspaces) of different "time helicity." The exceptional cases are

(1) the signature $(2m,0)$, where $B(\varphi_c,\varphi)$ has a definite sign; and

(2) the signature $(0,2m)$, where $\mathrm{Pin}_+(0,2m) = \mathrm{Spin}(0,2m)$ and the notions of "time helicity" and of helicity coincide.

Example 1: Space-time spinors.

Notice that the complexified approach does not make an essential distinction between the signatures (k,l) and (l,k) in contrast to a virtual purely real approach. For the Minkowski vector space, we choose (3,1). There is only one possibility for the real index, $r = 1$, and any MTN subspace is of the form

$$M(\varphi) = \operatorname{span}\{k_1, m_1\},$$

where $k_1 = \frac{1}{2}(\gamma_0 + \gamma_3)$, $m_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and φ is a Weyl two-component spinor. The space $K = \operatorname{span}\{k_1\}$ is sometimes called the flagpole and the space $\Re M = \operatorname{span}\{\gamma_1,\gamma_2\}$ the varying flag[14] of φ. The Kähler bivector is

$$j = 2i m_1 \wedge \bar{m}_{\dot{1}} = \gamma_1 \wedge \gamma_2$$

and nonvanishing multivectors $B_p(\varphi_c,\varphi)$ are $B_1(\varphi_c,\varphi) \sim k_1$ and $B_3(\varphi_c,\varphi) = -iB_1(\varphi_c,\varphi) \wedge j$.

Example 2: Twistors.[14,15]

Consider a six-dimensional real vector space V with $(k,l) = (4,2)$. The space of Weyl spinors with a given helicity is the *twistor space* T. An MTN subspace can have the real index $r = 0$ or 2.

In the first case, $r = 0$,

$$M(\varphi) = \operatorname{span}\{m_1, m_2, n_1\}.$$

The Kähler bivector $j = 2i m_1 \wedge \bar{m}_{\dot{1}} + 2i m_2 \wedge \bar{m}_{\dot{2}} - 2i n_1 \wedge \bar{n}_{\dot{1}}$ is uniquely determined by the *non-null* twistor φ, $B(\varphi_c,\varphi) \neq 0$. Remaining nonvanishing multivectors $B_p(\varphi_c,\varphi)$ are $B_2(\varphi_c,\varphi) = -iB(\varphi_c,\varphi)j$, $B_4(\varphi_c,\varphi) = -\frac{1}{2}B(\varphi_c,\varphi) j \wedge j$ and $B_6(\varphi_c,\varphi) = (i/6)B(\varphi_c,\varphi) j \wedge j \wedge j$.

In the second case, $r = 2$,

$$M(\varphi) = \operatorname{span}\{k_1, k_2, m_1\}.$$

The Kähler bivector $j = 2i m_1 \wedge \bar{m}_{\dot{1}}$ spans a "varying flag." The twistor φ is *null*, $B(\varphi_c,\varphi) = 0$. Nonvanishing multivectors $B_p(\varphi_c,\varphi)$ are $B_2(\varphi_c,\varphi) \sim k_1 \wedge k_2$ and $B_4(\varphi_c,\varphi) = -iB_2(\varphi_c,\varphi) \wedge j$.

Recall that the conformally compactified Minkowski space[16] $\widehat{M}$ is the projective space of all null vectors (i.e., a quadric Grassmannian[2]) in V. The intersection of $\widehat{M}$ with a real two-dimensional MTN subspace $K(\varphi) = M(\varphi) \cap V_\lambda$ associated with any null twistor, is a null geodesic in $\widehat{M}$ and every null geodesic in $\widehat{M}$ can be so represented. This is the basis of the *Penrose correspondence*:

$$\{\text{null directions in } T\} \leftrightarrow \{\text{null geodesics in } \widehat{M}\}.$$

Consider now two nonparallel null twistors φ and ω. According to the Lemma in Ref. 8, there are two possibilities for the intersection:

$$K(\varphi)\cap K(\omega)=\begin{cases}\{0\}, & \text{if } B(\varphi_c,\omega)\neq 0;\\ 1-\dim, & \text{if } B(\varphi_c,\omega)=0.\end{cases}$$

This leads immediately to the following conclusion: if φ and ω are nonparallel null twistors, then the associated null geodesics in the compactified Minkowski space $\hat{M}$ intersect (in a point) if, and only if, $B(\varphi_c,\omega)=0$.

IV. ARE SOME SIMPLE SPINORS SIMPLE-r THAN THE OTHERS?

The Lie algebra stabilizer of a TN subspace N of W under the action of the group $O(k,l)$ consists of those elements $c\in[V,V]$ = Lie algebra of $\mathrm{Pin}(k,l)$, which satisfy $[c,N]\subset N$. For a TN subspace that is not necessarily maximal, we have

$$V=K\oplus L\oplus\Re M\oplus(\Re N)^{\perp},$$

instead of (7). If (q_a), $a=1,\ldots,2(m-n)$ (where $n=r+n_0+n_++n_-=\dim N$), is a basis of $(\Re N)^{\perp}$, then a basis of the stabilizer consists of

$$\begin{aligned}&[k_i,k_j],\quad i<j,\\ &[k_i,l_j],\\ &\Re[k_i,m_A],\\ &\Im[k_i,m_A],\\ &[k_i,q_a],\\ &\Re[m_A,m_B],\quad A\leqslant B,\\ &\Im[m_A,m_B],\quad A<B,\\ &[q_a,q_b],\quad a<b.\end{aligned}$$

It is easily seen that the stabilizer is

$$\Re[N,\bar{N}]\oplus[K,(\Re N)^{\perp}]\oplus[(\Re N)^{\perp},(\Re N^{\perp}];$$

the second and the third term vanish if, and only if, N is MTN. Its dimension is

$$(r/2)(r-1)+r(2m-r)+(n-r)^2$$
$$+(m-n)(2m-2n-1).$$

Since the dimension of the group $O(k,l)$ is $m(2m-1)$, the dimension of the orbit of this group acting on the set of all TN subspaces of W and containing N is

$$d(m,n,r)=m(2m-1)-(r/2)(r-1)$$
$$-(n-r)^2-(m-n)(2m-2n-1).$$

In particular, if N is an MTN subspace, we have

$$d(m,m,r)=m(m-1)-(r/2)(r-1). \qquad (35)$$

This is also the dimension of the orbit of the group $\mathrm{Pin}(k,l)$ acting on the complex projective space of simple spinors and containing dir φ, where $N=M(\varphi)$.

There are two definitions of a complex simple spinor. The first requires maximality of the complex TN space $M(\varphi)$ associated with any spinor $\varphi\neq 0$. The second requires minimality of the dimension of the orbit of the group $\mathrm{Pin}(2m)$ acting on the projective space of spinors and containing dir φ. Wishing to adopt these definitions to the real case, we observe the following. First, the real TN subspace

$$K(\varphi)=\{u\in V\colon\ u\varphi=0\}=M(\varphi)\cap V$$

is maximal if, and only if, $r=\min(k,l)$; that means the real index coincides with the quantity called sometimes the index[2] of the scalar product. Second, according to (35), the dimension of the orbit of the group $\mathrm{Pin}(k,l)$ is minimal (at least when compared to that of other simple spinors) in such a case. These are sufficient reasons to call a (complex) simple spinor φ such that $r=\min(k,l)$, the *simple-r spinor*; simple-$r\simeq\frac{1}{2}$ (simple real + simple-r). The answer to the question posed as the title of this section is definitely yes (though it sounds tautological). The Weyl spinors of space-time and null twistors are basic examples of simple-r spinors.

If only the complex space W is given, and we are allowed to choose between its various real structures V, the simplest among the simple-r spinors are those that are simple and Majorana, $r=m$. For them the dimension of the orbit

$$d(m,m,m)=(m/2)(m-1)$$

is half of the dimension of the orbit of a simple spinor under the action of the complex Pin group.

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