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ON GRAVITATIONAL RADIATION FROM BOUNDED SOURCES

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ABSTRACT

Having observed that the Bondi-Sachs formulation of the theory of gravitational radiation from bounded sources does not explicitly include the Robinson-Trautman fields as a special case, we re-examine the theory in a coordinate system and with boundary conditions which ensure that the latter fields are manifestly present. We compare our results with those of Sachs by analysing the characteristic initial-value problem. We find that to obtain a solution of the vacuum field equations we must specify *five* arbitrary functions of three variables and three arbitrary functions of two variables, whereas in Sachs' formulation it is sufficient to specify *four* arbitrary functions of three variables and three arbitrary functions of two variables. In our formulation both shear-free *and* shearing gravitational radiation is manifestly present whereas in the Bondi-Sachs formulation only radiation with shear is explicit. If, in our formulation, we remove the shear-free radiation, then we can obtain a "mass-loss" formula of the type derived by Bondi and Sachs. If both shear-free and shearing radiation is present however, we fail to obtain a formula of this type.

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1. Introduction

In the years 1956-60, in a series of lectures given at several British universities, in Paris, Hamburg and at Leopold Infeld's seminar in Warsaw, Ivor Robinson put forward the *programme of characterizing simple* electromagnetic and gravitational *waves by the properties of* 'null' (optical, isotropic) *rays* associated with these waves. He discovered a new important property of congruences of such rays.

Let (F_{ij}) be the electromagnetic field in a four-dimensional space-time referred to local coordinates (x^i), $i = 1, 2, 3, 4$. The field is said to be null if there is a non-vanishing vector field $k = k^i \frac{\partial}{\partial x^i}$ such that

$$F_{ij}\, k^j = 0 \qquad \text{and} \qquad F_{[ij}\, k_{\ell]} = 0. \tag{1.1}$$

If $F_{ij} \neq 0$, then k is null,

$$g_{ij}\, k^i k^j = 0. \tag{1.2}$$

L. Mariot (1) showed that, if (F_{ij}) satisfies Maxwell's equations, then k generates a congruence of *null geodesics*,

$$k_{i;j}\, k^j = \mu\, k_i,$$

where μ is a function, $k_i = g_{ij}\, k^j$ and the semicolon followed by an index denotes covariant differentiation relative to the Levi-Civita connection associated with $g = g_{ij}\, dx^i dx^j$.

Ivor Robinson discovered a subtler property of the congruence: Maxwell's equations imply that it is non-shearing. If an affine parametrization of the congruence is chosen, then $\mu = 0$ and the *shear-free condition*, as given by Robinson, reads

$$k^{i;j}\,(k_{i;j} + k_{j;i}) = (k^i{}_{;i})^2. \tag{1.3}$$

Moreover, he proved the converse theorem (2),(3): given, in a conformal space-time, a congruence of non-shearing, null geodesics generated by k, one can find a solution (F_{ij}) of Maxwell's equations such that $F_{ij} \neq 0$ and conditions (1.1) hold.

Ivor Robinson was also the first to use the notion of *non-shearing null geodesics in the theory of gravitation.* He proved that multiple principal null directions of the Weyl tensor in an empty space-time are tangent to a congruence of such geodesics and studied the simplest case when k is covariantly constant ([4]). He called plane-fronted waves solutions of $R_{ij} = 0$ such that

$$R_{ijk\ell}\, k^{\ell} = 0 \;,\; k_{i;j} = 0 \;,\; g_{ij}\, k^i k^j = 0 \;,\; R_{ijk\ell} \neq 0 \;,\; k \neq 0,$$

and showed that the corresponding line-element is

$$c\,(u, x, y)\, du^2 + 2\, du\, dz - dx^2 - dy^2,$$

with

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0.$$

Since these solutions turned out to coincide with a class of metrics considered, in a purely geometrical context, by Brinkmann ([5]), Robinson did not publish his results. They had, however, an important influence on Kundt ([6]) and other authors.

In 1959 Ivor Robinson formulated the problem of finding 'spherical gravitational waves' and invited one of us (A.T.) to join him in this research. Those waves were defined as time-dependent *solutions of Einstein's equations admitting* a *shear-free* congruence of *null geodesics* which is *diverging,*

$$k^i{}_{;i} \neq 0, \tag{1.4}$$

and *hypersurface-orthogonal,*

$$k_{[i}\, k_{j;\ell]} = 0. \tag{1.5}$$

It easily follows from eqs. (1.2-1.5) and $R_{ij}\, k^i k^j = 0$ that (local) coordinates u, r, x and y can be chosen in space-time so that

$$du = k_i dx^i \, , \; k^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial r} \tag{1.6}$$

and the Robinson-Trautman (RT) line-element is

$$c\, du^2 + 2\, du\, dr - r^2\, p^{-2}\, \{(dx + a\, du)^2 + (dy + b\, du)^2\}, \frac{\partial p}{\partial r} = 0. \tag{1.7}$$

Moreover, the remaining field equations lead to the possibility of eliminating, by a charge of coordinates, both a and b and, after this has been achieved, they reduce to

$$c = -\frac{2m}{r} + K - 2\, p^{-1}\dot{p}\, r, \tag{1.8}$$

and

$$mp^{-1}\dot{p} - \frac{1}{3}\dot{m} + \frac{1}{12}\Delta K = 0, \tag{1.9}$$

where m is a function of u only, p depends on u, x and y, K is the Gaussian curvature of the 2-surface u = const., r = 1,

$$\Delta = p^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad \text{and} \quad \dot{p} = \frac{\partial p}{\partial u}.$$

Large classes of explicit solutions of this form have been found [7],[8]. Among them are waves which are 'spherical' in the sense of admitting wave fronts topologically equivalent to a two-sphere.

The null hypersurfaces u = const. are well-defined by the geometry of (1.7); in other words, the coordinate u can be replaced only by a smooth and monotonic function of itself. The existence of such a preferred family of null hypersurfaces results from the *algebraic degeneracy* of the Weyl tensor combined with (1.5). In generic, type I, in the sense of the Petrov classification [9], physically realistic space-times — as well as in conformally flat spaces — there are no such uniquely defined families of null hypersurfaces. It is known, however, that bursts of gravitational radiation, described by *discontinuities of curvature*, propagate along null hypersurfaces ([10]). The electromagnetic field produced by an accelerated point charge also defines a 'retarded-time', null coordinate u ([11]). Considerations such as these, have led Hermann Bondi to put forward, in 1960, a new approximate method of describing gravitational waves in general relativity ([12]),([13]). R. Sachs ([14]) extended the work of Bondi, van der Burg and Metzner by relaxing their assumption of axial symmetry and studying the characteristic initial-value problem.

The *Bondi-Sachs line-element* reads

$$\begin{aligned} ds^2 &= \varrho^{-1} V e^{2\beta} du^2 + 2e^{2\beta} du\, d\varrho \\ &- \varrho^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du), \end{aligned} \tag{1.10 a}$$

where

$$\begin{aligned} 2h_{AB}\, dx^A dx^B &= (e^{2\gamma} + e^{2\delta})\, d\theta^2 + 4 \sin\theta \,\mathrm{sh}\,(\gamma - \delta)\, d\theta\, d\phi \\ &+ (e^{-2\gamma} + e^{-2\delta}) \sin^2\theta\, d\phi^2. \end{aligned} \tag{1.10 b}$$

Here capital indices take values 1, 2 with $x^1 = \theta$, $x^2 = \phi$. The line-element depends on the six functions U^1, U^2, V, β, γ and δ of the four coordinates u, ϱ, θ and ϕ. It is thus more general than (1.7). A rather trivial difference between these line-elements is in the choice of the 'angular coordinates': the flat, spherically symmetric form of (1.7),

$$du^2 + 2\, du\, dr - r^2 P^{-2} (dx^2 + dy^2),$$

$$P = 1 + \frac{1}{4}(x^2 + y^2),$$

assumes the stereographic coordinates x and y whereas the Bondi-Sachs coordinates correspond to $u = t - \varrho$ and the polar variables ϱ, θ and ϕ. In both line-elements u is a null coordinate. Bondi's ϱ is a '*luminosity distance*': since $\det(h_{AB}) = \sin^2\theta$, the area of the two-surface u = const., ϱ = const. is $4\pi\varrho^2$.

The line-element (1.10) is supplemented with assumptions on the dependence of its coefficients on ϱ. They are justified, in part, by the requirement that (1.10) reduce to $du^2 + 2\,du\,d\varrho - \varrho^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$ at large distances. Somewhat subtler conditions are obtained by demanding that $(\log \varrho)/\varrho^3$ terms be absent from U^A (14). It is assumed that

$$V = \varrho - 2M + O(\varrho^{-1}), \tag{1.12}$$

$$U^A = \varrho^{-2} U^A_2 + O(\varrho^{-3}), \tag{1.13}$$

$$\delta + i\gamma = 2\varrho^{-1} n + O(\varrho^{-2}), \tag{1.14}$$

$$\beta = O(\varrho^{-1}), \tag{1.15}$$

where M, U^A_2 are real functions of u, θ and ϕ and n is a complex function of u, θ and ϕ (these expansions are discussed in greater detail in section 4 below). When the vacuum field equations are imposed one finds, for example, that (1.15) is strengthened to $\beta = O(\varrho^{-2})$, while another of the field equations gives

$$\dot{M} = -|\dot{n}|^2 + N, \tag{1.16}$$

where N is a function of u, θ and ϕ such that its average over the surface u, ϱ = const.,

$$\langle N \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} N \sin\theta\, d\theta,$$

vanishes. By integration of both sides of (1.16) Bondi and Sachs obtain

$$\frac{\partial}{\partial u} \langle M \rangle = -\langle |\dot{n}|^2 \rangle, \tag{1.17}$$

a result interpreted as giving the decrease of the total mass $\langle M \rangle$ of the system due to gravitational waves with a '*news function*' $\dot{n}$.

The null and geodesic congruence generated by the gradient of u is *shearing*. In fact, the leading term in the complex shear scalar σ is proportional to n. Therefore, Bondi's retarded time coordinate u cannot simply be identified with the one used by Robinson and Trautman.

R. Isaacson and J. Winicour ([15]) initiated the study of the *relation between the Robinson-Trautman solutions and the Bondi-Sachs approach to gravitational waves.* They noticed that the simple transformation

$$r = \varrho p/P, \tag{1.18}$$

where P is as in (1.11), brings the line-element (1.7), with a = b = 0 and c given by (1.8), to the form

$$\begin{aligned} &\left(K - \frac{2mP}{\varrho p}\right) du^2 + 2\frac{p}{P} du\, d\varrho + \\ &+ 2\varrho \left[\frac{\partial}{\partial x}\left(\frac{p}{P}\right) dx + \frac{\partial}{\partial y}\left(\frac{p}{P}\right) dy \right] du - \\ &- \varrho^2 P^{-2} (dx^2 + dy^2). \end{aligned} \tag{1.19}$$

This is closer to (1.10) with the assumption (1.12) than the original metric (1.7) which contains in c a term linear in r. However, the simple change (1.18) is not sufficient to satisfy all of the boundary conditions (1.12-15). In particular, as discussed by Bondi and van der Burgh in Part B § 4 of ref. 13, in order to have $g_{u\varrho} \to 1$ as $\varrho \to +\infty$ one has to apply to (1.19) a transformation of the form

$$\bar{u} = f(u, x, y) \tag{1.20}$$

which leads from a non-shearing u to a shearing $\bar{u}$. To our knowledge, no one has succeeded in performing a transformation, in a closed form, of the general line-element (1.7) to the Bondi-Sachs coordinates. It is clear, in any case, that the simplicity of the RT line-element will be lost under such a transformation ([16]).

One can argue that, at least in simple cases, the null coordinate u is well-defined by the physics and geometry of the systems under consideration. In addition to the examples mentioned before, consider the following: in a space-time diffeomorphic to $\mathbf{R}^4$ let there be a congruence of time-like, hypersurface-orthogonal world-lines. Let t = const. be the equation of the space-like hypersurfaces orthogonal to the congruence. Let S_0 be a convex surface, diffeomorphic to $\mathbf{S}_2$, and contained in the hypersurface t = 0. The curves of the congruence meeting S_0 span a

3-surface. Its intersection S_t with $t = \text{const.}$ is also diffeomorphic to $\mathbf{S}_2$. Consider future-oriented null conoids with vertices on S_t. Their envelope is a null hypersurface. By varying t, one obtains a foliation of space-time by a family of null hypersurfaces which define — at least locally — the required null coordinate (see figure 1).

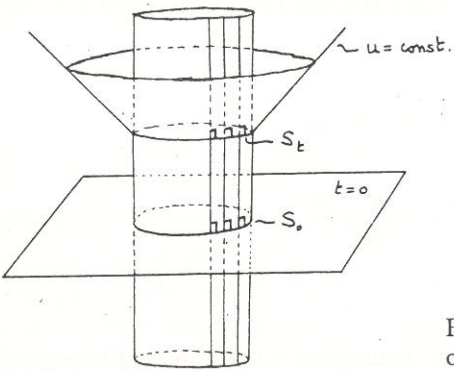

FIGURE 1: Geometrical construction of the null coordinate u.

In this paper we consider once more the problem of gravitational waves produced by a bounded system. We choose a line-element which is as general as the one investigated by Sachs: it contains six functions of four coordinates. The coordinates are, however, different from Bondi's: they are such as to include the RT line-element as a special case, without the necessity of performing transformations such as (1.18) or (1.20). Furthermore we assume expansions, in powers of a radial coordinate r, of the functions appearing in the metric, which preserve this property of our line-element. Having obtained the vacuum field equations to be satisfied by the coefficients in these expansions necessary to have a knowledge of the metric tensor components up to and including r^{-1}-terms, we compare our results with those of Sachs by analysing the characteristic initial-value problem. He found that to determine a solution of the vacuum field equations one must specify four arbitrary functions of three variables and three arbitrary functions of two variables. We find that one must specify *five* arbitrary functions of three variables and three arbitrary functions of two variables to obtain a solution. The extra function available to us is due to a weaker choice of boundary conditions than those chosen by Sachs. In our formulation of this problem both shear-free *and* shearing gravitational radiation is manifestly present whereas in the Bondi-Sachs formulation only radiation with shear is explicit. In our formulation if we remove the shear-free radiation then we can obtain a "mass-loss" formula similar to (1.17). On the other hand if both shearing and shear-free radiation is clearly present then we fail to obtain a formula of this type.

2. Line-Element and Field Equations

If (x^i), $i = 1, 2, 3, 4$ is a local coordinate system in terms of which the metric tensor of space-time has components $g_{ij}(x)$, and if $u(x)$ is a scalar function satisfying

$$g^{ij} u_{,i} u_{,j} = 0, \tag{2.1}$$

then the vector field $k = k^i \frac{\partial}{\partial x^i}$, with

$$k^i = g^{ij} u_{,j} \tag{2.2}$$

is null and has geodesic integral curves. Writing $k^i = \frac{\partial x^i}{\partial r}$ and thus $k = \frac{\partial}{\partial r}$, we see that r is an affine parameter along these curves. We shall assume that this congruence of null geodesics has non-zero expansion and is future-pointing. Choosing r and u as coordinates we write the line-element in the form

$$\begin{aligned} ds^2 = {} & c\,du^2 + 2\,du\,dr - \\ & - r^2 p^{-2} \{(e^{\alpha} \operatorname{ch}\beta\, dx + e^{-\alpha} \operatorname{sh}\beta\, dy + a\,du)^2 + \\ & + (e^{\alpha} \operatorname{sh}\beta\, dx + e^{-\alpha} \operatorname{ch}\beta\, dy + b\,du)^2\}, \end{aligned} \tag{2.3}$$

where c, p, α, β, a, b are six functions of the four coordinates u, r, x, y. The coordinates u, x, y are constant along each integral curve of $\frac{\partial}{\partial r}$ and the coordinates x, y are chosen so that the determinant of the metric tensor on the 2-surfaces u = const., r = const., induced by (2.3), is r^4p^{-4}. This line-element is completely general and has the property that if $\alpha = \beta = 0$ it takes the general form of the RT line-element (1.7). A significant difference between (2.3) and the Bondi-Sachs line-element (1.10) is the appearance of a non-constant coefficient of du dϱ in (1.10) and a constant coefficient of du dr in (2.3).

For the remainder of this section we shall be concerned only with the line-element (2.3). We shall consider the form (1.10) again in section 4 when we compare our results with those obtained by Sachs ([14]).

For ease of computation with (2.3) it is convenient to make use of a half-null tetrad defined via the basis 1-forms

$$\theta^1 = rp^{-1}\,(e^{\alpha}\,\mathrm{ch}\beta\,dx + e^{-\alpha}\,\mathrm{sh}\beta\,dy + a\,du), \tag{2.4 a}$$

$$\theta^2 = rp^{-1}\,(e^{\alpha}\,\mathrm{sh}\beta\,dx + e^{-\alpha}\,\mathrm{ch}\beta\,dy + b\,du), \tag{2.4 b}$$

$$\theta^3 = dr + \frac{1}{2}\,c\,du, \tag{2.4 c}$$

$$\theta^4 = du, \tag{2.4 d}$$

so that

$$ds^2 = -\,(\theta^1)^2 - (\theta^2)^2 + 2\theta^3\theta^4. \tag{2.5}$$

We shall require (2.3) to be a solution of Einstein's vacuum field equations. To simplify this task we assume that the six functions appearing in (2.3) can each be expanded in the following power series in r, for some range of the coordinates:

$$c = rc_{-1} + c_0 + \frac{c_1}{r} + \dots\,, \tag{2.6 a}$$

$$p = p_0\left(1 + \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{q_3}{r^3} + \dots\right), \tag{2.6 b}$$

$$\alpha = \frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \dots\,, \tag{2.6 c}$$

$$\beta = \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \dots, \tag{2.6 d}$$

$$a = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \dots, \tag{2.6 e}$$

$$b = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots. \tag{2.6 f}$$

The coefficients in these expansions are functions of u, x, y. We shall be satisfied to have the metric tensor components up to and including the r^{-1}-term in each and for this we require the twenty-one coefficients shown explicitly in (2.6) and only these coefficients. The expansions (2.6) determine our 'boundary' conditions on the six functions appearing in (2.3) as $r \to +\infty$. The linear term in r in (2.6 a) is a departure from the type of boundary conditions assumed by Bondi et al. ([13]) and by Sachs ([14]), and it, together with the remaining equations in (2.6), is chosen so as to allow the RT fields to be manifestly present as a special case.

The vacuum field equations will provide us with differential equations to be satisfied by the coefficients in (2.6). Before embarking upon the calculation of them it is helpful to note that Sachs' splitting of the field equations in the six "main equations", the one "trivial equation" and the three "supplementary conditions" is given in our case by

$$\text{Main Equations: } R_{33} = 0\ ,\ R_{A3} = 0\ ,\ R_{AB} = 0; \tag{2.7 a}$$

$$\text{Trivial Equation: } R_{34} = 0; \tag{2.7 b}$$

$$\text{Supplementary Conditions: } R_{44} = 0\ ,\ R_{A4} = 0. \tag{2.7 c}$$

Here again capital indices take values 1, 2. The trivial equation is a consequence of the main equations and thus provides a useful check on the calculation of the main equations. In addition it is helpful to examine the field equations in the order in which they are listed in (2.7) as the degree of complexity tends to increase in this ordering (see, for example ref. 1, 17).

Our calculation requires us to equate to zero the coefficients of various powers of r^{-1} in the expressions for the Ricci tensor components. In the table below the powers of r^{-1} required for this purpose (i.e. which involve the twenty-one functions needed to reconstruct the metric tensor components up to and including the r^{-1}-term, and only these functions), in each Ricci tensor component, are indicated by a shaded area. We have omitted R_{34}. A zero at r^{-1} in R_{33} in the figure, for example, indicates that there is no r^{-1}-term in the expansion of this Ricci tensor component.

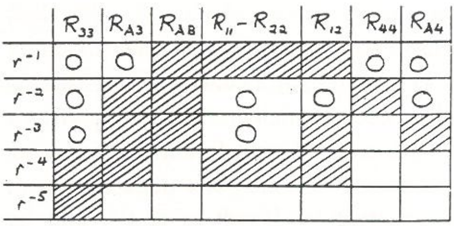

FIGURE 2: A guide to the expansion of the Ricci tensor components in inverse powers of r.

The complex shear σ and the expansion Z, of the null geodesic congruence tangent to $\frac{\partial}{\partial r}$ are given by

$$\sigma = -\frac{(\alpha_1 + i\beta_1)}{r^2} - 2\,\frac{(\alpha_2 + i\beta_2)}{r^3} - \frac{(2\alpha_1\beta_1^2 + 3\alpha_3 + 3i\beta_3)}{r^4} + \ldots, \quad (2.8\,a)$$

$$Z = \frac{1}{r} + \frac{q_1}{r^2} + \frac{2q_2 - q_1^2}{r^3} + \frac{q_1^3 - 3q_1q_2 + 3q_3}{r^4} + \ldots. \quad (2.8\,b)$$

The r^{-4}-term in $R_{33} = 0$ gives

$$q_2 - q_1^2 = \frac{1}{2}(\alpha_1^2 + \beta_1^2), \quad (2.9)$$

while the r^{-5}-term in the same field equation results in

$$2q_1^3 - 5q_1q_2 + 3q_3 = 2(\alpha_1\alpha_2 + \beta_1\beta_2). \quad (2.10)$$

R_{A3} (A = 1, 2) have r^{-2}-terms to begin with and the vanishing of their coefficients gives

$$a_1 = a_0\alpha_1 + b_0\beta_1, \tag{2.11 a}$$

$$b_1 = a_0\beta_1 - b_0\alpha_1. \tag{2.11 b}$$

Next R_{AB} begin with r^{-1}-terms from which we obtain

$$c_{-1} + 2p_0^{-1}\dot{p}_0 + 2p_0 a_0 \frac{\partial p_0^{-1}}{\partial x} + 2p_0 b_0 \frac{\partial p_0^{-1}}{\partial y} + \frac{3}{2}\frac{\partial a_0}{\partial x} + \frac{1}{2}\frac{\partial b_0}{\partial y} = 0, \tag{2.12 a}$$

$$c_{-1} + 2p_0^{-1}\dot{p}_0 + 2p_0 a_0 \frac{\partial p_0^{-1}}{\partial x} + 2p_0 b_0 \frac{\partial p_0^{-1}}{\partial y} + \frac{1}{2}\frac{\partial a_0}{\partial x} + \frac{3}{2}\frac{\partial b_0}{\partial y} = 0, \tag{2.12 b}$$

from $R_{11} = 0$ and $R_{22} = 0$ respectively. Here, as before, the dot indicates partial differentiation with respect to u. From (2.12) we have

$$\frac{\partial a_0}{\partial x} = \frac{\partial b_0}{\partial y}, \quad c_{-1} = -\,2H, \tag{2.13}$$

where

$$H = p_0^{-1}\dot{p}_0 + \frac{\partial a_0}{\partial x} - p_0^{-1}a_0\frac{\partial p_0}{\partial x} - p_0^{-1}b_0\frac{\partial p_0}{\partial y}\,. \tag{2.14}$$

The r^{-1}-term in $R_{12} = 0$ provides us with

$$\frac{\partial a_0}{\partial y} = -\,\frac{\partial b_0}{\partial x}. \tag{2.15}$$

Now the r^{-2}-term in R_{12} is identically zero while the vanishing of this term in R_{11} and R_{22} gives us the same equation, namely,

$$c_0 = \Delta \log p_0 - 2\left(\dot{q}_1 - Hq_1 - a_0\frac{\partial q_1}{\partial x} - b_0\frac{\partial q_1}{\partial y}\right), \tag{2.16}$$

where

$$\Delta = p_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{2.17}$$

is the Laplacian on the family of 2-surfaces with line-element

$$p_0^{-2}(dx^2 + dy^2). \tag{2.18}$$

At this stage of the calculation we can use some of the field equations we have calculated to remove some of the functions of u, x, y. If we make the transformation

$$r = r' + A(u, x, y) + O(r'^{-3}), \tag{2.19}$$

then the form of our line-element (up to and including r^{-1}-terms) remains invariant if the functions appearing in it (and here we only quote the functions of interest at this stage) undergo the following transformations:

$$p' = p_0 \left(1 + \frac{q_1'}{r'} + \frac{q_2'}{r'^2} + \frac{q_3'}{r'^3} + \dots \right), \tag{2.20 a}$$

where

$$\begin{aligned} q_1' &= q_1 - A\,,\ q_2' = q_2 - q_1^2 + (q_1 - A)^2, \\ q_3' &= q_3 - q_1^3 - 3\,(q_2 - q_1^2)\,A + (q_1 - A)^3, \end{aligned} \tag{2.20 b}$$

while

$$\begin{aligned} &\alpha_1' = \alpha_1\,,\ \beta_1' = \beta_1, \\ &\alpha_2' = \alpha_2 - \alpha_1 A\,,\ \beta_2' = \beta_2 - \beta_1 A, \\ &a_0' = a_0\,,\ b_0' = b_0\,,\ a_1' = a_1\,,\ b_1' = b_1, \\ &a_2' = a_2 - a_1 A - p_0^2 \frac{\partial A}{\partial x}\,,\ b_2' = b_2 - b_1 A - p_0^2 \frac{\partial A}{\partial y}, \end{aligned} \tag{2.21}$$

and also

$$c' = r' c_{-1}' + c_0' + \frac{c_1'}{r'} + \dots, \tag{2.22}$$

where

$$c'_{-1} = c_{-1} = -2H, \qquad (2.23)$$

$$c'_0 = c_0 + 2\left(\dot{A} - HA - a_0\frac{\partial A}{\partial x} - b_0\frac{\partial A}{\partial y}\right).$$

Hence we see from the first of (2.20 b) that if we choose $A = q_1$ then $q'_1 = 0$ and this value of A in the second of (2.23), coupled with the field equation (2.16), shows that $c'_0 = \Delta \log p_0$. We have given the transformations of α_2, β_2, a_2 and b_2 above since they are required to calculate c'_0. We note that the field equations (2.9)-(2.13), (2.15) are left invariant by (2.20 b), (2.21) and (2.23). Henceforth we can, without loss of generality, put $q_1 = 0$ in all our equations up to (2.16). Thus (2.9), (2.10) and (2.16) become

$$q_2 = \frac{1}{2}(\alpha_1^2 + \beta_1^2), \qquad (2.24\text{ a})$$

$$q_3 = \frac{2}{3}(\alpha_1\alpha_2 + \beta_1\beta_2), \qquad (2.24\text{ b})$$

$$c_0 = \Delta \log p_0. \qquad (2.24\text{ c})$$

Noting from (2.13) and (2.15) that a_0, b_0 satisfy the Cauchy-Riemann equations we can utilise this by making the transformation

$$x = f(x', y', u), \quad y = g(x', y', u) \qquad (2.25)$$

with $\dfrac{\partial f}{\partial x'} = \dfrac{\partial g}{\partial y'}$ and $\dfrac{\partial f}{\partial y'} = -\dfrac{\partial g}{\partial x'}$. The form of the line-element and the field equations remain invariant with

$$p' = pW, \qquad (2.26\text{ a})$$

$$\alpha'_1 = W^2\left\{\alpha_1\left[\left(\frac{\partial f}{\partial x'}\right)^2 - \left(\frac{\partial g}{\partial x'}\right)^2\right] + 2\beta_1\frac{\partial f}{\partial x'}\frac{\partial g}{\partial x'}\right\}, \qquad (2.26\text{ b})$$

$$\beta'_1 = W^2\left\{-2\alpha_1\frac{\partial f}{\partial x'}\frac{\partial g}{\partial x'} + \beta_1\left[\left(\frac{\partial f}{\partial x'}\right)^2 - \left(\frac{\partial g}{\partial x'}\right)^2\right]\right\}, \qquad (2.26\text{ c})$$

$$a_0' = W^2 \left\{(a_0 + \dot{f}\,)\frac{\partial f}{\partial x'} + (b_0 + \dot{g})\frac{\partial g}{\partial x'}\right\}, \tag{2.26 d}$$

$$b_0' = W^2 \left\{(b_0 + \dot{g})\frac{\partial f}{\partial x'} - (a_0 + \dot{f}\,)\frac{\partial g}{\partial x'}\right\}, \tag{2.26 e}$$

$$a_1' = W^2 \left\{(a_1 + \alpha_1\dot{f} + \beta_1\dot{g})\frac{\partial f}{\partial x'} + (b_1 - \alpha_1\dot{g} + \beta_1\dot{f}\,)\frac{\partial g}{\partial x'}\right\}, \tag{2.26 f}$$

$$b_1' = W^2 \left\{(b_1 - \alpha_1\dot{g} + \beta_1\dot{f}\,)\frac{\partial f}{\partial x'} - (a_1 + \alpha_1\dot{f} + \beta_1\dot{g})\frac{\partial g}{\partial x'}\right\}, \tag{2.26 g}$$

$$c' = c, \tag{2.26 h}$$

where

$$W = \left\{\left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial g}{\partial x'}\right)^2\right\}^{-1/2}. \tag{2.27}$$

We have included α_1', β_1' in the list above since they are needed to compute a_1' and b_1'. We can choose f and g so that $\dot{f} = - a_0$, $\dot{g} = - b_0$ and thus (2.26 d, e) imply $a_0' = 0$, $b_0' = 0$. We also have then

$$a_1 + \alpha_1\dot{f} + \beta_1\dot{g} = a_1 - \alpha_1 a_0 - \beta_1 b_0, \tag{2.28 a}$$

$$b_1 - \alpha_1\dot{g} + \beta_1\dot{f} = b_1 + \alpha_1 b_0 - \beta_1 a_0, \tag{2.28 b}$$

and both of these vanish on account of the field equations (2.11). Hence, by (2.26 f, g), we have $a_1' = 0$, $b_1' = 0$. Therefore, without loss of generality, we can put a_0, b_0, a_1, b_1 all equal to zero in our equations prior to (2.16). Thus, in addition to (2.24) we are left with

$$c_{-1} = - 2H\,,\ H = p_0^{-1}\dot{p}_0. \tag{2.29}$$

We turn now to the equations $R_{A3} = 0$ and the vanishing of the r^{-3}-term in each gives

$$a_2 = p_0^4 \left\{\frac{\partial}{\partial x}(p_0^{-2}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-2}\beta_1)\right\}, \tag{2.30 a}$$

$$b_2 = p_0^4 \left\{\frac{\partial}{\partial x}(p_0^{-2}\beta_1) - \frac{\partial}{\partial y}(p_0^{-2}\alpha_1)\right\}. \tag{2.30 b}$$

If we now compute the r^{-3}-terms in $R_{AB} = 0$ we find that they give us the equations

$$\frac{\partial}{\partial u}(p_0^{-2}\alpha_2) = 0 = \frac{\partial}{\partial u}(p_0^{-2}\beta_2). \tag{2.31}$$

The first equation here comes from both $R_{11} = 0$ and $R_{22} = 0$ while the second comes from $R_{12} = 0$. Now the r^{-4}-term in $R_{A3} = 0$ yields the two equations

$$\frac{\partial}{\partial x}(p_0^{-2}\alpha_2) = -\frac{\partial}{\partial y}(p_0^{-2}\beta_2), \tag{2.32 a}$$

$$\frac{\partial}{\partial x}(p_0^{-2}\beta_2) = \frac{\partial}{\partial y}(p_0^{-2}\alpha_2). \tag{2.32 b}$$

We can take advantage of the simplicity of (2.31) and (2.32) before proceeding with the calculation of the remaining field equations. Let us assume that the line-element (2.18) represents a family (parametrised by u) of closed 2-surfaces homeomorphic to $\mathbf{S}_2$. The coordinates x, y constitute a chart on $\mathbf{S}_2$ with a point removed and $p_0\,(u, x, y)$ is necessarily unbounded as a function of x, y. The equations (2.31) and (2.32) imply that, for any value of u, $p_0^{-2}\alpha_2$ and $p_0^{-2}\beta_2$ are harmonic functions on the closed 2-surface (2.18) and therefore they are constants. Thus $\alpha_2 = \text{const.} \times p_0^2$, $\beta_2 = \text{const.} \times p_0^2$ and in each case the constant must vanish otherwise α_2 and β_2 are unbounded. Thus we have

$$\alpha_2 = \beta_2 = 0. \tag{2.33}$$

We comment briefly in the next section on the relationship between (2.33) and an out-going radiation condition. We note from (2.24 b) that now

$$q_3 = 0. \tag{2.34}$$

We now compute the r^{-4}-term in $R_{12} = 0$. Putting $c_1 = -2m\,(u, x, y)$ from here on (this will facilitate comparison with the RT fields in the special case of vanishing shear of the integral curves of $\frac{\partial}{\partial r}$ discussed in the next section) we find

$$\frac{\partial a_3}{\partial y} + \frac{\partial b_3}{\partial x} = 8\,(\dot{\beta}_3 - 3H\beta_3) - 4m\beta_1 - 20\alpha_1\dot{\alpha}_1\beta_1$$

$$- 4\alpha_1^2\dot{\beta}_1 - 8\beta_1^2\dot{\beta}_1 + 8\beta_1\,(\beta_1^2 + 3\alpha_1^2)\,H$$

$$- \beta_1\frac{\partial a_2}{\partial x} - b_2\frac{\partial \alpha_1}{\partial x} + 3\alpha_1\frac{\partial b_2}{\partial x} - 3a_2\frac{\partial \beta_1}{\partial x}$$

$$- \beta_1\frac{\partial b_2}{\partial y} + a_2\frac{\partial \alpha_1}{\partial y} - 3\alpha_1\frac{\partial a_2}{\partial y} - 3b_2\frac{\partial \beta_1}{\partial y}$$

$$+ 4p_0^{-1}\beta_1\left(a_2\frac{\partial p_0}{\partial x} + b_2\frac{\partial p_0}{\partial y}\right) + 4p_0^{-2}a_2b_2. \tag{2.35}$$

The r^{-4}-terms in $R_{11} = 0$ and $R_{22} = 0$ will each introduce functions in addition to those twenty-one given explicitly in the expansions (2.6) and which therefore do not contribute to a knowledge of the metric tensor components up to and including the r^{-1}-terms. However the r^{-4}-term in $R_{11} - R_{22} = 0$ does not introduce such additional functions and provides us with the field equation

$$\frac{\partial a_3}{\partial x} - \frac{\partial b_3}{\partial y} = 8\,(\dot{\alpha}_3 - 3H\alpha_3) - 4m\alpha_1 + 12\alpha_1\beta_1\dot{\beta}_1 + 12\beta_1^2\dot{\alpha}_1$$

$$- 8\alpha_1^2\dot{\alpha}_1 + 8\alpha_1\,(\alpha_1^2 - 3\beta_1^2)\,H - \alpha_1\frac{\partial a_2}{\partial x}$$

$$- 3a_2\frac{\partial \alpha_1}{\partial x} - 3\beta_1\frac{\partial b_2}{\partial x} + b_2\frac{\partial \beta_1}{\partial x} - \alpha_1\frac{\partial b_2}{\partial y}$$

$$- 3b_2\frac{\partial \alpha_1}{\partial y} + 3\beta_1\frac{\partial a_2}{\partial y} - a_2\frac{\partial \beta_1}{\partial y}$$

$$+ 4p_0^{-1}\alpha_1\left(a_2\frac{\partial p_0}{\partial x} + b_2\frac{\partial p_0}{\partial y}\right)$$

$$+ 2p_0^{-2}\,(a_2^2 - b_2^2). \tag{2.36}$$

This completes the derivation of the main equations which are of interest to us and we turn now to the supplementary conditions.

In the case of $R_{44} = 0$ the first non-identically vanishing term is the r^{-2}-term from which we obtain the equation

$$\dot{M} - 3HM + |\dot{\gamma} - H\gamma|^2 = \frac{1}{4}\Delta c_0 + + 4\mathrm{Re}\left\{\gamma \frac{\partial}{\partial z}\left(p_0^2 \frac{\partial H}{\partial z}\right)\right\} \tag{2.37 a}$$

where

$$M = m - \dot{q}_2 + 2Hq_2 - \frac{1}{2}p_0^2\left\{\frac{\partial}{\partial x}(p_0^{-2}a_2) + \frac{\partial}{\partial y}(p_0^{-2}b_2)\right\}, \tag{2.37 b}$$

$$\gamma = \alpha_1 + i\beta_1, \tag{2.37 c}$$

$$z = x + iy, \tag{2.37 d}$$

and Re in (2.37 a) indicates the real part of the quantity in brackets following. Finally the first non-identically vanishing terms in R_{A4} are the r^{-3}-terms which give us the two field equations

$$\frac{3}{2}p_0^{-1}(\dot{A}_3 - 4HA_3) - p_0\frac{\partial m}{\partial x} = p_0^3\left\{\frac{\partial}{\partial x}(p_0^{-2}\alpha_1 c_0) + \frac{\partial}{\partial y}(p_0^{-2}\beta_1 c_0)\right\}$$

$$- b_2\frac{\partial^2 p_0}{\partial x \partial y} - a_2\frac{\partial^2 p_0}{\partial x^2} - (\dot{\beta}_1 - H\beta_1)p_0^2\left\{\frac{\partial}{\partial x}(p_0^{-1}\beta_1)\right.$$

$$\left. - \frac{\partial}{\partial y}(p_0^{-1}\alpha_1)\right\} + \frac{\partial p_0}{\partial y}\frac{\partial a_2}{\partial y} + 2\beta_1(\dot{\alpha}_1 - H\alpha_1)\frac{\partial p_0}{\partial y}$$

$$- (\dot{\alpha}_1 - H\alpha_1)p_0^2\left\{\frac{\partial}{\partial x}(p_0^{-1}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-1}\beta_1)\right\} - \frac{\partial p_0}{\partial x}\left(\dot{q}_2 - 2Hq_2\right.$$

$$\left. + \frac{\partial b_2}{\partial y} - b_2\frac{\partial}{\partial y}(\log p_0) - a_2\frac{\partial}{\partial x}(\log p_0)\right) - p_0\frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial a_2}{\partial y} - \frac{1}{2}\frac{\partial b_2}{\partial x}\right.$$

$$\left. + 2\beta_1(\dot{\alpha}_1 - H\alpha_1)\right) - \alpha_1\beta_1 p_0\frac{\partial H}{\partial y}$$

$$- (\dot{\alpha}_1 - H\alpha_1)p_0^{-1}a_2 - (\dot{\beta}_1 - H\beta_1)p_0^{-1}b_2 \tag{2.38 a}$$

and

$$\frac{3}{2}\, p_0^{-1}\,(\dot{B}_3 - 4HB_3) - p_0\,\frac{\partial m}{\partial y} = p_0^3\left\{\frac{\partial}{\partial x}\,(p_0^{-2}\beta_1 c_0) - \frac{\partial}{\partial y}\,(p_0^{-2}\alpha_1 c_0)\right\}$$

$$- a_2\,\frac{\partial^2 p_0}{\partial x \partial y} - b_2\,\frac{\partial^2 p_0}{\partial y^2} - (\dot{\beta}_1 - H\beta_1)\, p_0^2\left\{\frac{\partial}{\partial x}\,(p_0^{-1}\alpha_1)\right.$$

$$\left. + \frac{\partial}{\partial y}\,(p_0^{-1}\beta_1)\right\} + \frac{\partial p_0}{\partial x}\frac{\partial b_2}{\partial x} - 2\beta_1\,(\dot{\alpha}_1 - H\alpha_1)\,\frac{\partial p_0}{\partial x}$$

$$+ (\dot{\alpha}_1 - H\alpha_1)\, p_0^2\left\{\frac{\partial}{\partial x}\,(p_0^{-1}\beta_1) - \frac{\partial}{\partial y}\,(p_0^{-1}\alpha_1)\right\} - \frac{\partial p_0}{\partial y}\left(\dot{q}_2 - 2Hq_2\right.$$

$$\left. + \frac{\partial a_2}{\partial x} - a_2\,\frac{\partial}{\partial x}(\log p_0) - b_2\frac{\partial}{\partial y}\,(\log p_0)\right) - p_0\,\frac{\partial}{\partial x}\left(\frac{1}{2}\,\frac{\partial b_2}{\partial x} - \frac{1}{2}\,\frac{\partial a_2}{\partial y}\right.$$

$$\left. - 2\beta_1\,(\dot{\alpha}_1 - H\alpha_1)\right) + \alpha_1\beta_1 p_0\,\frac{\partial H}{\partial x}$$

$$+ (\dot{\alpha}_1 - H\alpha_1)\, p_0^{-1} b_2 - (\dot{\beta}_1 - H\beta_1)\, p_0^{-1} a_2, \qquad (2.38\text{ b})$$

where

$$A_3 = a_3 - \frac{1}{3}(b_2\beta_1 + a_2\alpha_1) - \frac{4}{3}\, q_2 p_0\,\frac{\partial p_0}{\partial x} - \frac{2}{3}\, p_0^3\,\frac{\partial}{\partial y}\,(p_0^{-1}\alpha_1\beta_1), \quad (2.38\text{ c})$$

$$B_3 = b_3 - \frac{1}{3}(a_2\beta_1 - b_2\alpha_1) - \frac{4}{3}\, q_2 p_0\,\frac{\partial p_0}{\partial y} + \frac{2}{3}\, p_0^3\,\frac{\partial}{\partial x}\,(p_0^{-1}\alpha_1\beta_1). \quad (2.38\text{ d})$$

This completes the calculation of all the field equations which bring into play the twenty-one functions shown explicitly in (2.6), and only these functions.

3. The Curvature Tensor

With the field equations satisfied we obtain the following expressions, in the notation of Newman and Penrose ([18]), for the leading terms in the expansions of the curvature tensor components in inverse powers of r:

$$\Psi_0 = -(R_{3131} + iR_{3132}) \tag{3.1 a}$$
$$= -\frac{1}{r^5}\left\{6(\alpha_3 + i\beta_3) - \frac{3}{2}(\gamma + \bar{\gamma})^2(\gamma - \bar{\gamma}) - 2\bar{\gamma}^3\right\} + \dots,$$

$$\Psi_1 = -\frac{1}{\sqrt{2}}(R_{3431} + iR_{3432}) \tag{3.1 b}$$
$$= -\frac{1}{r^4\sqrt{2}}\left\{\frac{3}{2}p_0^{-1}(a_3 + ib_3) + 3p_0^3\gamma\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right\} + \dots,$$

$$\Psi_2 = -\frac{1}{2}(R_{3434} + iR_{3412}) \tag{3.1 c}$$
$$= -\frac{1}{r^3}\left\{M + \gamma(\dot{\bar{\gamma}} - H\bar{\gamma}) + 2p_0^2\frac{\partial}{\partial\bar{z}}\left(p_0^2\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right)\right\} + \dots,$$

$$\Psi_3 = -\frac{1}{\sqrt{2}}(R_{3414} - iR_{3424}) \tag{3.1 d}$$
$$= -\frac{1}{r^2\sqrt{2}}\left\{p_0\frac{\partial c_0}{\partial z} + 2p_0\bar{\gamma}\frac{\partial H}{\partial\bar{z}} + 2p_0^2\frac{\partial}{\partial u}\left(p_0\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right)\right\} + \dots,$$

$$\Psi_4 = -(R_{4141} - iR_{4142}) \tag{3.1 e}$$
$$= -\frac{2}{r}\left\{\frac{\partial}{\partial z}\left(p_0^2\frac{\partial H}{\partial z}\right) + \frac{1}{2}p_0^2\frac{\partial}{\partial u}\left(p_0^{-1}\frac{\partial}{\partial u}(p_0^{-1}\bar{\gamma})\right)\right\} + \dots.$$

Had we not obtained the conditions (2.33), from the field equations and the assumed regularity of the family of 2-surfaces with line-element (2.18), we would have found a term $-2(\alpha_2 + i\beta_2)r^{-4}$ in Ψ_0. It would appear from the study by Sachs [19] that the absence of this term ensures that the radiation in our fields is out-going.

A glance at (2.8 a) indicates that if $\alpha = \beta = 0$ then the integral curves of $\frac{\partial}{\partial r}$ are shear-free (this is true exactly). We see from (3.1 e) that in this case gravitational radiation is still present in our fields (in contradistinction to the formulation of this problem by Bondi and Sachs whose fields do not manifestly contain shear-free radiation).

If we require that only this shear-free radiation be present then our equations undergo a great simplification. Retaining terms up to r^{-1} in the metric tensor components, the line-element (2.3) becomes

$$ds^2 = \left(c_{-1}r + c_0 - \frac{2m}{r}\right) du^2 + 2\,du\,dr$$

$$- r^2 p_0^{-2} \left(dx^2 + dy^2 + 2\frac{a_3}{r^3}\,dx\,du + 2\frac{b_3}{r^3}\,dy\,du\right), \tag{3.2}$$

where c_0 and c_{-1} are given by (2.24 c) and (2.29). The equations (2.35) and (2.36) reduce to the Cauchy-Riemann equations

$$\frac{\partial a_3}{\partial y} + \frac{\partial b_3}{\partial x} = 0\,, \quad \frac{\partial a_3}{\partial x} - \frac{\partial b_3}{\partial y} = 0. \tag{3.3}$$

The supplementary condition (2.37) is now

$$\dot{m} - 3Hm = \frac{1}{4}\Delta c_0, \tag{3.4}$$

and the supplementary conditions (2.38) simplify to

$$\frac{3}{2} p_0^{-2} (\dot{a}_3 - 4Ha_3) = \frac{\partial m}{\partial x}\,, \quad \frac{3}{2} p_0^{-2} (\dot{b}_3 - 4Hb_3) = \frac{\partial m}{\partial y}. \tag{3.5}$$

The transformations

$$x = x' + \frac{B(x', y', u)}{r'^3} + \ldots, \tag{3.6 a}$$

$$y = y' + \frac{C(x', y', u)}{r'^3} + \ldots, \tag{3.6 b}$$

$$r = r' - \frac{1}{r'^2}\left(\frac{\partial B}{\partial x'} - Bp_0^{-1}\frac{\partial p_0}{\partial x'} - Cp_0^{-1}\frac{\partial p_0}{\partial y'}\right) + \ldots, \tag{3.6 c}$$

where $\dfrac{\partial B}{\partial x'} = \dfrac{\partial C}{\partial y'}\,, \ \dfrac{\partial B}{\partial y'} = -\dfrac{\partial C}{\partial x'}$ leave the line-element (3.2) invariant in form (up to and including the r'^{-1}-term) and the functions of u, x, y appearing in (3.2) are replaced by the same functions of u, x′, y′ with the exception of a_3 and b_3 which undergo the transformations

$$a_3' = a_3(u, x', y') + \dot{B}\ , \ b_3' = b_3(u, x', y') + \dot{C}. \tag{3.7}$$

Thus, on account of the field equations (3.3), we may choose B and C so that $a_3' = 0$, $b_3' = 0$. Therefore putting, without loss of generality, $a_3 = b_3 = 0$ in (3.2)-(3.5), we obtain the equations which, together with (2.24 c) and (2.29), describe the RT vacuum gravitational fields. With $\alpha = \beta = a_3 = b_3 = 0$ in (3.1), the surviving leading terms in the curvature tensor are the *exact* expressions for these components of the curvature tensor of the RT fields, with the exception of (3.1 e) which, in addition, has an r^{-2}-term given by

$$\frac{1}{r^2}\frac{\partial}{\partial z}\left(p_0^2\,\frac{\partial c_0}{\partial z}\right).$$

4. The Characteristic Initial-Value Problem

We began our discussion of the vacuum field equations, resulting from the series expansions of the Ricci tensor components in powers of r^{-1}, with the twenty-one functions of u, x, y shown explicitly in (2.6). Using the field equations and allowable coordinate transformations we reduced these, by the end of section 2, to the thirteen functions

$$c_{-1},\ c_0,\ q_2,\ a_2,\ b_2,\ c_1 = -\,2m,\ p_0,\ \alpha_1,\ \alpha_3,\ \beta_1,\ \beta_3,\ a_3,\ b_3. \tag{4.1}$$

These however are not all independent for, by (2.24 c) and (2.29) both c_{-1} and c_0 can be derived from p_0; q_2 is derived from α_1 and β_1 as indicated in (2.24 a) while a_2 and b_2 are derived from p_0, α_1 and β_1 as in (2.30). Thus only the last eight functions listed in (4.1) are independent. To solve the characteristic initial-value problem we must determine what data must be specified on a characteristic hypersurface $u = u_0$ (say) so that the eight independent functions above are known for $u > u_0$. Our result may be summarised in the following.

Proposition: To determine a solution of the field equations one must specify five arbitrary functions of three variables, F (r, x, y), p_0 (u, x, y), $\dot{\alpha}_1$ (u, x, y), $\dot{\beta}_1$ (u, x, y), and three arbitrary functions of two variables, m (u_0, x, y), a_3 (u_0, x, y), b_3 (u_0, x, y), where

$$F(r, x, y) = (\alpha + i\beta)_{u=u_0}. \tag{4.2}$$

Proof: The supplementary condition (2.37 a) is a propagation equation for m (u, x, y) off the initial hypersurface $u = u_0$. It enables us to determine m (u, x, y) for $u > u_0$ from the given data. Using this, and the given data, in the supplementary conditions (2.38), which we regard as propagation equations for a_3 and b_3 off the initial hypersurface $u = u_0$, we obtain a_3 (u, x, y) and b_3 (u, x, y) for $u > u_0$. With m (u, x, y), a_3 (u, x, y), b_3 (u, x, y) for $u > u_0$ determined, we turn to the main equations (2.35) and (2.36). We regard these equations as propagation equations for β_3 (u, x, y) and α_3 (u, x, y) off $u = u_0$ and, with the data available to us now, these functions can be determined for $u > u_0$.

The proposition above should be compared with a similar result of Sachs ([14]) in which he found that to specify one solution of the field equations "one must specify four functions of three variables and three functions of two variables; these functions are not subject to constraints". This raises the question: why do we require an additional function of three variables to solve the characteristic initial-value problem when the line-element we began with, given by (2.3), is just as general as that chosen by Sachs (cf. (1.10)? The answer is that our boundary conditions as $r \to +\infty$, embodied in the expansions (2.6), are less restrictive than those of Sachs. In our formulation we thereby ensure the manifest appearance in (3.1 e) of both shear-free *and* shearing gravitational radiation. The extra function at our disposal is p_0 (u, x, y).

To make a more direct comparison between our and Sachs' results we note that his boundary conditions (eqs. (3.1) of ([14])) as $\varrho \to +\infty$ on the functions appearing in the line-element (1.10) are satisfied by assuming the following expansions in powers of his radial coordinate ϱ:

$$V = \varrho - 2M + O(\varrho^{-1}), \tag{4.3 a}$$

$$\beta = \frac{\beta_1}{\varrho} + \dots, \tag{4.3 b}$$

$$U^A = \frac{U_2{}^A}{\varrho^2} + \frac{U_3{}^A}{\varrho^3} + \ldots \qquad (A = 1, 2), \tag{4.3 c}$$

$$\gamma = \frac{\gamma_1}{\varrho} + \frac{\gamma_2}{\varrho^2} + \frac{\gamma_3}{\varrho^3} + \ldots, \tag{4.3 d}$$

$$\delta = \frac{\delta_1}{\varrho} + \frac{\delta_2}{\varrho^2} + \frac{\delta_3}{\varrho^3} + \ldots. \tag{4.3 e}$$

The twelve coefficients shown explicitly here are functions of u, θ, ϕ and are required to construct the metric tensor components up to and including the ϱ^{-1}-term. Using the vacuum field equations and an out-going radiation condition, which also ensures the absence of $(\log \varrho)/\varrho^3$ terms in U^A, making the expansion (4.3 c) possible, one can conclude (from eqs. (4.6) and (4.9 a) of ([14])) that $\beta_1 = 0$ and $\gamma_2 = \delta_2 = 0$. In addition (from (4.9 b) of ([14])) $U_2{}^A$ can be derived from γ_1 and δ_1 and so the number of independent functions of u, θ, ϕ remaining is seven, namely,

$$U_3{}^A, M, \gamma_1, \gamma_3, \delta_1, \delta_3. \tag{4.4}$$

This is one less than the number of independent functions resulting from our calculations. Upon writing out the line-element (1.10 a) in which the metric tensor components are calculated up to and including the ϱ^{-1}-term we find

$$\begin{aligned} ds^2 &= (1 - 2M\varrho^{-1})\, du^2 + 2\, du\, d\varrho \\ &- \varrho^2 h_{AB}\, [dx^A - (\varrho^{-2}U_2{}^A + \varrho^{-3}U_3{}^A)\, du] \times \\ &\times [dx^B - (\varrho^{-2}U_2{}^B + \varrho^{-3}U_3{}^B)\, du] \end{aligned} \tag{4.5 a}$$

where

$$h_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta\, d\phi^2 + O(\varrho^{-1}). \tag{4.5 b}$$

The ϱ^{-1}, ϱ^{-2} and ϱ^{-3}-terms in (4.5 b), which must be substituted into (4.5 a), are evaluated using γ_1, γ_3, δ_1, δ_3. Comparing this with our line-element at this stage of the computations we find that the luminosity distance ϱ differs from our radial coordinate r by terms of order r^{-1}. A comparison of the coefficient of du^2 in (4.5 a), with that given by (2.6 a) in (2.3), reveals that we must have

$$c_0 = \Delta \log p_0 = 1 \, , \; H = p_0^{-1}\dot{p}_0 = 0, \tag{4.6}$$

for agreement with Sachs. Since we are free to make a coordinate transformation of the type (2.25) with f and g independent of u (this would not bring back the functions a_0, b_0, a_1, b_1 which we used (2.25) to dispose of) we can use this freedom to transform the solution of (4.6) into

$$p_0 = 1 + \frac{1}{4}(x^2 + y^2). \tag{4.7}$$

Thus the extra function freely available to us is fixed by (4.7) in this case, and the 2-surfaces with line-element (2.18) are 2-spheres for all values of u.

5. Discussion

A number of questions demanding further study emerge from the results we have presented above. We have chosen to comment on three of them below.

If we substitute the value of p_0 given in (4.7) into the supplementary condition (2.37 a) we find the remarkably simple equation

$$\dot{M} + |\dot{\gamma}|^2 = 0, \tag{5.1}$$

where M (u, x, y) is obtained from (4.7) and (2.37 b). Upon averaging (5.1) over the 2-sphere, with line-element given by (2.18) and (4.7), and denoting this average by the brackets $\langle \; \rangle$, we have

$$\langle \dot{M} \rangle = - \langle |\dot{\gamma}|^2 \rangle. \tag{5.2}$$

This is a formula of the type (1.17) derived by Bondi and Sachs. For the general case, in which shear-free and shearing radiation is present in our formulation, it is an open question as to whether a formula comparable to (5.2) exists. Presumably the average now should be taken over the 2-surfaces with line-element (2.18). It is interesting to note that one cannot ascribe any physical significance to $\langle \dot{M} \rangle \neq 0$ in this case. If we make a coordinate transformation of the form

$$u = \tau(u') \, , \; r = r'/\dot{\tau} \, , \; \dot{\tau} = d\tau/du', \tag{5.3}$$

then our line-element and field equations remain invariant with substitutions such as

$$\begin{aligned} H' &= \dot{\tau}H + \ddot{\tau}/\dot{\tau} \, , \; c_0' = \dot{\tau}^2 c_0, \\ m' &= \dot{\tau}^3 m \, , \; p_0' = \dot{\tau} p_0 \, , \; q_2' = \dot{\tau}^2 q_2. \end{aligned} \tag{5.4}$$

From (3.27 b) these imply

$$M' = \dot{\tau}^3 M \tag{5.5}$$

and thus it is possible to choose $\tau(u')$ so as to reduce $\langle M \rangle$ to a constant or $\langle \dot{M} \rangle$ to zero. We note that the transformation (5.3) cannot be implemented in the fields considered by Bondi and Sachs since it would lead to a violation of their boundary conditions (cf. eqs. (3.1 a, b) of ([14]))

$$\lim_{\varrho \to +\infty} \varrho^{-1} V = 1 \, , \quad \lim_{\varrho \to +\infty} \beta = 0. \tag{5.6}$$

Nor can (5.3) be implemented in our formulation when the equations (4.6) hold and so the consequences of this transformation mentioned above do not apply to (5.2).

The question above is clearly related to the mysterious role that shear-free radiation plays in the Bondi-Sachs formulation. This also requires further consideration.

Finally, a study of the asymptotic symmetry group in our formulation and its relationship to the BMS group ([14]) also remains to be carried out.

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