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CAUCHY-RIEMANN STRUCTURES IN OPTICAL GEOMETRY*

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It is shown that the three-dimensional manifold of null geodesics forming a shear-free congruence has a natural Cauchy-Riemann structure, depending only on the optical geometry associated with the congruence.

1. INTRODUCTION

There is a natural geometry adapted to the study of null (isotropic, singular) Maxwell and Yang-Mills fields. It constitutes also the underlying structure of algebraically special gravitational fields. The origins of this geometry can be traced back to early papers by H. Bateman[1] and E. Cartan[2], to work on shear-free congruences of null geodesics[3-5] and to R. Penrose's twistor programme[6,7]. Recently, one of us proposed to use the name 'optical geometry' for this structure and listed its basic properties[8,9]. In this lecture we present a novel characterization of an optical geometry with shear-free rays: locally, such a geometry is a product of $\mathbb{R}$ by a 3-dimensional Cauchy-Riemann manifold[10-12].

In our work on this problem we have been influenced by conversations with, and/or papers by, R. Penrose[13], C.D. Hill[14], P. Sommers[15], J. Tafel[16], and R.O. Wells, Jr.[17]. The present text contains only a brief summary of the subject; a fuller account of our joint work is being published elsewhere[18-20]. We follow the terminology and notation prevalent in differential geometry and mathematical physics[21-23].

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2. FLAG GEOMETRY

A *flag geometry* on a 4-dimensional smooth orientable manifold M is a pair (K,L) of real line bundles such that

$$K \subset TM, \quad L \subset T^*M$$

and, if K_x and L_x denote, respectively, the fibres of K and L over $x \in M$, then

$$u \lrcorner \alpha = 0 \text{ for any } u \in K_x \text{ and } \alpha \in L_x.$$

A section k of $K \to M$ is a vector field on M, whereas a section λ of $L \to M$ is a field of 1-forms on M; for any such sections $k \lrcorner \lambda = 0$. A *metric* tensor g on M is said to be *adapted* to (K,L) if, for any such sections k and λ, one has $g(k) \wedge \lambda = 0$, where g(k) is the 1-form characterized by $l \lrcorner g(k) = g(k,l)$ for any vector field l. In other words, with respect to any adapted metric, and for any $x \in M$, the line K_x is null and $\ker \lambda(x)$ is the 3-space of all vectors orthogonal to K_x. The bundle $\bigcup_{x \in M} \ker\lambda(x)$ will be denoted kerL.

Similarly, a p-*form* (p=1,2 or 3) F on M is *adapted* to (K,L) if, for any k and λ defined as above, one has

(1) $k \lrcorner F = 0$ and $\lambda \wedge F = 0$.

For example, if g is an adapted metric, then g(k) is an adapted 1-form.

Let $(\phi_t(k))$ be the *flow* generated by the vector field k, section of $K \to M$. If L is invariant with respect to the flow $(\phi_t(k))$, then it is also invariant with respect to $(\phi_t(\rho k))$, where ρ is any function on M. It is meaningful, therefore, to define L as being invariant with respect to K if, for any sections k and λ, one has

(i) $\lambda \wedge \mathcal{L}_k \lambda = 0$

where $\mathcal{L}_k \lambda$ denotes the Lie derivative of λ in the direction of k. In Refs. 19 and 23 we have shown, that (i) is equivalent to any of the following conditions:

(ii) the 3-form $\lambda \wedge d\lambda$ is adapted;

(iii) the lines of the flow $(\phi_t(k))$ define a congruence of null geodesics with respect to any metric tensor adapted to (K,L);

(iv) if F is an adapted 2-form, then $\lambda \wedge dF = 0$.

A flag geometry which has any - and therefore all - of the properties (i)-

(iv) is said to be *geodetic*. In particular, a flag geometry corresponding to an *integrable* bundle $\ker L \subset TM$, i.e., such that

$$\lambda \wedge d\lambda = 0$$

is geodetic. If the bundle kerL is non-integrable, then the congruence of null curves defined by the flow is said - by physicists - to be *twisting*.

3. OPTICAL GEOMETRY

A flag geometry is sufficient to define a congruence of null geodesics and the notion of null (adapted) 2-forms. If any such form F is interpreted as an electromagnetic field, then it is possible to write one part of Maxwell's equations, namely $dF = 0$, but not the other. Roughly speaking, an optical geometry is the weakest structure needed on a 4-dimensional manifold M to write the full set of Maxwell's equations for null electromagnetic fields.

In a Lorentzian geometry based on a metric tensor g one introduces the Hodge dual $*_g F$ of F relative to g and some orientation on M. The other part of Maxwell's equations reads then $d*_g F = 0$.

Let us start again with a flag geometry (K,L) on M and let A be the set of all adapted Lorentzian metric tensor fields on M. If $g \in A$ and F is an adapted p-form, then $*_g F$ is an adapted (4-p)-form. For example, if the flag geometry is geodetic, then $*_g(\lambda \wedge d\lambda)$ is proportional to λ. If F is a nowhere vanishing 2-form on M adapted to (K,L) then

(2) $$g \underset{R}{\equiv} g' \Leftrightarrow *_g F = *_{g'} F\text{, where } g \text{ and } g' \in A,$$

defines an equivalence relation R in A. This equivalence relation does not depend on F; only at this point does the assumption of M being four-dimensional enter into our considerations.

An *optical geometry* on M consists of the pair (K,L) together with an element B of A/R and an orientation of the vector bundle (kerL)/K of fibre dimension 2. Equivalently, it can be defined as a flag geometry (K,L) supplemented by a complex structure on (kerL)/K, i.e. a linear bundle morphism

$$J: (\ker L)/K \to (\ker L)/K \text{ such that } J^2 = -\mathrm{id}.$$

This additional structure makes (kerL)/K into a complex line bundle over M.

It is easy to see that if $g \in B \subset A$ then $g' \in B$ if, and only if, there is a positive function ρ on M, and a 1-form μ such that

(3) $\quad g' = \rho g + 2\mu\lambda ,$

where $2\mu\lambda$ is an abbreviation for $\mu \otimes \lambda + \lambda \otimes \mu$.

Let M and M' be two 4-manifolds with optical geometries (K,L,B) and (K',L',B') respectively. A diffeomorphism $f: M \to M'$ is said to be an isomorphism of optical geometries if $f^*B' = B$, $f^*L' = L$ and $f_*K = K'$.

It is often convenient to define an optical geometry by (i) giving a Lorentzian metric g and a null vector field k, (ii) declaring that K and L are spanned by k and g(k), respectively, and (iii) specifying an orientation in (kerL)/K.

Given an optical geometry (K,L,B) on M, it is meaningful to consider solutions of Maxwell's equations

(4) $\quad dF = 0$ and $d{*_g}F = 0$

where F is assumed to be adapted and $g \in B$. Equations (1) and (4) imply

(5) $\quad \mathcal{L}_k F = 0$ and $\mathcal{L}_k {*_g}F = 0$

so that both F and ${*_g}F$ are invariant by the flow $(\phi_t(k))$. Therefore

$$*_g F = \phi_t(k)^*(*_g F) = *_{\phi_t(k)^* g} F, \quad \text{for any } t \in \mathbb{R} ,$$

and, if F vanishes nowhere, we obtain, by virtue of (2), that the flow $\phi_t(k)$ consists of optical automorphisms. The underlying flag geometry is then geodetic (because L is preserved); the remaining property implied by

(6) $\quad \phi_t(k)^* g \underset{R}{\equiv} g$, for any $t \in \mathbb{R}$,

is the shear-free nature of the null geodetic congruence. Indeed, in view of (3), condition (6) is equivalent to

(7) $\quad \mathcal{L}_k g = \sigma g + 2\nu\lambda$,

where σ is a function and ν is a 1-form on M. The last equation is known to be equivalent to the geodetic and shear-free property of the congruence of null curves defined by k[23].

An optical geometry satisfying any of the equivalent conditions (6) or (7) is said to be *shear-free;* the geodetic property is then implied. The relevance of optical geometry is apparent also from the following

THEOREM 1 (Bateman[1], Trautman[9]). An optical isomorphism transforms an adapted Maxwell field into another such field.

4. THE CAUCHY-RIEMANN SPACE ASSOCIATED WITH A SHEAR-FREE OPTICAL GEOMETRY

Consider first a geodetic flag geometry (K,L) on M and assume that the equivalence relation S defined on M by the congruence of null geodesics is regular so that the quotient $N = M/S$ has a manifold structure and the canonical map $\pi: M \to N$ is a submersion[24]. Since the bundle L is invariant with respect to the flow $(\phi_t(k))$, it projects to a line bundle $L/S \subset T^*N$. If λ is a section of $L/S \to N$, then $\pi^*\lambda$ is a section of $L \to M$. Since π is canonical, there can be no confusion if, from now on, we omit pull-backs and say that λ is a section of $L \to M$. The vector bundle $\ker(L/S) \subset TN$ is of fibre dimension 2; it defines a field of 2-planes in the 3-space N. Assume now that M is endowed with a shear-free optical geometry based on the flag structure (K,L). The complex structure on $(\ker L)/K$ is invariant with respect to the flow and, therefore, projects to a complex structure J on $\ker(L/S)$. The complex line bundle $H = \ker(L/S) \subset TN$ makes N into a *Cauchy-Riemann 3-manifold*[17]. For brevity, we shall say that N is a *CR space*. Our considerations are summarized in

THEOREM 2 Any point of a manifold with a shear-free optical geometry has a neighbourhood optically isomorphic to the Cartesian product of $\mathbb{R}$ by a CR space.

From the point of view of local differential geometry, the study of optical manifolds is thus reduced to that of CR spaces. There are, however, interesting global phenomena[19] and subtleties at the frontier between smooth and real-analytic structures[13,16].

If the bundle $\ker L$ is integrable, then so is the bundle H; the latter defines a foliation of N by surfaces with complex structure. In this case, around any point of N one can find a system of local coordinates (u,x,y) such that $\lambda = du$ is a local section of L/S, the vector fields $\partial/\partial x$ and $\partial/\partial y$ span H and

$$J(\partial/\partial x) = \partial/\partial y \quad .$$

The quadratic form $dx^2 + dy^2$ defines a conformal structure in the leaves of the foliation, compatible with their complex structure. Let r be a fibre coordinate along the fibres of $\pi: M \to N$ restricted to a suitable neighbourhood of a point in M, as in Theorem 2. The optical geometry in that neighbourhood can be described as follows: K is spanned by the vector field $k = \partial/\partial r$, L is spanned by $\lambda = du$ and B consists of all metric tensors of the form

(8) $$2du(h\, dr + \mu) - P^2(dx^2 + dy^2)$$

where the functions h and P vanish nowhere and μ is a 1-form linear in du, dx and dy[25].

It is in the non-integrable case that the CR-structure of N comes really into play. Locally, one can now find vector fields X and Y on N which span H and are such that

$$J(X) = Y\ ,$$

but $w \neq 0$, where

$$w = [X,Y] \lrcorner\, \lambda$$

is a measure of the 'twist'. Let $(\xi, \eta, \lambda/w)$ be a field of coframes dual to the field of frames $(X, Y, [X,Y])$: $X \lrcorner\, \xi = 1 = Y \lrcorner\, \eta$, $X \lrcorner\, \eta = 0 = Y \lrcorner\, \xi$, etc. The tensor $\xi^2 + \eta^2$ defines a conformal structure in the fibres of H, compatible with J and the analogue of formula (8) reads now[16]

(9) $$2\lambda(hdr + \mu) - P^2(\xi^2 + \eta^2).$$

where the meaning of the symbols is as before.

If there is a complex function $z = x+iy$ on N such that

(T) $$dz \wedge \zeta \wedge \lambda = 0 \text{ and } d\bar{z} \wedge dz \wedge \lambda \neq 0\ ,$$

where

$$\zeta = \xi + i\eta\ ,$$

then the latter form is a linear combination of λ and dz. In this case, the line-element (9) can be reduced to[26]

(10) $$2\lambda(hdr + \mu) - P^2(dx^2 + dy^2)$$

It is worth noting that in all three cases (8)-(10) the function h can be reduced to 1 by a rescaling of the coordinate r.

J. Tafel[16] has pointed out that the differential equation (T) is of the Lewy type[27] and, as shown by Jacobowitz and Treves[31], need not be solvable, even if

λ and ζ are of class C^∞. There is always a solution - at least locally - if the CR space is real-analytic.

Most important examples of CR spaces are provided by real hypersurfaces in $\mathbb{C}^2$. E. Cartan[10] has classified all such CR spaces admitting a transitive group of automorphisms. Among all CR spaces, the sphere $S_3 \subset \mathbb{C}^2$ has a CR structure with the highest dimension of the symmetry group. In this case, the bundle H consists of all vectors tangent to S_3 and orthogonal to the fibres of the Hopf map $S_3 \to S_2$. This CR structure is real-analytic and twisting. Locally, the optical geometry on $S_1 \times S_3$ induced from the CR structure of S_3 is equivalent to the one in Minkowski space $\mathbb{R}^4$ associated with the 'Robinson congruence'[6,7,19]. This optical geometry may be described as follows: K and L are generated by

$$k = \partial/\partial r \tag{11}$$

and

$$\lambda = du + xdy - ydx, \tag{12}$$

respectively, whereas B contains the flat metric

$$2\lambda dr - (r^2 + 1)(dx^2 + dy^2). \tag{13}$$

The complex structure of (kerL)/K is given by the Hans Lewy operator[27]

$$X + iY = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} - i(x+iy)\,\frac{\partial}{\partial u} \quad . \tag{14}$$

The optical geometry underlying both the Taub-NUT metric[28,29] and Hauser's null gravitational field[30] is isomorphic to the one given by (11)-(14).

Our description of shear-free optical geometries generalizes the twistor formulation[6,32] of the Kerr theorem. The generalization is essential in the sense that, as made clear by Penrose[13], the CR spaces corresponding to shear-free congruences of null geodesics in Minkowski space form a 'small' subset of the set of all CR spaces. Robert Bryant asked the following question: are there any algebraically special, Ricci-flat, Lorentzian 4-manifolds whose underlying local CR structure does not come from the Kerr-Penrose construction?

REFERENCES

1) H. Bateman, Proc. Lond. Math. Soc. 8 (1910) 469.
2) E. Cartan, Comptes Rendus Acad. Sci. (Paris) 174 (1922) 857.
3) I. Robinson, J. Math. Phys. 2 (1961) 290.

4) J.N. Goldberg and R.K. Sachs, Acta Phys. Polon. 22 Suppl. (1962) 13.

5) I. Robinson and A. Schild, J. Math. Phys. 4 (1963) 484.

6) R. Penrose, J. Math. Phys. 8 (1967) 345.

7) R. Penrose, On the origins of twistor theory, in: Gravitation and Geometry, W. Rindler and A. Trautman, eds. (Bibliopolis, Napoli, 1986), in print.

8) A. Trautman, J. Geometry and Physics (Florence) 1 (1984) 85.

9) A. Trautman, Optical structures in relativistic theories, lecture at Colloque International "Elie Cartan et les mathématiques d'aujourd'hui", Lyon, June 1984, Astérisque, in print.

10) E. Cartan, Ann. Math. Pura Appl. (4) 11 (1932) 17 and Ann. Scuola Norm. Sup. Pisa (2) 1 (1932) 333.

11) A. Andreotti and C.D. Hill, Ann. Scuola Norm. Sup. Pisa 26 (1972) 294, 325 and 747.

12) S.S. Chern and J.K. Moser, Acta Math. 133 (1975) 219.

13) R. Penrose, Bull. Amer. Math. Soc. (N.S.) 8 (1983) 427.

14) C.D. Hill, Indiana Univ. Math. J. 22 (1972) 339.

15) P. Sommers, GRG Journal 8 (1977) 855.

16) J. Tafel, Lett. Math. Phys. 10 (1985) 33.

17) R.O. Wells, Jr., Bull. Amer. Math. Soc. (N.S.) 6 (1982) 187.

18) I. Robinson and A. Trautman, Lett. Math. Phys. 10 (1985), in print.

19) I. Robinson and A. Trautman, Proc. Roy. Soc. (London), in print.

20) I. Robinson and A. Trautman, Optical geometry, in preparation.

21) R. Abraham and J.E. Marsden, Foundations of mechanics, 2nd ed. (Addison-Wesley, Reading, Mass., 1978).

22) A. Trautman, Differential Geometry for Physicists: Stony Brook Lectures (Bibliopolis, Napoli, 1984).

23) I. Robinson and A. Trautman, J. Math. Phys. 24 (1983) 1425.

24) J.P. Serre, Lie Algebras and Lie Groups (Benjamin, New York, 1965) LG § 12.

25) I. Robinson and A. Trautman, Proc. Roy. Soc. (London) A265 (1962) 463.

26) I. Robinson and A. Trautman, Exact degenerate solutions of Einstein's equations, in: Proceedings on Theory of Gravitation, L. Infeld, ed. (Gauthier-Villars and PWN, Paris and Warsaw, 1964) pp. 107-114.

27) H. Lewy, Ann. of Math. 66 (1957) 155.

28) A.H. Taub, Ann. of Math. 53 (1951) 472.

29) E.T. Newman et al., J. Math. Phys. 4 (1963) 915.

30) I. Hauser, J. Math. Phys. 19 (1978) 661.

31) H. Jacobowitz and F. Treves, Bull. Amer. Math. Soc. (N.S.) 8 (1983) 467.

32) R. Penrose and W. Rindler, Spinors and space-time, vol. 2 (Cambridge University Press, Cambridge, 1985), Ch. 7.