

CAUCHY-RIEMANN STRUCTURES IN OPTICAL GEOMETRY\*

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It is shown that the three-dimensional manifold of null geodesics forming a shear-free congruence has a natural Cauchy-Riemann structure, depending only on the optical geometry associated with the congruence.

1. INTRODUCTION

There is a natural geometry adapted to the study of null (isotropic, singular) Maxwell and Yang-Mills fields. It constitutes also the underlying structure of algebraically special gravitational fields. The origins of this geometry can be traced back to early papers by H. Bateman<sup>1</sup> and E. Cartan<sup>2</sup>, to work on shear-free congruences of null geodesics<sup>3-5</sup> and to R. Penrose's twistor programme<sup>6,7</sup>. Recently, one of us proposed to use the name 'optical geometry' for this structure and listed its basic properties<sup>8,9</sup>. In this lecture we present a novel characterization of an optical geometry with shear-free rays: locally, such a geometry is a product of  $\mathbb{R}$  by a 3-dimensional Cauchy-Riemann manifold<sup>10-12</sup>.

In our work on this problem we have been influenced by conversations with, and/or papers by, R. Penrose<sup>13</sup>, C.D. Hill<sup>14</sup>, P. Sommers<sup>15</sup>, J. Tafel<sup>16</sup>, and R.O. Wells, Jr.<sup>17</sup>. The present text contains only a brief summary of the subject; a fuller account of our joint work is being published elsewhere<sup>18-20</sup>. We follow the terminology and notation prevalent in differential geometry and mathematical physics<sup>21-23</sup>.

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## 2. FLAG GEOMETRY

A *flag geometry* on a 4-dimensional smooth orientable manifold  $M$  is a pair  $(K, L)$  of real line bundles such that

$$K \subset TM, \quad L \subset T^*M$$

and, if  $K_x$  and  $L_x$  denote, respectively, the fibres of  $K$  and  $L$  over  $x \in M$ , then

$$u \lrcorner \alpha = 0 \text{ for any } u \in K_x \text{ and } \alpha \in L_x.$$

A section  $k$  of  $K \rightarrow M$  is a vector field on  $M$ , whereas a section  $\lambda$  of  $L \rightarrow M$  is a field of 1-forms on  $M$ ; for any such sections  $k \lrcorner \lambda = 0$ . A *metric* tensor  $g$  on  $M$  is said to be *adapted* to  $(K, L)$  if, for any such sections  $k$  and  $\lambda$ , one has  $g(k) \wedge \lambda = 0$ , where  $g(k)$  is the 1-form characterized by  $l \lrcorner g(k) = g(k, l)$  for any vector field  $l$ . In other words, with respect to any adapted metric, and for any  $x \in M$ , the line  $K_x$  is null and  $\ker \lambda(x)$  is the 3-space of all vectors orthogonal to  $K_x$ . The bundle  $\bigcup_{x \in M} \ker \lambda(x)$  will be denoted  $\ker L$ .

Similarly, a  $p$ -form ( $p=1, 2$  or  $3$ )  $F$  on  $M$  is *adapted* to  $(K, L)$  if, for any  $k$  and  $\lambda$  defined as above, one has

$$(1) \quad k \lrcorner F = 0 \quad \text{and} \quad \lambda \wedge F = 0.$$

For example, if  $g$  is an adapted metric, then  $g(k)$  is an adapted 1-form.

Let  $(\phi_t(k))$  be the *flow* generated by the vector field  $k$ , section of  $K \rightarrow M$ . If  $L$  is invariant with respect to the flow  $(\phi_t(k))$ , then it is also invariant with respect to  $(\phi_t(\rho k))$ , where  $\rho$  is any function on  $M$ . It is meaningful, therefore, to define  $L$  as being invariant with respect to  $K$  if, for any sections  $k$  and  $\lambda$ , one has

$$(i) \quad \lambda \wedge \mathcal{L}_k \lambda = 0$$

where  $\mathcal{L}_k \lambda$  denotes the Lie derivative of  $\lambda$  in the direction of  $k$ . In Refs. 19 and 23 we have shown, that (i) is equivalent to any of the following conditions:

(ii) the 3-form  $\lambda \wedge d\lambda$  is adapted;

(iii) the lines of the flow  $(\phi_t(k))$  define a congruence of null geodesics with respect to any metric tensor adapted to  $(K, L)$ ;

(iv) if  $F$  is an adapted 2-form, then  $\lambda \wedge dF = 0$ .

A flag geometry which has any - and therefore all - of the properties (i)-

(iv) is said to be *geodetic*. In particular, a flag geometry corresponding to an *integrable* bundle  $\ker L \subset TM$ , i.e., such that

$$\lambda \wedge d\lambda = 0$$

is geodetic. If the bundle  $\ker L$  is non-integrable, then the congruence of null curves defined by the flow is said - by physicists - to be *twisting*.

### 3. OPTICAL GEOMETRY

A flag geometry is sufficient to define a congruence of null geodesics and the notion of null (adapted) 2-forms. If any such form  $F$  is interpreted as an electromagnetic field, then it is possible to write one part of Maxwell's equations, namely  $dF = 0$ , but not the other. Roughly speaking, an optical geometry is the weakest structure needed on a 4-dimensional manifold  $M$  to write the full set of Maxwell's equations for null electromagnetic fields.

In a Lorentzian geometry based on a metric tensor  $g$  one introduces the Hodge dual  $*_g F$  of  $F$  relative to  $g$  and some orientation on  $M$ . The other part of Maxwell's equations reads then  $d*_g F = 0$ .

Let us start again with a flag geometry  $(K,L)$  on  $M$  and let  $A$  be the set of all adapted Lorentzian metric tensor fields on  $M$ . If  $g \in A$  and  $F$  is an adapted  $p$ -form, then  $*_g F$  is an adapted  $(4-p)$ -form. For example, if the flag geometry is geodetic, then  $*_g(\lambda \wedge d\lambda)$  is proportional to  $\lambda$ . If  $F$  is a nowhere vanishing 2-form on  $M$  adapted to  $(K,L)$  then

$$(2) \quad g \equiv_R g' \Leftrightarrow *_g F = *_g' F, \text{ where } g \text{ and } g' \in A,$$

defines an equivalence relation  $R$  in  $A$ . This equivalence relation does not depend on  $F$ ; only at this point does the assumption of  $M$  being four-dimensional enter into our considerations.

An *optical geometry* on  $M$  consists of the pair  $(K,L)$  together with an element  $B$  of  $A/R$  and an orientation of the vector bundle  $(\ker L)/K$  of fibre dimension 2. Equivalently, it can be defined as a flag geometry  $(K,L)$  supplemented by a complex structure on  $(\ker L)/K$ , i.e. a linear bundle morphism

$$J: (\ker L)/K \rightarrow (\ker L)/K \text{ such that } J^2 = -id.$$

This additional structure makes  $(\ker L)/K$  into a complex line bundle over  $M$ .

It is easy to see that if  $g \in B \subset A$  then  $g' \in B$  if, and only if, there is a

