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# Remarks on Pure Spinors

PAOLO BUDINICH and ANDRZEJ TRAUTMAN
*Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoza 69, 00–681 Warszawa, Poland and Scuola Internazionale Superiore di Studi Avanzati, Strada Costiera 11, 34014 Trieste, Italy*

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**Abstract.** General remarks on the significance of spinors are followed by a brief description of spinor connections on low-dimensional spheres and their interpretation as gauge configurations. Cartan's notion of pure spinors is related to the general problem of classification of orbits of the spin group in projective spinor space. There is a nontrivial bundle of pure spinor directions over the conformal compactification of any space with a metric of suitable signature. In higher dimensions, pure spinors introduce natural nonlinearities and lead to topologically nontrivial configurations. It is shown how the constraint defining pure spinors may induce a 'mass term' in the Weyl equation for such spinors in a space of signature (3, 4).

## 1. Introduction

Spinors – and structures associated with them – are among the geometrical notions whose importance was recognized as a result of research in physics. Spinors are implicit in the early work of Olinde Rodrigues [1] on the Euler angles and in the discovery of quaternions by Hamilton [2]. By means of quaternions. Cayley represented rotations in three and four dimensions [3]. These representations were generalized to higher dimensional spaces by Lipschitz [4] who used the algebras of Clifford [5] for this purpose. The discovery of spinors may be attributed to Elie Cartan: he realized that the Lie algebras of orthogonal groups have representations which do not lift to linear representations of the groups themselves [6]. In a later work [7], Cartan gave a beautiful geometrical interpretation of spinors: he showed that in a low-dimensional vector space with a 'neutral' quadratic form, there is a one-to-one correspondence between spinor directions and maximal, totally null subspaces of the vector space. If the dimension of the vector space exceeds 6, then the dimension of the projective spinor space is larger than that of the space $\mathcal{N}$ of all such null subspaces. As shown by Cartan, the bijective correspondence between $\mathcal{N}$ and spinor directions can be extended to higher dimensions, provided that one restricts the spinors to being *pure*. (In the French text [7], the adjective *simple* is used, which seems more appropriate, but, for the time being, we follow the English terminology of Chevalley [8] and Penrose and Rindler [9].) The space $\mathcal{N}$ is a manifold homogeneous with respect to the action of the appropriate orthogonal group. Unless the dimension of the underlying space is smaller than 7, the set of all pure spinors is a line bundle over $\mathcal{N}$ rather than a vector space: there is a natural nonlinearity associated with the concept of pure spinors.

For a long time, the interest of physicists in spinors was restricted to low-dimensional spaces, where all (Weyl) spinors are pure so that there is no need to introduce them as a separate notion. Recent research on fundamental interactions and their unification makes essential use of geometries of more than four dimensions. For this reason, spinor structures in higher dimensions and, in particular, pure spinors, have now more chance of becoming relevant to physics than they had at the time of the appearance of the article by Brauer and Weyl [10] and Cartan's lecture notes [7]. Somewhat unexpectedly, spinors have proved to be very useful in the theory of gravitation [11, 12]. There are also interesting coincidences between spinor connections on homogeneous spaces and simple, topologically nontrivial gauge configurations [13, 14]. Considerations such as these motivate us in our work on Clifford algebras and spinor structures.

In this Letter, which reviews the first stage of our research, we briefly (i) describe the relation between spinor connections on low-dimensional spheres and simple gauge configurations, (ii) outline an approach to the notion of pure spinors in spaces of nonneutral signature and, in particular, in conformal extensions of spacetime, (iii) indicate some topologically nontrivial configurations associated with pure spinors, and (iv) show how the 'purity constraint' may lead to a 'mass term' in the Weyl equation for pure spinors in a seven-dimensional space.

## 2. Spinor Structures on Spheres

An $n$-dimensional sphere $S_n$ admits, for any $n \geqslant 2$, a unique spinor structure which may be described as follows. Assume $S_n$ to be oriented and given its standard Riemannian metric $g_n$. The bundle of orthonormal and coherently oriented frames on $S_n$ can be identified with $\mathrm{SO}(n+1)$ and the spinor structure is given by the sequence of maps

$$\mathrm{Spin}(n+1) \xrightarrow{\rho_{n+1}} \mathrm{SO}(n+1) \rightarrow S_n . \tag{1}$$

Let $\bar{\rho}_n$: $\mathrm{spin}(n) \rightarrow \mathrm{so}(n)$ denote the isomorphism of Lie algebras derived from $\rho_r$ and let $\theta_n$ be the Levi–Civita connection form associated with $g_n$. The form

$$\overline{\omega}_n = \bar{\rho}_n^{-1} \circ \theta_n \circ T\rho_{n+1} \tag{2}$$

defines a spinor connection on $S_n$. The curvatures $\Theta_n$ and $\Omega_n$ corresponding, respectively, to $\theta_n$ and $\omega_n$, are related to each other by an analogue of (2). For low values of $n$, the spinor curvature $\Omega_n$ can be interpreted as a gauge field [13]. For example, $\Omega_2$ gives the restriction to $S_2$ of the field of a Dirac magnetic pole of lowest strength. Since $\mathrm{Spin}(3) = \mathrm{SU}(2)$, the spinor connection on $S_3$ can be interpreted as a Yang–Mills configuration: it is the 'meron' solution [15]. In this case, the bundles (1) are trivial and there are no true topological invariants, but there is a Chern–Simons conformal invariant [16]. The case $n = 4$ corresponds to the instanton and anti-instanton solutions of Belavin, Polyakov, Schwartz, and Tyupkin [17], whereas $\Omega_8$ has been recently shown [18] to coincide with a Spin(8) gauge field [19]. Similarly, one can interpret $\Omega_5$ and $\Omega_6$ as Sp(2) and SU(4) gauge fields on $S_5$ and $S_6$, respectively. These observations may be easily extended to other homogeneous spaces.

## 3. Clifford Algebras, Pure Spinors, and the Problem of Orbit Classification

Clifford algebras and the spinor spaces associated with them exhibit a wealth and variety of structures which are not apparent from a superficial study of the underlying vector spaces. The Clifford algebra $A$ corresponding to a vector space $V$ over a field $K = \mathbb{R}$ or $\mathbb{C}$ contains, as subsets, the vector space itself and its Grasmann algebra $\wedge V$, the pin and spin groups, their Lie algebras and also their irreducible representation (spinor) spaces, which may be identified with minimal left ideals in $A$. There is a natural place in $A$ for a $\mathbb{Z}_2$-grading and the Hodge duality.

Any Clifford algebra $A$ admits a faithful, but not necessarily irreducible, representation in a vector space $S$ over $K$; the elements of $S$ are then called spinors. Let $\gamma: A \to \mathrm{End}_K S$ be such a representation. Denoting by $g$ the scalar product in $V$ one has

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u, v)$$

for any $u, v \in V$. Let $\varphi \neq 0$ be a spinor and $\Phi = K\varphi \subset S$ its direction. The set

$$\Phi' = \{u \in V : \gamma(y)\varphi = 0\}$$

is a totally null vector subspace of $V$. If $N$ is a totally null vector subspace of $V$, then

$$N' = \{\psi \in S : \gamma(u)\psi = 0 \text{ for any } u \in N\}$$

is a linear subspace of $S$. Since $\Phi \subset \Phi'' = \Psi$ and $\Phi' = \Phi'''$, a nonzero spinor $\varphi$ defines a triplet $(\Phi, \Psi, N)$ such that $N = \Phi'$ and $N' = \Psi$. The Lie algebra of the spin group can be identified with $\wedge^2 V$ and also with $[V, V] \subset A$ by $u \wedge v \mapsto [u, v]$. An element $S$ of the spin group preserves the direction of $\varphi$ if $\gamma(s)\varphi = \lambda\varphi$ for some $\lambda \in K$. If $\lambda = 0$ then $\varphi$ itself is preserved. For any $\Sigma \subset S$ and $\varphi \in S$

$$L(\Sigma) = \{a \in \wedge^2 V : \psi \in \Sigma \Rightarrow \gamma(a)\psi \in \Sigma\}$$

and

$$L(\varphi) = \{a \in \wedge^2 V : \gamma(a)\varphi = 0\}$$

and the Lie algebras of the stability groups of $\Sigma$ and $\varphi$, respectively. It is clear that $[L(\Phi), L(\Phi)] \subset L(\varphi) \subset L(\Phi)$, and $L(\Phi)/L(\varphi)$ is the Lie algebra of dilatations of $\varphi$. If $N$ is totally null, then

$$N^{\perp} = \{u \in V : v \in N \Rightarrow g(u, v) = 0\}$$

contains $N$ and there exists a totally null subspace $P \subset V$ of the same dimension as $N$ such that $V = N^{\perp} \oplus P$ [20]. If $V_1$ and $V_2$ are vector subspaces of $V$, then $[V_1, V_2] \subset [V, V]$ is spanned by all elements of the form $[V_1, V_2]$, where $v_1 \in V_1$ and $v_2 \in V_2$. Information about the Lie algebras of the stability groups of $\Phi$ and $\Psi$ can be obtained, in part, from

PROPOSITION 1. *Let $(\Phi, \Psi, N)$ be a triplet defined by a spinor $\varphi$, then*

$$[N, V] \subset L(\Phi) \quad \textit{and} \quad [N, V] + [N^{\perp}, N^{\perp}) \subset L(\Psi)\,. \tag{3}$$

Assume now $K = \mathbb{C}$ and $\dim V = n = 2\nu$ or $2\nu + 1$ so that $\nu = 1, 2, \dots$ is the dimension of maximal totally null subspaces of $V$. The space of spinors $S$ is then of dimension $2^{n-\nu}$. Let $(e_1, \dots, e_n)$ be an orthonormal frame in $V$ and $\varepsilon = e_1 \dots e_n$ the corresponding 'volume element'. Since $\varepsilon^2 = (-1)^\nu$, it is convenient to introduce the complex volume element $\eta = i^\nu \varepsilon$. As a vector space, the Clifford algebra $A$ can be identified with the exterior algebra $\wedge V$ and the Hodge dual $* : \wedge V \to \wedge V$ is then obtained from Clifford multiplication by $\eta$,

$$*a = a\eta\,, \tag{4}$$

so that $**a = a$ for any $a \in \wedge V$. Since $\wedge V^*$ can be identified with $(\wedge V)^*$, one can define the Hodge dual of a form $\omega \in \wedge V^*$ by

$$\langle a, *\omega \rangle = \langle *a, \omega \rangle\,, \quad \text{any } a \in \wedge V\,, \tag{5}$$

where the angular brackets denote the evaluation map.

The endomorphism $\Gamma = \gamma(\eta)$ of $S$ is involutory and defines a decomposition of $S$ into a direct sum, $S = S_+ \oplus S_-$, corresponding to the decomposition of a spinor,

$$\varphi = \varphi_+ + \varphi_-\,, \quad \text{where } \varphi_\pm = \tfrac{1}{2}(I \pm \Gamma)\varphi\,.$$

If $n$ is *odd*, $n = 2\nu + 1$, then $\varepsilon$ is in the centre of $A$ and the representation $\gamma: A \to \operatorname{End} S$ decomposes, $\gamma = \gamma_+ \oplus \gamma_-$, where $\gamma_\pm : A \to \operatorname{End} S_\pm$ is given by $\gamma_\pm(a) = \frac{1}{2}(I \pm \Gamma)\gamma(a)$. Each of the two spaces $S_+$ and $S_-$ is $2^\nu$-dimensional and its elements are *Cartan spinors*. If $n$ is *even*, $n = 2\nu$, then $\varepsilon$ is in the centre of the even part $A^+$ of the Clifford algebra. The restriction of $\gamma$ to $A^+$ decomposes, $\gamma | A^+ = \gamma_+ \oplus \gamma_-$, in a similar manner as before. The spaces $S_+$ and $S_-$ are $2^{\nu-1}$-dimensional and their elements are *Weyl spinors* [21], sometimes also called reduced [9] or semi-spinors [7, 20].

Let $\alpha$ and $\beta$ denote, respectively, the main involution and anti-involution of $A$ [20]. For any linear map $f: U \to V$ its transpose $f^*: V^* \to U^*$ is defined by $\langle u, f^*(v^*) \rangle = \langle f(u), v^* \rangle$, where $u \in U$ and $v^* \in V^*$. Given the representation $\gamma$ of $A$ in $S$ described above, one can define another representation $\gamma'$ of $A$ in $S^*$ by setting

$$\gamma'(a) = (\gamma \circ \beta(a))^* \quad \text{if } \nu \text{ is even}$$

and

$$\gamma'(a) = (\gamma \circ \alpha \circ \beta(a))^* \quad \text{if } \nu \text{ is odd}\,.$$

The representations $\gamma$ and $\gamma'$ are equivalent: there exists an isomorphism $C: S \to S^*$ such that

$$\gamma'(a) = C \circ \gamma(a) \circ C^{-1}\,. \tag{6}$$

One shows that [7]

$$C^* = \begin{cases} C & \text{for } \nu \equiv 0, 3 \bmod 4\,, \\ -C & \text{for } \nu \equiv 1, 2 \bmod 4\,. \end{cases} \tag{7}$$

The decomposition $S = S_+ \oplus S_-$ induces a corresponding decomposition of the dual space, $S^* = S^*_+ \oplus S^*_-$. If $n$ is odd or $\equiv 0 \bmod 4$, then $C$ maps $S_\pm$ onto $S^*_\pm$. If $n \equiv 2 \bmod 4$, then $C$ maps $S_\pm$ onto $S^*_\mp$. By rescaling one can obtain $CC^* = I$. The Cartan isomorphism $C$ defines a bilinear form in $S$ by

$$S \times S \ni (\varphi, \psi) \mapsto \langle \varphi, C\psi \rangle .$$

This form is invariant in the sense that, for any unit vector $u \in V \subset A$ and any spinors $\varphi, \psi \in S$ one has

$$\langle \gamma(u)\varphi, C\gamma(u)\psi \rangle = (-1)^\nu \langle \varphi, C\psi \rangle . \tag{8}$$

With the identification $A = \wedge V$ in mind, we define the *Cartan map*

$$k: S \to \wedge V^*$$

by putting, for any $\varphi \in S$ and $a \in A$,

$$\langle a, k(\varphi) \rangle = \langle \varphi, C\gamma(a)\varphi \rangle . \tag{9}$$

If $\varphi \in S_\pm$ is a Cartan ($n$ odd) or Weyl ($n$ even) spinor, then $\Gamma\varphi = \pm\varphi$ and

$$*k(\varphi) = \pm k(\varphi) . \tag{10}$$

We denote $k_q(\varphi)$ the component of degree $q$ of $k(\varphi)$. If $n = 2\nu$ is even, then $k_\nu$ is self- or anti-self dual,

$$*k_\nu(\varphi) = \pm k_\nu(\varphi) . \tag{11}$$

We can now prove

PROPOSITION 2. *Let* $0 \neq \varphi \in S_\pm$ *and* $(\Phi, \Psi, N)$ *be the corresponding triplet, as in Proposition* 1. *If* $N$ *is* $\mu$*-dimensional, then, for any* $\psi \in \Psi$,

$$k_q(\psi) = 0 \quad \textit{for } q < \mu \textit{ and } q > n - \mu .$$

Indeed, by virtue of (10), it is sufficient to consider $q > n - \mu$. If $a \in \wedge^q V$, $q > n - \mu$, then $a$ is a sum of terms each containing as a factor at least one element of $N$. Such an element annihilates $\psi$; therefore, $k_q(\psi) = 0$.

According to Cartan, a spinor $\varphi \in S_\pm$ is *pure* if the dimension of $N$ is maximal, i.e., equal to $\nu$. In this case $\Psi = \Phi$ and $k_q(\varphi) = 0$ for any $q \neq \nu$ ($n = 2\nu$) or $q \neq \nu$ and $\nu + 1$ ($n = 2\nu + 1$). There is a bijective correspondence between the set of all pure spinor directions and the set of all maximal, totally null planes given by

$$\Phi \mapsto \{u \in V : u \lrcorner k_{n-\nu}(\varphi) = 0\}$$

where $\varphi \in \Phi$ is a pure spinor and $u \lrcorner \omega$ denotes the contraction of the vector $u$ with the form $\omega$. The action of the spin group – and also of the corresponding special orthogonal group – is transitive on the manifold of pure spinor directions in both $S_+$ and $S_-$. In fact, the manifold $\mathcal{N}$ of pure spinor directions is an orbit of the action of the spin group in the projective spinor space $P(S)$ characterized by having the least dimension among all orbits. The co-dimension of the orbit is a 'measure of purity' of the spinor directions it contains. This point of view allows an extension of Cartan's notion of pure spinors to real vector spaces with a nonneutral quadratic form. Even in the positive-definite case, for $n \geqslant 10$, there are nontrivial orbits in the projective spinor space [22]. We leave the general problem of the classification of orbits of the spin group to future work and restrict ourselves to a few remarks on pure spinors associated with real vector spaces (cf. also [22–25]).

The structure of the Clifford algebra of a real vector space depends on the signature $(\mu, \nu)$ of its fundamental quadratic form $y_1^2 + \cdots + y_\mu^2 - x_1^2 - \cdots - x_\nu^2 = y^2 - x^2$. The square of the volume element $\varepsilon$ is now

$$\varepsilon^2 = (-1)^{1/2(\mu-\nu)(\mu-\nu-1)}.$$

Whenever $\varepsilon^2 = 1$ the representation $\gamma$ of $A$ ($\mu + \nu$ odd) or $A^+$ ($\mu + \nu$ even) decomposes and one has real Cartan or Weyl spinors, respectively. In particular, if $\mu = \nu + 1$, then the spaces $S_\pm$ of Cartan spinors are each real $2^\nu$-dimensional; if $\mu = \nu$ (neutral signature), then the spaces $S_\pm$ of Weyl spinors are each real $2^{\nu-1}$-dimensional. In both these cases $V$ contains $\nu$-dimensional totally null planes. It is easy to describe the manifold $\mathcal{N}_{\mu,\nu}$ of all null planes of dimension $\nu$ in a space $V$ of signature $(\mu, \nu)$ with $\mu \geqslant \nu$. If $N$ is any such plane and $(y, x) \in N$, then $y = ax$, where $a: \mathbb{R}^\nu \to \mathbb{R}^\mu$ is a linear map such that $a^*a = \mathrm{id}$. The orthogonal group $\mathrm{O}(\mu)$ acts transitively in $\mathcal{N}_{\mu,\nu}$ by sending the plane associated with $a$ into the plane associated with $ba$, where $b \in \mathrm{O}(\mu)$. The stability subgroup at $a_0$ = standard injection of $\mathbb{R}^\nu$ into $\mathbb{R}^\mu$ is easily seen to be isomorphic to $\mathrm{O}(\mu - \nu)$ so that

$$\mathcal{N}_{\mu,\nu} = \mathrm{O}(\mu)/\mathrm{O}(\mu-\nu). \tag{12}$$

In particular, $\mathcal{N}_{\nu,\nu} = \mathrm{O}(\nu)$ and $\mathcal{N}_{\nu+1,\nu} = \mathrm{SO}(\nu+1)$. In both these cases Cartan's definition and description of pure spinors can be carried over from the complex to the real domain in a straightforward manner. For example, in signature $(\nu, \nu)$ a spinor $\varphi \in S_\pm$ is pure if and only if

$$k_q(\varphi) = 0 \quad \text{for any } q \neq \nu. \tag{13}$$

The two connected components of $\mathrm{O}(\nu)$ correspond to opposite 'helicities' given by the signs in Equation (10), and also to the $\alpha$-planes and $\beta$-planes of classical projective geometry (cf., for example, [26, 27]).

The action of the group $\mathrm{SO}(\nu, \nu)$ on $\mathrm{SO}(\nu)$ identified with one of the connected components of $\mathcal{N}_{\nu,\nu}$ can be described as follows. Let $c \in \mathrm{SO}(\nu)$ define the $\nu$-plane $N(c) = \{(cx, x) : x \in \mathbb{R}^\nu\}$ and let

$$\begin{pmatrix} a & d \\ e & b \end{pmatrix} \in \mathrm{SO}(\nu, \nu) \tag{14}$$

where $a$, $b$, $d$, and $e$ are endormorphisms of $\mathbb{R}^\nu$. The element (14) transforms $N(c)$ into another null $\nu$-plane $N(c')$, where

$$c' = (ac + d)(ec + b)^{-1}. \tag{15}$$

The group $O(\mu + 1, \nu + 1)$ is known to act conformally on the compactification of $\mathbb{R}^{\mu+\nu}$ endowed with a metric of signature $(\mu, \nu)$, cf. § 4. The case $\mu - \nu = 2$ is of special interest for the conformal structure of spacetime. For example, $O(3, 1)$ acts conformally in $S_2$, whereas $O(4, 2)$ yields the group of conformal automorphisms of compactified Minkowski space. The Clifford algebra of a real vector space with a quadratic form of signature $(\nu + 1, \nu - 1)$ is isomorphic to the algebra $\mathrm{End}_{\mathbb{R}} S$ of all endormorphisms of a $2^\nu$-dimensional real spinor space $S$. Since $\varepsilon^2 = -1$, the volume element defines two complex structures $\pm\gamma(\varepsilon)$ in $S$; each of them makes $S$ into a complex space $S_\pm$ of complex dimension $2^{\nu-1}$. These two spaces of 'complex Weyl spinors' carry faithful and irreducible representations of $A^+$. For $\nu \geqslant 2$, the notion of a pure spinor can be extended to this case by reference to Equation (13). Instead of (11), one now has $*k_\nu(\varphi) = \pm ik_\nu(\varphi)$. This generalizes the well-known correspondence between null bivectors and two-component spinors in Minkowski space [9].

## 4. Bundles of Pure Spinors and Conformal Compactification

Consider, for simplicity, the real vector space $\mathbb{R}^{2\nu}$ with a quadratic form of signature $(\nu, \nu)$. If $\nu < 4$, then any Weyl spinor associated with such a space is pure. Therefore, if $\nu < 4$, then the set of all pure spinors belonging to the same eigenvalue of $\varepsilon$ forms a linear space. If $\nu \geqslant 4$ then it is no longer so: pure spinors are elements of a line bundle over $\mathcal{N}_{\nu,\nu}$. It is convenient to regard Weyl spinors as elements of such a line bundle also for $\nu < 4$. Let us restrict our attention to pure spinors belonging to $S_+$, say. The manifold of directions of such pure spinors is diffeomorphic to $SO(\nu)$. The line bundle $E_\nu \to SO(\nu)$ of pure spinors is the bundle associated with the principal $\mathbb{Z}_2$-bundle $\mathrm{Spin}(\nu) \to SO(\nu)$ by the obvious action of $\mathbb{Z}_2$ in $\mathbb{R}$. For example, $E_2 \to SO(2)$ is the Möbius band and $E_3 \to SO(3) = \mathbb{R}P_3$ is the canonical line bundle of the real projective three-space.

Any real (vector) space $V = \mathbb{R}^{\mu+\nu}$ with a scalar product $g$ of signature $(\mu, \nu)$ admits a conformal compactification $Q_{\mu,\nu}$ which may be described as follows. Consider $W = V \times \mathbb{R}^2$ with a scalar product $h$ of signature $(\mu + 1, \nu + 1)$ given by

$$h(w, w) = g(v, v) - xy\,,$$

where $x, y \in \mathbb{R}$, $v \in V$ and $w = (v, x, y)$. Define the projective quadric $Q_{\mu,\nu}$ as the submanifold of the projective space

$$P(W) = \{\mathrm{dir}\, w : 0 \neq w \in W\}$$

given by

$$Q_{\mu,\nu} = \{\mathrm{dir}\, w \in P(W) : h(w, w) = 0\}$$

where $\mathrm{dir}\, w$ is the direction containing $w$. The quadratic $Q_{\mu,\nu}$ inherits from $W$ a conformal geometry of signature $(\mu, \nu)$ and

$$l : \mathbb{R}^{\mu+\nu} \to Q_{\mu,\nu} \quad \text{given by } l(v) = \mathrm{dir}(v, g(v, v), 1)$$

is a conformal immersion. The complement of the image of $l$ consists of the null cone $\{v \in V : g(v, v) = 0\}$ and the projective $\{\operatorname{dir} v \in P(V) : g(v, v) = 0 \neq v \in V\}$ 'at infinity'.

It is easily seen (cf., for example [22], Prop. 12.20) that the quadric $Q_{\mu, \nu}$ is diffeomorphic to the quotient space $(S_\mu \times S_\nu)/\mathbb{Z}_2$. In particular, $Q_{\mu, 0} = S_\mu$ (one-point compactification of $\mathbb{R}^\mu$), $Q_{2\mu+1, 1} = S_{2\mu+1} \times S_1$ (this includes, for $\mu = 1$, the conformal compactification of Minkowski space), $Q_{4\mu-1, 3} = S_{4\mu-1} \times \mathbb{R}P_3$ and $Q_{8\mu-1, 7} = S_{8\mu-1} \times \mathbb{R}P_7$.

The action of the orthogonal group $\mathrm{O}(W, h)$ in $Q_{\mu, \nu}$ given by $(A, \operatorname{dir} w) \mapsto \operatorname{dir}(Aw)$, where $A \in \mathrm{O}(W, h)$ and $h(w, w) = 0$, is conformal and this group provides a double cover of the group of all conformal automorphisms of the projective quadric. Spinors associated with the Clifford algebra of $(W, h)$ are called *twistors* of $Q_{\mu, \nu}$ [9]. If $\mu = \nu$, then a pure twistor defines a totally null $(\nu + 1)$-plane in $W$ which projects to an $\mathbb{R}P_\nu$ embedded in the projective quadric.

The case $\mu = \nu = 4$ is especially simple and interesting because the *triality* associated with it [7, 22, 28]. The spaces of Weyl spinors are both eight-dimensional and $C$ defines in each of them a neutral quadratic form $k_0$. A Weyl spinor $\varphi$ is now pure if and only if $k_0(\varphi) = 0$. Consider the quadric

$$Q = (S_4 \times S_4)/\mathbb{Z}_2$$

obtained by conformal compactification of $\mathbb{R}^8$ equipped with a neutral quadratic form. The quadric has a conformal geometry and with each of its tangent spaces there is associated the manifold of null four-planes. The collection of all such manifolds defines the total space of a fibre bundle over $Q$. This bundle may be identified with the *bundle of pure spinor directions* on $Q$. We now proceed to describe this bundle.

The group $G = \mathrm{O}(5) \times \mathrm{O}(5) \subset \mathrm{O}(5, 5)$ acts transitively on $Q$ and the stability subgroup $H$ at a point is isomorphic to $\mathbb{Z}_2 \times \mathrm{O}(4) \times \mathrm{O}(4)$ embedded in $\mathrm{O}(5) \times \mathrm{O}(5)$ by

$$(\sigma, a, b) \mapsto \left( \begin{pmatrix} \sigma & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} \sigma & 0 \\ 0 & b \end{pmatrix} \right),$$

where $\sigma = \pm 1$ and $a, b \in \mathrm{O}(4)$. The kernel $N$ of the tangent action of the stability subgroup is $\mathbb{Z}_2$ embedded by

$$\sigma \mapsto (\sigma, \sigma I, \sigma I)$$

where $I$ is the unit of $\mathrm{O}(4)$. From these data one can find a restriction $F = G/N$ of the bundle of linear frames of $Q$ to the group $H/N$ [14]:

$$F = \mathrm{O}(5) \times \mathrm{SO}(5) \quad \text{and} \quad H/N = \mathrm{O}(4) \times \mathrm{O}(4).$$

The corresponding action of $H/N$ in $F$ is as follows: let $a, b \in \mathrm{O}(4)$, $A \in \mathrm{O}(5)$ and $B \in \mathrm{SO}(5)$, then

$$(A, B)(a, b) = (Aa \det b, Bb \det b).$$

The bundle $F$ has two connected components; therefore, $Q$ is orientable and, if an orientation is singled out, the bundle of frames can be further restricted to the group

$$S(\mathrm{O}(4) \times \mathrm{O}(4)) = \{(a, b) \in \mathrm{O}(4) \times \mathrm{O}(4) : \det a = \det b\}.$$

The total space of the restricted bundle is

$$SF = \mathrm{SO}(5) \times \mathrm{SO}(5)\,.$$

Since each fibre of $SF$ has two connected components and $SF$ is connected, the bundle $\pi\colon SF \to Q$ is not trivial. The bundle $E \to Q$ of pure spinor directions of coherent helicity (i.e., corresponding to pure spinors in either $S_+$ or $S_-$) is a bundle associated with $\pi$ by the action of $S(\mathrm{O}(4) \times \mathrm{O}(4))$ in $\mathrm{SO}(4)$ given, in accordance with (15), by

$$(a, b)c = acb^{-1}\,,$$

where $(a, b) \in S(\mathrm{O}(4) \times \mathrm{O}(4))$ and $c \in \mathrm{SO}(4)$. The total space of this bundle is

$$E = \mathrm{SO}(5) \times \mathbb{R}P_4\,.$$

These simple examples illustrate the importance of projective notions in the study of conformal properties of spaces, massless systems, and of the associated spinors [29].

## 5. On the Weyl Equation for Pure Spinors

Pure spinors are defined by equations quadratic in their components, such as

$$\langle \varphi, C\varphi \rangle = 0\,. \tag{16}$$

Such nonlinear constraints are, in general, incompatible with linear differential equations. If the equations in question can be derived from a variational principle, then a standard procedure to account for the constraints is to introduce Lagrange multipliers and modify the action integrand with suitable terms.

We now show how this procedure can be applied to the Weyl differential equation for a spinor field in $\mathbb{R}^7$ with a quadratic form of signature (3, 4). In this case, the even and full Clifford algebras are isomorphic to $\mathbb{R}(8)$ and $\mathbb{C}(8)$, respectively. Let $(e_\mu)$, $\mu = 1, \ldots, 7$ be a frame in $\mathbb{R}^7$, orthonormal with respect to the scalar product of signature (3, 4). The corresponding gamma matrices, $\gamma_\mu = \gamma(e_\mu)$, can be chosen to be pure imaginary and, according to (7), the isomorphism $C$ is symmetric, $C = C^*$. In this case, Equation (16) is the only constraint defining pure spinors. Since $\gamma_\mu^* = -C\gamma_\mu C^{-1}$, we have, for any spinor fields $\varphi$ and $\psi$,

$$\langle \psi, C\gamma_\mu\varphi \rangle + \langle \varphi, C\gamma_\mu\psi \rangle = 0\,.$$

Therefore, the Lagrangian

$$\frac{i}{2}\,\langle \varphi, C\gamma^\mu\, \partial\varphi/\partial x^\mu \rangle + \lambda\,\langle \varphi, C\varphi \rangle$$

yields, by variation with respect to $\lambda$ and $\varphi$, the constraint (16) and the Weyl equation with a 'mass term',

$$i\gamma^\mu\, \partial\varphi/\partial x^\mu + \lambda\varphi = 0\,. \tag{17}$$

The Lagrange multiplier $\lambda$, determined by solving the system of Equations (16) and (17), is a function of the coordinates $(x^\mu)$, in general. Since $i\gamma^\mu$ is real, this system of equations admits real-valued solutions.

An alternative approach to formulating the Weyl equation for pure spinors may be found in [23].

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