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A generalization of the Mariot theorem on congruences of null geodesics

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It is shown that the property of a congruence of curves to consist of null geodesics can be defined in terms of a distribution of a co-dimension one, without reference to the conformal structure of the underlying differentiable manifold: if k is the vector field tangent to the congruence and κ is a 1-form characterizing the distribution, then the congruence is said to be null if $k \lrcorner \kappa = 0$ and geodesic if, and only if, $\kappa \wedge \pounds_k \kappa = 0$. The geodesic property of the congruence, on an n-dimensional manifold, means that if F is an $(n-2)$-form such that $k \lrcorner F = 0$ and $\kappa \wedge F = 0$, then $\kappa \wedge \mathrm{d}F = 0$. A twisting geodesic null congruence on $S_1 \times S_{2l+1}$, associated with the Hopf fibration $S_{2l+1} \to \mathbb{C}\mathbb{P}_l$, is constructed as an illustration.

1. Introduction

Geodesics play a fundamental role in differential geometry and theoretical physics, where they provide mathematical models for freely falling particles and the propagation of light. There are several distinct, but related, definitions of geodesics. Given a smooth manifold with a linear connection Γ, one defines a geodesic as an auto-parallel line, that is a curve such that all its tangent vectors are parallel to each other with respect to Γ. On a Riemannian manifold, geodesics may be defined as curves which render stationary the 'energy integral'. These two definitions are compatible: a geodesic in the latter sense is an auto-parallel line with respect to the Levi-Civita connection associated with the metric. On a pseudo-Riemannian space there are also null (optical, light-like) geodesics: they are auto-parallel lines whose tangent vectors have vanishing squares. It is known that the property of being a null geodesic depends only on the conformal structure of the pseudo-Riemannian space.

A null (optical, simple) electromagnetic field in a Lorentzian space defines a congruence of null geodesics (Mariot 1954) which is also shear free. Conversely, with any congruence of null, shear-free geodesics there is associated a null and non-zero solution of Maxwell's equations (Robinson 1961). But what if a congruence of null geodesics fails to be shear free? Presumably the associated fields of null 2-forms would satisfy something weaker than Maxwell's equations. One might also expect that the weaker equations would require for their formulation rather less than the conformal structure underlying Maxwell's theory. We propose to consider these questions for a space of n dimensions, with $n \geqslant 3$.

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2. Notation and algebraic preliminaries

The standard terminology and notation of algebra and differential geometry are used in this paper. The Grassmann algebra of an n-dimensional, real vector space V is denoted by

$$\Lambda V^* = \bigoplus_{l=0}^{n} \Lambda^l V^*,$$

where $\Lambda^0 V = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ is the dual of V. If $u \in V$, then

$$i(u): \Lambda V^* \to \Lambda V^*$$

is the (anti)derivation of degree -1 defined by

$$i(u)\,\alpha = \langle u, \alpha \rangle \quad \text{for any} \quad \alpha \in V^*$$

and
$$i(u)\,(\beta \wedge \gamma) = (i(u)\,\beta) \wedge \gamma + (-1)^l \beta \wedge i(u)\,\gamma$$

for any $\beta \in \Lambda^l V^*$ and $\gamma \in \Lambda V^*$. One often writes $u \lrcorner \alpha$ instead of $i(u)\,\alpha$. A metric tensor on V is defined as a map $g: V \times V \to \mathbb{R}$ which is bilinear, symmetric and non-singular. For any $u \in V$, we denote by $g(u)$ the 1-form such that $v \lrcorner g(u) = g(u, v)$ for any $v \in V$.

If L is a vector subspace of V, then

$$L^0 = \{\alpha \in V^*: \text{if } u \in L \text{ then } u \lrcorner \alpha = 0\}$$

is a vector subspace of V^*. If K is another subspace of V and

$$K \subset L$$

then
$$L^0 \subset K^0$$

and the vector spaces
$$(L/K)^* \quad \text{and} \quad K^0/L^0$$

are isomorphic to one another in a natural manner. If $u \in L$, then $u + K$ denotes its image in L/K.

In particular, we define a *flag* in V to consist of a pair (K, L) of subspaces such that $K \subset L$ and both K and L^0 are one-dimensional. Let

$$k \in K \quad \text{and} \quad \kappa \in L^0$$

be non-zero elements. A metric tensor g is said to be *adapted* to the flag (K, L) if $\kappa \wedge g(k) = 0$; this implies $g(k, k) = 0$.

An element α of the Grassmann algebra of V is said to be *simple* with respect to the flag (K, L) if

$$k \lrcorner \alpha = 0 \quad \text{and} \quad \kappa \wedge \alpha = 0. \tag{1}$$

Clearly, L^0 consists of simple 1-forms, and the space L^m of simple $(m+1)$-forms is isomorphic to the tensor product (Trautman 1985)

$$L^0 \otimes \Lambda^m (L/K)^*$$

so that its dimension is $\binom{n-2}{m}$. Any simple $(n-2)$-form is decomposable.

Let $(e^1, \ldots, e^{n-1})$ be a linear basis on K^0 and

$$(n-1)!\,\eta = \epsilon_{\mu_1 \ldots \mu_{n-1}} e^{\mu_1} \wedge \ldots \wedge e^{\mu_{n-1}},$$

where the μs range from 1 to $n-1$. For any $u \in L$, the $(n-2)$-form $u \lrcorner \eta$ is simple. Moreover, if $u' - u \in K$, then $u' \lrcorner \eta = u \lrcorner \eta$. Therefore, the map

$$L/K \to L^{n-3}$$

given by

$$u + K \mapsto u \lrcorner \eta$$

is an isomorphism of vector spaces.

All manifolds, maps and tensor fields considered in this paper are assumed to be of class C^∞. If $f: M \to N$ is a diffeomorphism and T is a tensor field on N, then the pull-back f^*T of T by f is a tensor field on M. A vector field k on a manifold M generates a flow, i.e. a local, one-parameter group (ϕ_t), $t \in \mathbb{R}$, of local diffeomorphisms of M. The Lie derivative of a tensor field T on M, with respect to k, is defined by

$$\pounds_k T = (d/dt)\, \phi_t^* T|_{t=0}.$$

Since

$$(d/dt)\, \phi_t^* T = \phi_t^* \pounds_k T$$

the condition

$$\phi_t^* T = T \text{ for any } t \text{ is equivalent to } \pounds_k T = 0. \tag{2}$$

If α is a differential form on M, then

$$\pounds_k \alpha = k \lrcorner d\alpha + d(k \lrcorner \alpha).$$

3. Geodesic flag structures

A congruence of null curves on a real manifold M with a conformal structure defines, at each point p of M, a flag in the vector space $T_p M$ tangent to M at p. It turns out that such a distribution of flags on M is sufficient to define the geodesic property of the congruence without any further reference to the conformal structure (Robinson & Trautman 1983).

We say that the manifold M has a flag structure if two smooth distributions $M \ni p \mapsto K_p \subset T_p M$ and $M \ni p \mapsto L_p \subset T_p M$ are given such that (K_p, L_p) is a flag in $T_p M$ for each $p \in M$. In other words, a flag structure on M is a pair $(\mathscr{K}, \mathscr{L})$ of vector bundles over M such that

$$\mathscr{K} \subset \mathscr{L} \subset TM,$$

$\mathscr{K}$ is a line bundle and $\mathscr{L}$ is of codimension 1. The fibres of $\mathscr{K}$ and $\mathscr{L}$ at $p \in M$ are K_p and L_p, respectively. Any notion related to flags can be pointwise extended to flag structures; for example, a metric tensor field on M is said to be adapted to $(\mathscr{K}, \mathscr{L})$ if its restriction to $T_p M$ is adapted to (K_p, L_p) at each point $p \in M$. The line bundle $\mathscr{K} \to M$ defines on M a congruence of curves, i.e. a one-dimensional foliation. $\mathscr{K}^0$ and $\mathscr{L}^0$ are vector bundles whose fibres at p are K_p^0 and L_p^0, respectively. Clearly,

$$\mathscr{L}^0 \subset \mathscr{K}^0 \subset T^*M$$

and $\mathscr{L}^0$ is a line bundle.

PROPOSITION. *The following properties of a flag structure on* M *are equivalent:*

(i) *the bundle* $\mathscr{L}\to \mathrm{M}$ *is invariant under the flow* (ϕ_t) *corresponding to any (local) section* k *of* $\mathscr{K}\to\mathrm{M}$;

(ii) *for any local section* κ *of* $\mathscr{L}^0\to\mathrm{M}$ *one has*

$$\kappa\wedge £_k\kappa = 0; \tag{3}$$

(iii) *the 3-form* $\kappa\wedge \mathrm{d}\kappa$ *is simple;*

(iv) *the congruence of curves defined by* $\mathscr{K}$ *consists of null geodesics with respect to any metric tensor adapted to the flag structure.*

Indeed, the bundle $\mathscr{L}\to\mathrm{M}$ may be determined locally by a nowhere vanishing section κ of $\mathscr{L}^0\to\mathrm{M}$, $\mathrm{L}_p=\ker\kappa(p)$. Therefore, the invariance of $\mathscr{L}$ under (ϕ_t) is characterized by

$$\kappa\wedge\phi_t^*\kappa = 0,\quad t\in\mathbb{R},$$

and this, by virtue of (2), is equivalent to (3). If g is a metric tensor field adapted to $(\mathscr{K},\mathscr{L})$, then $g(k)$ is a section of $\mathscr{L}^0\to\mathrm{M}$ and

$$g(k)\wedge £_k\, g(k) = 0$$

is equivalent to

$$g(k)\wedge g(\nabla_k k) = 0,$$

where ∇_k is the covariant derivative in the direction of k with respect to the Levi-Civita connection associated with g. Clearly, the last equation is equivalent to the geodesic property of the congruence.

A flag structure which has any – and therefore all – of properties (i)–(iv) is said to be geodesic. In particular, a flag structure corresponding to an integrable bundle $\mathscr{L}\to\mathrm{M}$, i.e. such that

$$\kappa\wedge\mathrm{d}\kappa = 0 \tag{4}$$

is geodesic. On a 3-manifold, conversely, the bundle $\mathscr{L}$ of a geodesic flag structure is integrable. In general, an integrable bundle $\mathscr{L}\to\mathrm{M}$ defines a foliation of co-dimension one. If $\mathscr{L}$ is non-integrable, then the congruence defined by $\mathscr{K}$ is said (by physicists) to be twisting.

4. The theorem

The geodesic property of a flag structure on an n-manifold M may also be characterized in terms of simple $(n-2)$-forms on M.

Let $(\mathscr{K},\mathscr{L})$ be a geodesic flag structure on the n-manifold M, $n\geqslant 3$. For any $p\in\mathrm{M}$ there is a neighbourhood N of p and a system of local coordinates $(x^1,\dots,x^{n-1},x^n=t)$ with domain N such that $k=\partial/\partial t$ is a section of the bundle $\mathscr{K}$ restricted to N. If κ is a section of $\mathscr{L}^0$ restricted to N, then one can find $n-1$ functions n_μ $(\mu=1,\dots,n-1)$ on N such that $\kappa=n_\mu\,\mathrm{d}x^\mu$. The geodesic property (3) is equivalent to

$$\kappa\wedge\dot\kappa = 0, \tag{5}$$

where $\dot\kappa=\dot n_\mu\,\mathrm{d}x^\mu$ and the dot denotes a derivative with respect to time.

According to §2, any simple $(n-2)$-form F on M may be written as

$$F = m \lrcorner\, \eta,$$

where $$(n-1)!\,\eta = \epsilon_{\mu_1 \ldots \mu_{n-1}}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{n-1}}$$

and m is a section of the bundle $\mathscr{L} \to \mathrm{N}$, i.e. a vector field on N such that

$$m \lrcorner \kappa = 0.$$

Since m and $m + ak$ give rise to the same F, one can assume m to have a vanishing component along $\partial/\partial t$,

$$m = m^{\mu}\, \partial/\partial x^{\mu}.$$

This being so, let $\operatorname{div} m$ denote the divergence of m with respect to η,

$$\operatorname{div} m = \partial m^{\mu}/\partial x^{\mu}$$

and $$\dot{m} = \dot{m}^{\mu}\, \partial/\partial x^{\mu}.$$

The exterior derivative of F is

$$dF = \eta \operatorname{div} m + dt \wedge \dot{F},$$

where $$\dot{F} = \dot{m} \lrcorner \eta. \tag{6}$$

Since $\kappa \wedge \eta = 0$ and $$\kappa \wedge \dot{F} = (\dot{m} \lrcorner \kappa)\, \eta = -(m \lrcorner \dot{\kappa})\, \eta$$

it is clear that the geodesic condition (5) implies

$$\kappa \wedge dF = 0. \tag{7}$$

Conversely, if (7) holds for any simple $(n-2)$-form F on M, then

$$m \lrcorner \dot{\kappa} = 0$$

for any section m of $\mathscr{L} \to \mathrm{M}$, thus implying (5).

This may be summarized in the following theorem.

THEOREM. *A flag structure on an n-manifold* M *is geodesic if, and only if, any simple* $(n-2)$*-form* F *on* M *satisfies the differential equation* (7).

Remark 1. The form F satisfies (7) if, and only if, $£_k F$ is simple. In conformal spacetime, of course, a simple 2-form is a linear combination of F and its Hodge dual.

Remark 2. In general, the form dF is not simple because

$$k \lrcorner dF = \dot{F}$$

need not be zero. It is possible, however, to choose, at least locally, $n-2$ vector fields m_a $(a = 1, \ldots, n-2)$ such that

$$\dot{m}_a = 0$$

and, at each point of M, the sequence $(m_1, \ldots, m_{n-2}, k)$ is linearly independent. The corresponding $(n-2)$-forms,

$$F_a = m_a \lrcorner \eta$$

have simple exterior derivatives and span the space of all simple $(n-2)$-forms.

5. An example

Consider $M' = S_1 \times \mathbb{C}^{l+1}$ as a real, $(2l+3)$-manifold, choose a standard coordinate ϕ on S_1 and denote by $\bar{z}_\alpha$ the complex conjugate of z^α, the αth complex coordinate on $\mathbb{C}^{l+1}$, where $\alpha = 0, 1, \ldots, l$. The vector field

$$k = \partial/\partial\phi + \mathrm{i}(\bar{z}_\alpha \partial/\partial\bar{z}_\alpha - z^\alpha \partial/\partial z^\alpha) \tag{8}$$

and the 1-form

$$\kappa = z^\alpha \bar{z}_\alpha \,\mathrm{d}\phi + \tfrac{1}{2}\mathrm{i}(z^\alpha \,\mathrm{d}\bar{z}_\alpha - \bar{z}_\alpha \,\mathrm{d}z^\alpha) \tag{9}$$

define on M′ a geodesic flag structure. A metric adapted to this structure is

$$g = \mathrm{d}z^\alpha \,\mathrm{d}\bar{z}_\alpha - z^\alpha \bar{z}_\alpha \,\mathrm{d}\phi^2. \tag{10}$$

The vector field k is tangent to the submanifold $M = S_1 \times S_{2l+1}$ of M′, given by

$$z^\alpha \bar{z}_\alpha = 1. \tag{11}$$

The restrictions of k and κ to M, which will be denoted by the same letters, define on M a geodesic flag structure. Let $\eta = \kappa \wedge \nu$, where $\nu = \mathrm{d}\kappa \wedge \ldots \wedge \mathrm{d}\kappa$ (l factors); then any simple $2l$-form on M may be written as

$$F = m \,\lrcorner\, \eta,$$

where the vector field

$$m = m^\alpha \partial/\partial z^\alpha + \bar{m}_\alpha \partial/\partial \bar{z}_\alpha$$

is a section of $\mathscr{L} \rightarrow M$, i.e. $m \,\lrcorner\, \kappa = 0$. For $\mathrm{d}F$ to be simple, the components of m must be restrictions to M of functions on M′ depending on

$$z^\alpha \mathrm{e}^{\mathrm{i}\phi} \quad \text{and} \quad \bar{z}_\alpha \mathrm{e}^{-\mathrm{i}\phi}$$

only.

The restriction of (9) to S_{2l+1}, given by $\phi = \text{const.}$ and (11), defines a connection form on the Hopf bundle

$$U(1) \rightarrow S_{2l+1} \rightarrow \mathbb{CP}_l$$

corresponding to a solution of Maxwell's equations (Trautman 1977). This restriction defines also a non-integrable and analytic distribution of co-dimension one on S_{2l+1}. According to Haefliger (1958), a compact manifold with a finite fundamental group has no real-analytic foliation of co-dimension one. It may, however, have a smooth – but non-analytic – foliation of co-dimension one; Reeb (1952) constructed such a foliation on S_3.

For $l = 1$, the 4-manifold $M = S_1 \times S_3$ can be identified with the compactified Minkowski space, and the lines generated by (8) form a Robinson congruence (Penrose 1967, 1985). This may be seen by identifying M with the manifold of the unitary group U(2) and considering an embedding of the Minkowski space $\mathbb{R}^4$ into U(2) given as follows (Uhlmann 1963). Let a point in Minkowski space be represented by the Hermitian matrix

$$X = \begin{pmatrix} v & \bar{w} \\ w & u \end{pmatrix}, \quad \text{where} \quad u, v \in \mathbb{R} \quad \text{and} \quad w \in \mathbb{C}.$$

If I denotes the unit 2×2 matrix, then the matrix

$$Y = f(X) \tag{12}$$

given by
$$f(X) = (I+\mathrm{i}X)(I-\mathrm{i}X)^{-1} \tag{13}$$
is unitary. The map
$$f: \mathbb{R}^4 \to \mathrm{U}(2)$$
is an analytic diffeomorphism of $\mathbb{R}^4$ on the image.

The complement of the image of f in $\mathrm{U}(2)$,
$$\{A \in \mathrm{U}(2) : \det(I+A) = 0\},$$
is the 'light-cone at infinity' (Penrose 1962).

If (ϕ, z^0, z^1) are local coordinates on $S_1 \times S_3$ such that
$$z^0 \bar{z}_0 + z^1 \bar{z}_1 = 1,$$
then
$$Y = \mathrm{e}^{\mathrm{i}\phi}\begin{pmatrix} z^0 & \bar{z}_1 \\ -z^1 & \bar{z}_0 \end{pmatrix}$$
belongs to $\mathrm{U}(2)$, and any unitary 2×2 matrix can be so represented. The matrix
$$Y^{-1}\,\mathrm{d}Y = \mathrm{i}\begin{pmatrix} \mathrm{d}\phi + \mu & \bar{\lambda} \\ \lambda & \mathrm{d}\phi - \mu \end{pmatrix}$$
of Maurer–Cartan forms on $\mathrm{U}(2)$ is anti-Hermitian,
$$\mu = \mathrm{i}(z^0\,\mathrm{d}\bar{z}_0 + z^1\,\mathrm{d}\bar{z}_1), \quad \lambda = z^1\,\mathrm{d}z^0 - z^0\,\mathrm{d}z^1,$$
and
$$\det(Y^{-1}\,\mathrm{d}Y) = \lambda\bar{\lambda} + \mu^2 - \mathrm{d}\phi^2$$
$$= \mathrm{d}z^0\,\mathrm{d}\bar{z}_0 + \mathrm{d}z^1\,\mathrm{d}\bar{z}_1 - \mathrm{d}\phi^2 \tag{14}$$
coincides with the metric (10) restricted to $S_1 \times S_3$.

From (12) and (13) follows that
$$Y^{-1}\,\mathrm{d}Y = \tfrac{1}{2}\mathrm{i}(I+Y^{-1})\,\mathrm{d}X(I+Y) \tag{15}$$
and since
$$\det \mathrm{d}X = \mathrm{d}u\,\mathrm{d}v - \mathrm{d}w\,\mathrm{d}\bar{w} \tag{16}$$
it is clear that $f: \mathbb{R}^4 \to \mathrm{U}(2)$ is a conformal map relative to the metrics (14) and (16).

The form (9) is now
$$\kappa = \mathrm{d}\phi + \mu$$
and its pull-back to Minkowski space, $f^*\kappa$, may be evaluated from (15),
$$af^*\kappa = \mathrm{d}v - \bar{\zeta}\,\mathrm{d}w - \zeta\,\mathrm{d}\bar{w} + \zeta\bar{\zeta}\,\mathrm{d}u$$
$$= \mathrm{d}\sigma + \mathrm{i}(\zeta\,\mathrm{d}\bar{\zeta} - \bar{\zeta}\,\mathrm{d}\zeta), \tag{17}$$
where
$$\zeta = w/(u+\mathrm{i}), \quad \sigma = v - uw\bar{w}/(u^2+1), \tag{18}$$
and
$$a = \tfrac{1}{2}\{(1-uv+w\bar{w})^2 + (u+v)^2\}/(u^2+1).$$
The Minkowski line-element (16) takes the form
$$af^*\kappa\,\mathrm{d}u - (u^2+1)\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}.$$
The vector field corresponding to the 1-form $f^*\kappa$ was first discovered in the coordinates $u, \sigma, \zeta, \bar{\zeta}$. According to the Haefliger theorem, any null congruence which extends to an analytic congruence on the compactified Minkowski space is twisting, since the associated two-dimensional distribution on S_3 is non-integrable.

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