

Integrable Optical Geometry

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Abstract. It is shown that a Lorentzian 4-manifold admitting a congruence of optical (null) geodesics without shear and twist defines an optical geometry which is integrable (locally flat) in the sense of the theory of G -structures. The existence of a symmetric linear connection compatible with the optical geometry is another condition equivalent to the integrability of the optical G -structure.

1. Introduction and Heuristic Remarks

There is an 'optical geometry' in spacetime associated with purely radiative electromagnetic fields. An isomorphism of this geometry can be used to transform one such purely radiative field into another [1]. It has been known for a long time that a purely radiative solution of Maxwell's equations defines a congruence of null (optical) geodesics without shear [2]. This and related notions have led to effective methods of solving Einstein's equations for algebraically special metrics (cf., for example, [3] and the numerous references listed there). Recently [4], but under the influence of old ideas due to Bateman [1] and Cartan [5], optical geometry has been defined and studied in terms of a G_0 -structure, where G_0 , the 'optical group', is a suitable Lie subgroup of $GL(4, \mathbb{R})$. This Letter presents the fundamental theorem of optical geometry which characterizes integrable G_0 -structures. It is preceded by a few heuristic remarks and a summary of the required notions of optical geometry. A comprehensive account of this subject may be found in [6] and [7].

Consider the Minkowski space \mathbb{R}^4 with coordinates (x, y, r, u) and the line element

$$dx^2 + dy^2 + 2 du dr. \quad (1)$$

If p and $q: \mathbb{R} \rightarrow \mathbb{R}$ are smooth, then the 2-form

$$F = du \wedge (p(u) dx + q(u) dy) \quad (2)$$

represents the electromagnetic field of a *plane* wave propagating in the direction perpendicular to the (x, y) plane. If the orientation in \mathbb{R}^4 is given by the volume form $dx \wedge dy \wedge dr \wedge du$, then the Hodge dual of F is

$$*F = du \wedge (p(u) dy - q(u) dx).$$

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