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Integrable Optical Geometry

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Abstract. It is shown that a Lorentzian 4-manifold admitting a congruence of optical (null) geodesics without shear and twist defines an optical geometry which is integrable (locally flat) in the sense of the theory of G-structures. The existence of a symmetric linear connection compatible with the optical geometry is another condition equivalent to the integrability of the optical G-structure.

1. Introduction and Heuristic Remarks

There is an 'optical geometry' in spacetime associated with purely radiative electromagnetic fields. An isomorphism of this geometry can be used to transform one such purely radiative field into another [1]. It has been known for a long time that a purely radiative solution of Maxwell's equations defines a congruence of null (optical) geodesics without shear [2]. This and related notions have led to effective methods of solving Einstein's equations for algebraically special metrics (cf., for example, [3] and the numerous references listed there). Recently [4], but under the influence of old ideas due to Bateman [1] and Cartan [5], optically geometry has been defined and studied in terms of a G_0-structure, where G_0, the 'optical group', is a suitable Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$. This Letter presents the fundamental theorem of optical geometry which characterizes integrable G_0-structures. It is preceded by a few heuristic remarks and a summary of the required notions of optical geometry. A comprehensive account of this subject may be found in [6] and [7].

Consider the Minkowski space $\mathbb{R}^4$ with coordinates (x, y, r, u) and the line element

$$\mathrm{d}x^2 + \mathrm{d}y^2 + 2\,\mathrm{d}u\,\mathrm{d}r\,. \tag{1}$$

If p and q: $\mathbb{R} \to \mathbb{R}$ are smooth, then the 2-form

$$F = \mathrm{d}u \wedge (p(u)\,\mathrm{d}x + q(u)\,\mathrm{d}y) \tag{2}$$

represents the electromagnetic field of a *plane* wave propagating in the direction perpendicular to the (x, y) plane. If the orientation in $\mathbb{R}^4$ is given by the volume form $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}r \wedge \mathrm{d}u$, then the Hodge dual of F is

$$*F = \mathrm{d}u \wedge (p(u)\,\mathrm{d}y - q(u)\,\mathrm{d}x)\,.$$

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This dual does not change if the line element (1) is replaced by

$$P^2(\mathrm{d}x^2 + \mathrm{d}y^2) + 2\,\mathrm{d}u(\mathrm{d}r + \xi) \tag{3}$$

where P is a nowhere vanishing, smooth function on $\mathbb{R}^4$ and ξ is any smooth 1-form on $\mathbb{R}^4$. Therefore, the 2-form (2) is a solution of Maxwell's equations $\mathrm{d}F = 0$ and $\mathrm{d}*F = 0$ on any 4-manifold with line element (3). Many solutions of Einstein's equations admit local coordinates relative to which their metric tensor is of this form. For example, if $P = 2r/(1 + x^2 + y^2)$ and $2\xi = (1 - 2m/r)\,\mathrm{d}u$, then (3) is a local expression of the Schwarzchild metric. In this case, Equation (2) represents outgoing, pure electromagnetic radiation with *spherical* wavefronts, propagating on the Schwarzchild background. The electromagnetic field (2) is optical (null, purely radiative) in the sense that there exists a nowhere vanishing vector field k such that $k \lrcorner \mathrm{F} = 0$ and $k \lrcorner *F = 0$.

Optical geometry has been developed to put the above observations into a broader perspective.

2. Optical Geometry

Let K and L denote, respectively, the line $x_1 = x_2 = x_4 = 0$ and the 3-plane $x_4 = 0$ in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). The quotient vector space L/K may be given a complex structure defined by the isomorphism $\mathbb{C} \leftarrow\!\shortmid L/K$ such that

$$x + iy \mapsto (x, y, 0, 0) \bmod K.$$

The *optical group* G_0 is defined as the group of all linear automorphisms of $\mathbb{R}^4$ which preserve K, L, and the complex structure in L/K. It is a nine-dimensional closed Lie subgroup of $\mathrm{GL}(4, \mathbb{R})$ admitting $\mathrm{SO}(2)$ as a maximal compact subgroup.

An *optical geometry* on a four-dimensional smooth manifold M is defined as a G_0-structure [8], i.e., a restriction P of the bundle of linear frames of M to the group G_0. To alleviate the language, we assume these bundles to be trivial. For any global section $e = (e_1, e_2, e_3, e_4)$ of $P \to M$, let $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ be the corresponding (dual) section of the bundle of linear coframes, $\langle e_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu}$, where $\mu, \nu = 1, \ldots, 4$. Any such section defines on M a metric tensor

$$g = \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_3\varepsilon_4 \tag{4}$$

of Lorentz signature. If e' is another section of $P \to M$, then the corresponding metric tensor g' is related to g as follows:

$$g' = \mu^2 g + 2\kappa\xi\,. \tag{5}$$

Here μ is a nowhere vanishing function, ξ is a 1-form on M and κ is the 1-form on M such that, for any vector field u, one has

$$\langle u, \kappa \rangle = g(u, k) \tag{6}$$

where $k = e_3$. Clearly, $\kappa = \varepsilon_4$.

Let $\mathscr{E}$ denote the set of all metric tensors on M which can be obtained from sections of $P \to M$ according to Equation (4). Any two elements g and g' of $\mathscr{E}$ are related to each other by (5). It is now clear that an optical geometry on a 4-manifold M may be equivalently defined by giving (a) a line-bundle $\mathscr{K} \subset TM$, i.e., a field of directions on M; (b) a set $\mathscr{E}$ of Lorentzian metric tensors on M such that if k is a nowhere vanishing section of $\mathscr{K} \to M$, $g \in \mathscr{E}$, and κ is as in (6), then $\langle k, \kappa \rangle = 0$ and, for any $g' \in \mathscr{E}$, there is a function of μ and a 1-form ξ on M such that (5) holds.

A diffeomorphism f of M is said to be an *optical automorphism* if it preserves P. Equivalently, such an automorphism preserves $\mathscr{K}$ and $\mathscr{E}$. According to [9], the flow generated by the section k: $M \to \mathscr{K}$ consists of optical automorphisms if and only if the geometry is that of *optical geodesics without shear* [2].

An optical geometry on M defines the 3-plane bundle $\mathscr{L} = \ker \kappa$. In other words, for any section e of $P \to M$, the bundle $\mathscr{L}$ is spanned by the vector fields e_1, e_2, and e_3. The bundle $\mathscr{L}$ is integrable if and only if

$$\kappa \wedge \mathrm{d}\kappa = 0 . \tag{7}$$

In this case, the integral curves of $\mathscr{K}$ are said to be *free of twist*. They are (optical) geodesics with respect to the Levi–Civita connection associated with any $g \in \mathscr{E}$.

3. Integrable G_0-Structures

Recall that a G-structure P on M is said to be locally flat [10] or *integrable* [11] if, for any point of M, there is a system of local coordinates x around that point such that the field of natural frames associated with x is a local section of the bundle $P \to M$. For example, an $\mathrm{O}(n)$-structure on an n-manifold – i.e., a Riemannian geometry – is integrable if and only if its curvature vanishes. A 1-structure – i.e., a field of frames – is integrable if and only if the field of frames is holonomic.

Any principal bundle over a (paracompact) manifold admits a connection; in particular, this applies to a G-structure on M. Such a connection will, in general, be asymmetric. An integrable G-structure admits a symmetric linear connection, but the converse need not be true, as is apparent from the example of Riemannian geometry.

An optical geometry is said to be integrable if it is integrable as a G_0-structure. An optical geometry on M is thus integrable if and only if around any point of M there is a system of local coordinates (x, y, r, u) such that the vector field $k = \partial/\partial r$ spans $\mathscr{K}$ and the metric tensor (1) belongs to $\mathscr{E}$.

THEOREM. *The following properties of an optical geometry on M are equivalent to one another*:

(*i*) *the optical geometry is integrable*;
(*ii*) *M admits a symmetric linear connection compatible with its optical geometry*;
(*iii*) *the optical geometry on M is that of optical geodesics without shear and twist.*

Proof. The implication (i)$\Rightarrow$(ii) is straightforward: it is enough to take local coordinates (x, y, r, u), $k = \partial/\partial r$, $g \in \mathscr{E}$ given by (1), and a connection with vanishing coefficients in this coordinate system. To prove (ii)$\Rightarrow$(iii), consider a section k of

$\mathcal{K} \to M$, $g \in \mathcal{E}$, and let ∇ be the covariant derivative corresponding to a symmetric connection. This connection is compatible with the optical geometry defined by $\mathcal{K}$ and $\mathcal{E}$ if and only if there exist tensor fields l, m, n such that

$$\nabla_\mu k_\nu = l_\mu k_\nu \tag{8}$$

and

$$\nabla_\rho g_{\mu\nu} = m_\rho g_{\mu\nu} + n_{\rho\mu} k_\nu + n_{\rho\nu} k_\mu \tag{9}$$

where $x = (x^\mu)$ are local coordinates on M, $k = k^\mu \partial/\partial x^\mu$, $k_\mu = \partial/\partial x^\mu \lrcorner \kappa = g_{\mu\nu} k^\nu$, $g = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, etc. Since the connection is symmetric, Equation (8) implies (7). If both sides of (9) are concentrated with k^ρ and (8) is used, then one obtains

$$(L_k g)_{\mu\nu} = \alpha g_{\mu\nu} + \beta_\mu k_\nu + \beta_\nu k_\mu \tag{10}$$

with suitable α and β. Equation (10) for the Lie derivative of g with respect to k is known to characterize a geometry of optical geodesics without shear [9]. Finally, the proof of (iii) $\Rightarrow$ (i) is implicit in our early work [12] where we showed that the metric tensor of a Lorentzian geometry admitting a congruence of optical geodesics without shear and twist can be brought to the form (3). Together with $k = \partial/\partial r$, that metric defines the same optical geometry as (1).

Among the metrics corresponding to the integrable optical geometry there are many nontrivial solutions of Einstein's field equations.

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