

OPTICAL STRUCTURES IN RELATIVISTIC THEORIES

BY

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ABSTRACT. — In 1922 Élie CARTAN discovered the existence, in any Lorentzian and not conformally flat 4-manifold, of four privileged directions which are optical in the sense that they belong to the light cone defined by the fundamental form. In special cases — as for the Schwarzschild spacetime — some of these directions may coincide. These observations were later rediscovered and used by physicists in the study of purely radiative Maxwell fields and of ‘algebraically special’ Einstein metrics. The present article reviews some of these developments and outlines the underlying ‘optical geometry.’ This geometry is defined as a G_0 -structure on a 4-manifold, where G_0 , the ‘optical group’, is a suitable 9-dimensional Lie subgroup of $GL(4, \mathbf{R})$. It is shown that the G_0 -structure is integrable if and only if the optical geometry is that of rays without shear and twist. By an extension of the Robinson theorem, optical (purely radiative) solutions to sourceless Maxwell and Yang-Mills equations exist in geometries of rays without shear. An isomorphism of optical geometries is shown to transform one such solution into another. For example, one such isomorphism — which is not a conformal map — transforms plane waves into spherical waves of a special kind.

1. Introduction

Relativistic theories of spacetime and of interactions between particles and fields are based on geometrical models which include, as an essential element, a metric tensor of Lorentz (normal hyperbolic) signature. As a result of the indefinite character of the metric, relativistic models admit directions — and other geometric elements — which are isotropic (null, light-like) in the sense of being associated with non-zero vectors of vanishing square. Such isotropic elements are of considerable interest from the point of view of both geometry and physics. Élie CARTAN has shown that totally isotropic maximal planes can be used to define spinors over Euclidean vector spaces of any number of dimensions [1].

For a physicist, a light-like vector in Minkowski space may be identified with the energy-momentum vector of a particle of zero rest-mass (photon, neutrino).

Null hypersurfaces in a Lorentz 4-manifold M with metric tensor g are represented by solutions $u : M \rightarrow \mathbf{R}$ of the eikonal equation

$$(1) \quad g^{\mu\nu} \frac{\partial u}{\partial x^\mu} \frac{\partial u}{\partial x^\nu} = 0$$

where (x^μ) , $\mu = 1, 2, 3, 4$, are local coordinates on M . Let M be oriented; Maxwell's equations for the 2-form F of electromagnetism are

$$(2) \quad dF = 0 \quad \text{and} \quad d * F = 0,$$

where $*F$ is the Hodge dual of F defined in terms of g and the orientation. Assume now F to be of the form

$$(3) \quad F = \text{Re}(F_0 \exp iu/\lambda)$$

where F_0 is a (complex) 2-form representing the amplitude of an electromagnetic wave with phase $u : M \rightarrow \mathbf{R}$. In the limit of wave optics ($\lambda \rightarrow 0$), Maxwell's equations (2) imply

$$(4) \quad du \wedge F_0 = 0 \quad \text{and} \quad du \wedge *F_0 = 0$$

so that (1) is a necessary and sufficient condition for the existence of a nowhere vanishing F_0 subject to (4). The subtler question of whether there exists a non-vanishing F_0 , solution to (2) and (4) is considered in § 5 [3].

The study of isotropic elements and of the associated classical fields — such as gravitation, electromagnetism and Yang-Mills — resulted in much progress in the area of finding exact solutions and establishing their properties. The study has been particularly fruitful in the theory of general relativity where it led to the notion of algebraically special spacetimes. Large classes of exact, explicit solutions of Einstein's equations have been found in this manner. Among them are plane-fronted and sphere-fronted waves [4], and the Kerr solution representing the exterior of a rotating black hole [5]. A good summary of this research, with many references, is given in [6]. Almost all explicitly known exact solutions of Einstein's equations in four dimensions and with Lorentz signature belong to one of three classes :

1. algebraically special metrics,
2. stationary metrics with axial symmetry,
3. metrics with cylindrical symmetry.

The consideration of isotropic elements in relativistic theories has also led to a number of general results, such as

1. The Robinson theorem on the existence of purely radiative (isotropic, null) solutions of Maxwell's [3] and Yang-Mills [7] equations associated with congruences of null, shear-free geodesics; *cf.* § 5.

2. The Goldberg-Sachs [8] theorem and its generalizations [9, 10] : a space-time subject to a suitably weakened form of Einstein's equations is algebraically degenerate if and only if it contains a congruence of shear-free null geodesics.

3. The Sachs-Newman-Penrose theory of asymptotic behaviour of the curvature tensor of spacetimes with isolated sources (the 'peeling-off' theorem) [11-15].

4. The Xanthopoulos theorem [16] on space-times of the Kerr-Schild family [17, 33].

In spite of much research, the mathematics underlying purely radiative Maxwell fields, algebraically special metrics, and similar structures, has not, until recently, been formulated in the language of modern differential geometry. The purpose of the present paper is to outline such a formulation. A fuller account is being published elsewhere [18, 19, 58-60].

2. Élie Cartan on optical directions

In a short note published in 1922 in *Comptes Rendus* Élie CARTAN wrote [2] :

“Au point de vue géométrique, la propriété suivante mérite d'être signalée. Il existe en chaque point A quatre directions *optiques* (c'est-à-dire annulant le ds^2) privilégiées... Dans le cas du ds^2 d'une seule masse attirante (ds^2 de Schwarzschild), ces quatre directions optiques privilégiées se réduisent à deux (doubles) : les deux rayons lumineux qui leur correspondent iraient au centre d'attraction ou en viendraient.”

The four optical (isotropic) directions referred to by CARTAN are defined by the tensor of conformal curvature. Possible coincidences among the four directions constitute the basis of an algebraic classification of conformal curvature tensors. A space-time is said to be *algebraically special* if its conformal curvature tensor has at least a double preferred optical direction. This classification was elaborated only in the 1950's, without reference to CARTAN's remarks on the subject, which seem to have gone unnoticed for almost 50 years. The algebraic classification of conformal curvature tensors has been developed by A.Z. PETROV [20] and later refined, related to physics, and used by F.A.E. PIRANI [21], R. DEBEVER [22], A. LICHNEROWICZ [23], L. BEL [24], R.K. SACHS [11], R. PENROSE [25], and many other mathematicians and physicists [6, 28-37].

Following a suggestion by R. DEBEVER made at *Journées Relativistes* 1983 at Turin, I propose to go back to CARTAN's terminology of 1922 and use the adjective *optical* instead of the somewhat confusing 'isotropic' or 'null'.

H. BATEMAN [26] may also be considered as an avatar of optical geometry. Recently, R.A. d'INVERNO and J. STACHEL [27] recognized the role of conformal, two-dimensional geometry in four-dimensional spacetimes for the description of the degrees of freedom of gravitation.

Optical geometry is closely related to the picture of spacetime and relativistic physics which is being developed by Roger PENROSE [13, 25, 47, 54]; in a sense, this geometry is a step-child of his twistor programme [41, 42, 55].

3. Vector spaces with optical structure

Consider once more the algebraic properties of the 2-form given by (3) and subject to conditions (4). Let F and κ denote, respectively, the values of this 2-form and of the 1-form du at $x \in M$. Assume that the tangent vector space $T_x M$, over which these forms are defined, is oriented and endowed with a scalar product of Lorentz signature. There then exist 1-forms α and β which are of equal length, orthogonal to each other and to κ , and such that

$$F = \kappa \wedge \alpha \quad \text{and} \quad *F = \kappa \wedge \beta.$$

Only the direction of κ is well-defined by F . The forms α and β are defined up to a common factor and addition of multiples of κ . By duality, similar structures can be identified in the tangent space itself: there is a direction $K \subset T_x M$ such that, for any $k \in K$,

$$k \lrcorner F = 0 \quad \text{and} \quad k \lrcorner *F = 0,$$

and $L = \ker \kappa$ contains K .

Considerations such as these lead to the following

Definition. — An *optical structure* in a real 4-dimensional vector space V consists of

(A) an optical *flag*, i.e. a pair (K, L) of vector subspaces of V , of dimension 1 and 3, respectively, and such that

$$K \subset L \subset V;$$

(B) an orientation and a conformal scalar product in the 2-dimensional vector space L/K .

Note that V itself is not assumed to have a preferred scalar product.

Clearly, condition (B) is equivalent to giving (B') a complex structure in L/K , i.e. a linear map

$$J : L/K \rightarrow L/K \quad \text{such that} \quad J^2 = -\text{id}.$$

An optical structure in V defines, in a natural manner, a similar structure in the dual space V^* . Its flag is (L^0, K^0) , where, for any vector subspace $W \subset V$, $W^0 \subset V^*$ is the space of all forms vanishing on W . Since the space K^0/L^0 is isomorphic, in a natural manner, to $(L/K)^*$, the transpose of J defines a complex structure in K^0/L^0 . If $k \in V$ and $\kappa \in V^*$ are non-zero and $\langle k, \kappa \rangle = 0$, then these elements define a flag by $K = \mathbf{R}k$ and $L = \ker \kappa$.

If (V, K, L, J) and (V', K', L', J') are two vector spaces with optical structure, then $f : V \rightarrow V'$ is an optical isomorphism if it is an isomorphism of vector spaces such that

$$f(K) = K', \quad f(L) = L' \quad \text{and} \quad J' \circ \bar{f} = \bar{f} \circ J$$

where

$$\bar{f} : L/K \rightarrow L'/K' \quad \text{is given by} \quad \bar{f}(l \bmod K) = f(l) \bmod K'$$

for any $l \in L$.

The standard optical structure in $V_0 = \mathbf{R}^4$ is defined by

$$K_0 = \{x \in \mathbf{R}^4 : x_1 = x_2 = x_4 = 0\},$$

$$L_0 = \{x \in \mathbf{R}^4 : x_4 = 0\},$$

and

$$J_0[(x_1, x_2, 0, 0)] = [(-x_2, x_1, 0, 0)]$$

where $x = (x_\mu)$, $\mu = 1, 2, 3, 4$, and square brackets denote an equivalence class modulo K_0 .

The optical group $G_0 \subset GL(4, \mathbf{R})$ is defined as the group of all optical automorphisms of the standard optical structure. It is a 9-dimensional Lie group consisting of all matrices of the form

$$\begin{pmatrix} \rho \cos \varphi & -\rho \sin \varphi & 0 & p \\ \rho \sin \varphi & \rho \cos \varphi & 0 & q \\ a & b & \sigma & r \\ 0 & 0 & 0 & \tau \end{pmatrix},$$

where $0 \leq \varphi < 2\pi$, $\rho > 0$, $\sigma, \tau \neq 0$ and $a, b, p, q, r \in \mathbf{R}$.

Let (e_μ^0) be the canonical frame in \mathbf{R}^4 : e_μ^0 is the vector whose μ -th component is 1 and all other components are zero. An optical frame $e = (e_\mu)$ in a vector space V with optical structure is the image of (e_μ^0) by an optical isomorphism of the standard structure onto V . In other words, a frame e in V is optical if, and only if,

L is spanned by e_1, e_2 and $e_3 \in K$;

$J(e_1 \bmod K) = e_2 \bmod K$.

If (ϵ_μ) is the frame dual to an optical frame (e_μ) , $\langle e_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$, then K^0 is spanned by ϵ_1, ϵ_2 and $\epsilon_4 \in L^0$. The group G_0 acts, in a free and transitive manner, in the set $\text{Opt}(V)$ of all optical frames of a vector space V with optical structure; ea denotes the result of the action of $a \in G_0$ on $e \in \text{Opt}(V)$.

An optical structure does not define a scalar product in its underlying vector space, but it is convenient to consider the map

$$(5) \quad \text{Opt}(V) \ni e \mapsto g_e \in S(V)$$

from the set of optical frames to the set $S(V) \subset V^* \otimes V^*$ of scalar products in V given by

$$g_e = \epsilon_1 \otimes \epsilon_1 + \epsilon_2 \otimes \epsilon_2 + \epsilon_3 \otimes \epsilon_4 + \epsilon_4 \otimes \epsilon_3.$$

I shall follow the tradition of classical differential geometry according to which one omits the symbol of the symmetric tensor product in all formulae for the fundamental quadratic form. Therefore, the last equation becomes

$$g_e = \epsilon_1^2 + \epsilon_2^2 + 2\epsilon_3\epsilon_4.$$

Clearly, g_e is a scalar product of Lorentz signature. If $a \in G_0$, then

$$g_{ea} = \rho^{-2}g_e + 2\xi\epsilon_4$$

for some form $\xi \in V^*$. For any $e \in \text{Opt}(V)$, the direction K is orthogonal to L with respect to g_e . The restriction of g_e to L induces a scalar product in L/K compatible with its conformal structure implied by (B) or (B'). Let $E \subset S(V)$ be the image of $\text{Opt}(V)$ by the map (5). If g and $g' \in E$, then there exist $\rho \neq 0$ and $\xi \in V^*$ such that

$$(6) \quad g' = \rho^{-2}g + 2\xi\kappa$$

where $0 \neq \kappa \in L^0$. Condition (B) in the definition of optical structure in V may be replaced by :

(B'') an orientation in L/K and an equivalence class $E \subset S(V)$ of scalar products, of Lorentz signature, such that for any of them K is orthogonal to L , two scalar products g and g' being considered as equivalent if and only if they are related as in (6).

Let $0 \neq k \in K$ and $0 \neq \kappa \in L^0$. A form $\alpha \in \bigwedge V^*$ such that

$$k \lrcorner \alpha = 0 \quad \text{and} \quad \kappa \wedge \alpha = 0$$

is said to be optical. (N.B. The property of being optical depends only on the flag structure). For example, the real part of the 2-form F_0 representing the amplitude of an electromagnetic wave in the limit of wave optics (*cf.* § 1) is optical. Let L^n ($n = 0, 1, 2$) denote the space of optical $(n+1)$ -forms. The vector space $L^0 \oplus L^1 \oplus L^2$ of all optical forms is isomorphic, in a natural manner, to the 4-dimensional vector space $L^0 \otimes \bigwedge(K^0/L^0)$.

Assume V to be oriented and, for any $g \in S(V)$, let

$$\sigma(g) : \bigwedge V^* \rightarrow \bigwedge V^*$$

be the Hodge dual corresponding to g and the orientation. If $g \in E$ then $\sigma(g)$ maps optical forms into optical forms,

$$\sigma(g) : L^n \rightarrow L^{2-n}, \quad n = 0, 1, 2.$$

Moreover, if g and $g' \in E$ are related to each other as in (6), then

$$(7) \quad \sigma(g') \mid L^n = \rho^{2n-2} \sigma(g) \mid L^n.$$

Of special interest are optical 2-forms. The vector space L^1 of all such forms is isomorphic to $L^0 \otimes (K^0/L^0)$ and may be identified with it. Under this identification, the Hodge map applied to L^1 becomes a rotation by 90° in K^0/L^0 ,

$$\sigma(g)(\kappa \otimes \gamma) = \kappa \otimes {}^t J(\gamma),$$

where $g \in E$, $\gamma \in K^0/L^0 = (L/K)^*$ and ${}^t J$ is the transpose of J . Moreover, eq. (7) for $n = 1$ may be replaced by the somewhat sharper

PROPOSITION 1. — *If $F \neq 0$ is an optical 2-form, $g \in E$ and g' is a scalar product of Lorentz signature, then*

$$\sigma(g)F = \sigma(g')F \quad \text{if and only if} \quad g' \in E.$$

4. Optical Geometry

Let M be a smooth, 4-dimensional, oriented manifold. An *optical geometry* on M may be defined in several equivalent ways, as

- (i) a smooth distribution of optical structures in the tangent spaces to M ;
- (ii) a G_0 -structure on M ;
- (iii) a complex line bundle $\mathcal{L}/\mathcal{K} \rightarrow M$, where \mathcal{K} and \mathcal{L} are real vector sub-bundles of TM of fibre dimensions 1 and 3, respectively.

If the real line bundle \mathcal{K} is orientable — and hence trivial — one can characterize the corresponding optical geometry by giving

(α) an equivalence class $[(k, g)]$ of pairs consisting of a nowhere vanishing vector field k — a section of \mathcal{K} — and a Lorentzian metric g on M , such that k is optical with respect to g , $g(k, k) = 0$; two such pairs (k, g) and (k', g') are considered as equivalent if, and only if, there are two nowhere vanishing functions λ and ρ on M , and a (differential) 1-form ξ on M , such that

$$(8) \quad k' = \lambda k \quad \text{and} \quad g' = \rho^{-2}g + 2\xi\kappa,$$

where κ is the 1-form on M defined by $\langle v, \kappa \rangle = g(v, k)$ for any vector field v on M ;

(β) an orientation in the bundle $\ker \kappa/\mathcal{K}$.

Whenever the bundle \mathcal{K} is trivial, as is the case in most applications, it is convenient to use the characterization of an optical geometry given under (α) and (β). This is often done in the sequel even though most of the results could have been derived without the assumption of triviality of \mathcal{K} . The set of all Lorentz metrics described under (α) is denoted by \mathcal{E} .

The general theory of G -structures (*cf.*, for example, [38] and [39]) provides us with ready definitions of isomorphisms and automorphisms of optical geometries. Let the equivalence classes $[(k, g)]$ and $[(k', g')]$ define optical geometries in the manifolds M and M' , respectively. A diffeomorphism $f: M \rightarrow M'$ is said to be an isomorphism of optical geometries if $[(k, g)] = [(f^*k', f^*g')]$, where f^* denotes the pull-back of tensor fields from M' to M defined by f . With some reluctance, I shall use the barbarism *optomorphism* instead of the longer expression 'automorphism of optical geometry'.

Consider an optical geometry in M , let $(\varphi_t)_{t \in \mathbb{R}}$ denote the flow generated by the vector field k and let κ be a 1-form on M , defined as in (α). By extending the results presented in [40] one obtains

PROPOSITION 2. — *The following conditions are equivalent to each other :*

- (a) the 3-form $\kappa \wedge d\kappa$ is optical,
- (b) $\kappa \wedge L_k \kappa = 0$, where L_k is the Lie derivative with respect to k ,
- (c) the vector bundle $\mathcal{L} \rightarrow M$ is invariant by (φ_t) ,
- (d) the trajectories of k are optical geodesics for any $g \in \mathcal{E}$.

The last condition admits a 'physical interpretation': if it is satisfied, the congruence of trajectories of k may be associated with the rays of light. This motivates the following

Definition. — A *geometry of rays* is an optical geometry for which any (and therefore all) of the conditions (a)–(d) are satisfied.

There are optical geometries which are not geometries of rays. For example, let $M = \{(x_\mu) \in \mathbf{R}^4 : x_1 > 0\}$ be endowed with the optical geometry defined by $[(k, g)]$, where

$$k = \frac{\partial}{\partial x_4} + x_1^{-1} \frac{\partial}{\partial x_2} \quad \text{and} \quad g = dx_1^2 + x_1^2 dx_2^2 + dx_3^2 - dx_4^2,$$

then $\kappa \wedge L_k \kappa = (x_1^{-1} dx_4 - dx_2) \wedge dx_1$ is nowhere zero.

From now on only geometries of rays will be considered. According to § 3, for any $g \in \mathcal{E}$, the Hodge dual of the optical 3-form $\kappa \wedge d\kappa$ is an optical 1-form, i.e. a multiple of κ . There thus exists a function ω on M such that

$$(9) \quad \sigma(g)(\kappa \wedge d\kappa) = \omega \kappa.$$

Eq. (7) shows that ω does not depend on g though it changes under a replacement of k by λk ; in a region where $\omega \neq 0$ one can always choose k so as to have $\omega = 1$. If $\omega = 0$, then the bundle (distribution of 3-dimensional vector spaces on M) $\mathcal{L} \rightarrow M$ is integrable; ω thus measures the 'aholonomy' or 'twist' of the distribution. From the point of view of ray (also conformal) geometry only two values (0 and 1) of twist are relevant.

Since $d\kappa \wedge d\kappa = 0$, one obtains from (9) the 'conservation law'

$$d(\omega \sigma(g) \kappa) = 0.$$

The following example of ray geometry with twist is based on Robinson congruences [29, 41] which played a major rôle in the development of twistors [42]. Consider the compactified Minkowski space M [43–48] with its natural conformal geometry and suitable orientation. Define first M as the semi-Riemannian product $S_1 \times S_3$ [31]. Let g_n denote the standard Riemannian metric on S_n and let π_n be the projection of M on S_n ($n = 1$ or 3). Put

$$g = \pi_3^* g_3 - \pi_1^* g_1$$

and let k be the global optical vector field on M with S_1 -component the standard unit vector field on S_1 and with S_3 -component a unit vector field tangent to the Hopf fibration of S_3 . The optical geometry defined by the pair (k, g) has non-zero twist. The complex line bundle $\mathcal{L}/\mathcal{K} \rightarrow M$ is obtained, in this case, by pulling back to M the vector bundle TS_2 considered as a complex line bundle over S_2 .

According to part (c) of PROPOSITION 2, in a geometry of rays, the flow generated by a section $k : M \rightarrow \mathcal{K}$ preserves the bundle $\mathcal{L} \rightarrow M$, but not necessarily the complex (conformal) structure of the bundle $\mathcal{L}/\mathcal{K} \rightarrow M$. To see this, consider, as in a previous example, $M = \{(x_\mu) \in \mathbf{R}^4 : x_1 > 0\}$, $g = dx_1^2 + x_1^2 dx_2^2 + dx_3^2 - dx_4^2$, and choose now $k = \partial/\partial x_4 + \partial/\partial x_1$ so that the corresponding optical 1-form is $\kappa = dx_1 - dx_4$. The pair (k, g) defines in M a geometry of rays without twist, $\omega = 0$. The bundle \mathcal{L} is spanned by k , $x_1^{-1}\partial/\partial x_2$, and $\partial/\partial x_3$ so that the complex structure in \mathcal{L}/\mathcal{K} may be defined by

$$J\left(ax_1^{-1}\frac{\partial}{\partial x_2} + b\frac{\partial}{\partial x_3} \bmod k\right) = a\frac{\partial}{\partial x_3} - bx_1^{-1}\frac{\partial}{\partial x_2} \bmod k.$$

This complex structure is not preserved by the flow generated by k , as may be inferred from

$$L_k\left(ax_1^{-1}\frac{\partial}{\partial x_2} + b\frac{\partial}{\partial x_3}\right) = -ax_1^{-2}\frac{\partial}{\partial x_2},$$

where $a, b \in \mathbf{R}$. The flow generated by k is 'shearing': it changes the relative length of vectors in \mathcal{L}/\mathcal{K} .

5. Geometries of rays without shear

Definition. — A geometry of rays without shear is an optical geometry such that the flow generated by k consists of optomorphisms.

For simplicity, let us assume that the vector field k defining, together with $g \in \mathcal{E}$, the optical geometry in M is complete so that its flow is a one-parameter group of global transformations of M . It is not difficult, but slightly cumbersome, to reformulate subsequent paragraphs so as to allow a k which need not be complete.

PROPOSITION 3. — Let the pair (k, g) define an optical geometry in M . The following conditions are equivalent to each other :

- (a) the optical geometry is a geometry of rays without shear;
- (b) for any $t \in \mathbf{R}$,

$$\varphi_t^* \mathcal{E} = \mathcal{E},$$

where (φ_t) is the flow generated by k and \mathcal{E} is the set of all metrics on M defining the same optical geometry as g ;

(c) there is a function μ and a 1-form ν on M such that

$$(10) \quad L_k g = \mu g + \nu \otimes \kappa + \kappa \otimes \nu.$$

The equivalence of (a) and (b) follows from the definition of optomorphisms, whereas the equivalence of (b) and (c) follows from properties of Lie derivation. Eq. (10) implies

$$L_k \kappa = (\mu + k \lrcorner \nu) \kappa$$

so that $\kappa \wedge L_k \kappa = 0$ and the geometry of rays without shear is a particular case of a geometry of rays.

The importance of geometries of rays without shear results from the fundamental

THEOREM 1 (Ivor ROBINSON [3], cf. also [30] and [40]). — *An optical geometry of class C^ω admits a nowhere vanishing, optical solution of sourceless Maxwell's equations if and only if it is a geometry of rays without shear.*

Let F be such a solution on M , then

$$(11) \quad k \lrcorner F = 0, \quad k \lrcorner \sigma(g)F = 0$$

and

$$(12) \quad dF = 0, \quad d\sigma(g)F = 0$$

so that

$$(13) \quad L_k F = 0 \quad \text{and} \quad L_k \sigma(g)F = 0.$$

By integration, the last two equations imply, for any $t \in \mathbf{R}$,

$$(14) \quad \varphi_t^* F = F \quad \text{and} \quad \varphi_t^* \sigma(g)F = \sigma(g)F.$$

By naturality of the Hodge dual,

$$\varphi_t^* \sigma(g)F = \sigma(\varphi_t^* g) \varphi_t^* F.$$

Using both equations (14) one obtains

$$\sigma(g)F = \sigma(\varphi_t^* g)F.$$

Finally, PROPOSITION 1 implies $\varphi_t^*g \in \mathcal{E}$ and PROPOSITION 3 shows that the flow (φ_t) consists of optomorphisms. This completes the proof of the "only if" part of the theorem. The "if" part is proved by constructing suitable initial data on a hypersurface transversal to k and extending them to the manifold M by means of the flow [49].

It is often claimed that such a solution can be found in any smooth geometry of rays without shear. However, as pointed out by J. TAFEL [56], the constraint equations on the initial data, in the case of rays with twist, are of the Hans Lewy type [57]. For this reason, the geometry has been assumed here to be of class C^ω rather than C^∞ . The latter assumption is sufficient when the rays are without twist.

The structure of the set of all optical solutions of Maxwell's equations depends crucially on whether the rays are twisting or not. This may be seen by considering the optical geometry in \mathbf{R}^4 , with coordinates $x_1 = x$, $x_2 = y$, $x_3 = r$ and $x_4 = u$, induced by the pair (k, g) , where

$$k = \frac{\partial}{\partial r}, \quad g = dx^2 + dy^2 + 2\kappa dr$$

and

$$\kappa = du + \frac{1}{2}\omega(xdy - ydx), \quad \omega = 0 \text{ or } 1.$$

Since $L_k g = 0$, the geometry is that of rays without shear. It is convenient to introduce the complex variable

$$z = x + iy$$

and the complex electromagnetic field,

$$F + i * F$$

where the dual, for an optical F , depends only on the optical geometry. One has

$$*(\kappa \wedge d\kappa) = \omega\kappa, \quad *(\kappa \wedge dz) = -i\kappa \wedge dz.$$

The most general optical electromagnetic field corresponds to

$$F + i * F = C\kappa \wedge dz$$

where $C : \mathbf{R}^4 \rightarrow \mathbf{C}$. Maxwell's equations (2) are now equivalent to $\partial C / \partial r = 0$ and

$$\frac{\partial C}{\partial \bar{z}} = \frac{1}{4}i\omega z \frac{\partial C}{\partial u}.$$

If $\omega = 0$, then the complex function C depends arbitrarily on u and is analytic in z . For $\omega = 1$, the last equation reduces to the one put forward by Hans LEWY in his construction of a smooth linear differential equation without solution [57].

6. The integrable G_0 -structure

Recall that, if M is an m -dimensional smooth manifold and G is a Lie subgroup of $GL(m, \mathbf{R})$, then a G -structure on M is a restriction P of the bundle of linear frames LM to G . The G -structure P on M is said to be *integrable* if, for any point of M there is a system of local coordinates (x_1, \dots, x_m) around that point, such that the local section $(\partial/\partial x_1, \dots, \partial/\partial x_m)$ of LM is a section of the bundle $P \rightarrow M$ [39].

THEOREM 2. — *An optical geometry, considered as a G_0 -structure, is integrable if and only if it is a geometry of rays without shear and twist.*

Indeed, assume first that $P \subset LM$ is an integrable G_0 -structure on a 4-dimensional manifold M . Let (x, y, r, u) be a system of local coordinates on M such that $(\partial/\partial x, \partial/\partial y, \partial/\partial r, \partial/\partial u)$ is a local section of $P \rightarrow M$. The vector field $k = \partial/\partial r$ is then a local section of $\mathcal{K} \rightarrow M$ and the metric tensor

$$(15) \quad g = dx^2 + dy^2 + 2dr du$$

belongs to \mathcal{E} . Since $L_k g = 0$ and $\kappa = du$, the optical geometry defined by this G_0 -structure is a geometry of rays without shear and twist. Conversely, for any $g \in \mathcal{E}$ defining, together with k , a geometry of rays without shear and twist there is a system of local coordinates (x, y, r, u) such that

$$(16) \quad k = \frac{\partial}{\partial r}, \quad \kappa = du$$

and

$$(17) \quad g = P^{-2}[(dx - adu)^2 + (dy - bdu)^2] + 2du dr + cdu^2$$

where P , a , b , and c are arbitrary functions [4]. Clearly, the metric tensors (15) and (17) define, together with k , the same optical geometry.

Let the manifold $M_0 = \mathbf{R}^4$ be endowed with the integrable optical structure given, in terms of coordinates (x, y, r, u) by (15) and (16). Let Γ_0 be the pseudogroup of local optomorphisms of M_0 . A locally defined vector field v on M_0 generates a flow of local optomorphisms if and only if

$$v = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + R \frac{\partial}{\partial r} + U \frac{\partial}{\partial u}$$

where $X + iY$ is an arbitrary function of u , analytic in $z = x + iy$, U depends on u only, and R is an arbitrary function on M_0 . Therefore, the Lie algebra of Γ_0 is infinite-dimensional and Γ_0 cannot be obtained by localizing the elements of a Lie group of transformations of M_0 . This observation is

related to the fact that the Lie algebra of the optical group G_0 is of infinite type [39].

The set \mathcal{E}_0 of all metrics on M_0 , which define, together with (16), the same optical geometry as (15), contains large classes of inequivalent solutions of Einstein's equations. In some cases these metrics are defined only on open submanifolds of M_0 . The following solutions are in \mathcal{E}_0 : plane gravitational waves and their generalizations, the Schwarzschild metric, outgoing gravitational waves with spherical wave-fronts [4] and a solution which has been interpreted [50] as corresponding to a uniformly accelerating charged material point. All these and many other solutions are known in an explicit form [6, 51].

7. Application to Yang-Mills theory

The notion of an optical field can be easily extended to Yang-Mills configurations [7, 37]. Consider a Lie group of matrices, $G \subset GL(N, \mathbf{R})$, and its Lie algebra $\mathcal{G} \subset \text{End}(\mathbf{R}^N)$. Let M be a 4-dimensional, smooth, oriented manifold with a metric tensor g of Lorentz signature. A Yang-Mills potential A is a \mathcal{G} -valued 1-form on M . The corresponding Yang-Mills field strength

$$F = dA + \frac{1}{2}[A, A]$$

satisfies the Bianchi identity

$$(18) \quad dF + [A, F] = 0$$

where brackets denote the commutator in \mathcal{G} and the exterior product of differential forms. If $S : M \rightarrow G$, then

$$A' = S^{-1}AS + S^{-1}dS$$

is said to be obtained from A by a gauge transformation. The field strength F' corresponding to A' is

$$(19) \quad F' = S^{-1}FS.$$

The sourceless Yang-Mills equations are

$$(20) \quad d\sigma(g)F + [A, \sigma(g)F] = 0.$$

Let a vector field k define, together with g , an optical geometry on M . The Yang-Mills configuration defined by A is said to be optical if

$$(21) \quad k \lrcorner F = 0$$

and

$$(22) \quad k \lrcorner \sigma(g)F = 0.$$

Both these conditions are invariant under gauge transformations (19). The Bianchi identity (18), together with (21), implies

$$(23) \quad L_k F = [F, k \lrcorner A].$$

If (20) and (22) are used, then a similar equation is obtained for the dual $\sigma(g)F$,

$$(24) \quad L_k \sigma(g)F = [\sigma(g)F, k \lrcorner A].$$

LEMMA. — Let $\alpha : \mathbf{R} \rightarrow \mathcal{G}$ be a curve in \mathcal{G} and let $s : \mathbf{R} \rightarrow G$ be the solution of

$$\frac{ds(t)}{dt} = s(t)\alpha(t), \quad t \in \mathbf{R},$$

such that

$$s(0) = \text{id}.$$

The function (curve) $f : \mathbf{R} \rightarrow \mathcal{G}$

$$(25) \quad f(t) = s(t)^{-1}f(0)s(t)$$

is then the solution of

$$(26) \quad \frac{df(t)}{dt} = [f(t), \alpha(t)].$$

The proof of the LEMMA is straightforward.

PROPOSITION 4. — Let (φ_t) be the flow generated by a complete vector field k on M and let F be a Yang-Mills field strength such that (21) holds. Then there exists a map $\mathbf{R} \times M \ni (t, x) \mapsto S_t(x) \in G$ such that

$$(27) \quad \varphi_t^* F = S_t^{-1} F S_t.$$

Indeed, for any fixed $x \in M$, let

$$\alpha(t) = \varphi_t^*(k \lrcorner A)(x), \quad f(t) = \frac{d}{dt} \varphi_t^* F(x),$$

and put $S_t(x) = s(t)$, where s is determined as in LEMMA. Since

$$\frac{d}{dt}\varphi_t^*F = \varphi_t^*L_kF$$

equation (23) implies (26); finally, (27) at x follows from (25).

PROPOSITION 5. — *If F is the field strength of an optical and sourceless Yang-Mills configuration, then*

$$\sigma(g)F = \sigma(\varphi_t^*g)F$$

so that, if $F \neq 0$, the optical geometry is that of rays without shear.

In fact, in PROPOSITION 4 one can replace the field strength F by its dual $\sigma(g)F$ and obtain

$$\varphi_t^*\sigma(g)F = S_t^{-1}\sigma(g)FS_t.$$

An argument similar to the one used in proving THEOREM 1 leads now to the conclusion of PROPOSITION 5. An analysis of the initial data on a hypersurface transversal to k can be used to prove the local existence of an optical Yang-Mills configuration in any real analytic geometry of rays without shear [7].

THEOREM 3. — *Consider two optical geometries M and M' corresponding to the pairs (k, g) and (k', g') , respectively. Let A' be the potential of an optical, sourceless Yang-Mills configuration on M' . If $h : M \rightarrow M'$ is an isomorphism of optical geometries, then h^*A' is the potential of an optical, sourceless Yang-Mills configuration on M .*

Indeed, the field strength F corresponding to $A = h^*A'$ is $F = h^*F'$. Let κ and κ' be, respectively, 1-forms on M and M' , defined as in (α) of § 4. From the definition of an isomorphism of optical geometries it follows, firstly, that h^*k' and $h^*\kappa'$ are proportional to k and κ , respectively. Therefore, F is an optical 2-form. Secondly, by PROPOSITION 1, one has $\sigma(g)F = \sigma(h^*g')F$ so that the Yang-Mills equation (20) is satisfied on M .

Remark. — The theorem is true, in particular, for electromagnetic fields and must have been known, for this case, to H. BATEMAN [26].

In particular, consider the Lorentzian manifold M , with metric tensor (17), and Minkowski space M' referred to coordinates (x', y', r', u') and with metric

$$g' = dx'^2 + dy'^2 + 2dr'du'.$$

The vector fields $k = \partial/\partial r$ and $k' = \partial/\partial r'$ define on M and M' , respectively, optical geometries which are isomorphic with respect to the (local) diffeomorphism $h : M \rightarrow M'$ mapping into each other points with the same values

of the coordinate functions. Therefore, any optical Maxwell or Yang-Mills field can be transferred from Minkowski space to M . In particular, one can take for M Minkowski space referred to a system of curvilinear coordinates built around a time-like curve [52,53]. In this case, h transforms a plane wave in M into a spherical wave emanating from the particle whose worldline coincides with the curve. Such an h provides the example of an isomorphism of optical geometries which is not a conformal transformation.

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