

OPTICAL STRUCTURES IN RELATIVISTIC THEORIES

BY

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ABSTRACT. — In 1922 Élie CARTAN discovered the existence, in any Lorentzian and not conformally flat 4-manifold, of four privileged directions which are optical in the sense that they belong to the light cone defined by the fundamental form. In special cases — as for the Schwarzschild spacetime — some of these directions may coincide. These observations were later rediscovered and used by physicists in the study of purely radiative Maxwell fields and of ‘algebraically special’ Einstein metrics. The present article reviews some of these developments and outlines the underlying ‘optical geometry.’ This geometry is defined as a G_0 -structure on a 4-manifold, where G_0 , the ‘optical group’, is a suitable 9-dimensional Lie subgroup of $GL(4, \mathbf{R})$. It is shown that the G_0 -structure is integrable if and only if the optical geometry is that of rays without shear and twist. By an extension of the Robinson theorem, optical (purely radiative) solutions to sourceless Maxwell and Yang-Mills equations exist in geometries of rays without shear. An isomorphism of optical geometries is shown to transform one such solution into another. For example, one such isomorphism — which is not a conformal map — transforms plane waves into spherical waves of a special kind.

1. Introduction

Relativistic theories of spacetime and of interactions between particles and fields are based on geometrical models which include, as an essential element, a metric tensor of Lorentz (normal hyperbolic) signature. As a result of the indefinite character of the metric, relativistic models admit directions — and other geometric elements — which are isotropic (null, light-like) in the sense of being associated with non-zero vectors of vanishing square. Such isotropic elements are of considerable interest from the point of view of both geometry and physics. Élie CARTAN has shown that totally isotropic maximal planes can be used to define spinors over Euclidean vector spaces of any number of dimensions [1].

For a physicist, a light-like vector in Minkowski space may be identified with the energy-momentum vector of a particle of zero rest-mass (photon, neutrino).

Null hypersurfaces in a Lorentz 4-manifold M with metric tensor g are represented by solutions $u : M \rightarrow \mathbf{R}$ of the eikonal equation

$$(1) \quad g^{\mu\nu} \frac{\partial u}{\partial x^\mu} \frac{\partial u}{\partial x^\nu} = 0$$

where (x^μ) , $\mu = 1, 2, 3, 4$, are local coordinates on M . Let M be oriented; Maxwell's equations for the 2-form F of electromagnetism are

$$(2) \quad dF = 0 \quad \text{and} \quad d * F = 0,$$

where $*F$ is the Hodge dual of F defined in terms of g and the orientation. Assume now F to be of the form

$$(3) \quad F = \text{Re}(F_0 \exp iu/\lambda)$$

where F_0 is a (complex) 2-form representing the amplitude of an electromagnetic wave with phase $u : M \rightarrow \mathbf{R}$. In the limit of wave optics ($\lambda \rightarrow 0$), Maxwell's equations (2) imply

$$(4) \quad du \wedge F_0 = 0 \quad \text{and} \quad du \wedge *F_0 = 0$$

so that (1) is a necessary and sufficient condition for the existence of a nowhere vanishing F_0 subject to (4). The subtler question of whether there exists a non-vanishing F_0 , solution to (2) and (4) is considered in § 5 [3].

The study of isotropic elements and of the associated classical fields — such as gravitation, electromagnetism and Yang-Mills — resulted in much progress in the area of finding exact solutions and establishing their properties. The study has been particularly fruitful in the theory of general relativity where it led to the notion of algebraically special spacetimes. Large classes of exact, explicit solutions of Einstein's equations have been found in this manner. Among them are plane-fronted and sphere-fronted waves [4], and the Kerr solution representing the exterior of a rotating black hole [5]. A good summary of this research, with many references, is given in [6]. Almost all explicitly known exact solutions of Einstein's equations in four dimensions and with Lorentz signature belong to one of three classes :

1. algebraically special metrics,
2. stationary metrics with axial symmetry,
3. metrics with cylindrical symmetry.

The consideration of isotropic elements in relativistic theories has also led to a number of general results, such as

