

Deformations of the Hodge map and optical geometry

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Abstract. A simple formula is derived for the infinitesimal change of the Hodge dual of a k -form, induced by a deformation of the scalar product in the underlying vector space. By considering deformations due to a flow generated by a vector field on a differential manifold, one obtains an expression for the commutator of the Hodge dual with the Lie derivation with respect to the vector field, acting on differential forms. This formula is useful in proving theorems on optical solutions of Maxwell's and Yang-Mills equations. The optical geometry underlying such solutions is defined as a restriction of the bundle of linear frames of a 4-dimensional manifold to a 9-dimensional optical group. This geometry provides a natural framework for the study of shearfree, optical and geodesic congruences and of the associated fields.

1. INTRODUCTION

In theoretical physics one often considers mathematical models of the following type. There is given an n -dimensional smooth manifold M , a Lie group G , a principal G -bundle $P \rightarrow M$ and a representation of G in a finite-dimensional, real or complex vector space V . Physical histories (classical fields, wave-functions, etc.) are described by V -valued k -forms on P , equivariant under the action of G . For example, a connection on P is described by a \mathfrak{g} -valued 1-form ω which corresponds

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to the adjoint representation of G in its Lie algebra \mathfrak{g} and, moreover, is a left inverse for the map $\mathfrak{g} \rightarrow TP$ defined by the action of G in P . Of special interest are horizontal k -forms; the curvature 2-form

$$\Omega = d\omega + \frac{1}{2} [\omega, \omega]$$

is horizontal and, if ϕ is a horizontal k -form, then its covariant exterior derivative $D\phi$ is a horizontal $(k+1)$ -form. If M is oriented and has a (Riemannian or Lorentzian) metric tensor g , then the Hodge dual map $*$ can be applied to horizontal forms on P : if ϕ is a horizontal V -valued k -form on P , then $*\phi$ is a similar $(n-k)$ -form. Many fundamental equations of physics have the following structure

$$(1) \quad D * \phi = * j,$$

where ϕ and j are horizontal, V -valued k - and $(k-1)$ -forms, respectively. For example, if $G = U(1)$, ϕ is the curvature 2-form and j is the \mathbb{R} -valued 1-form of electric current, then (1) is simply the Maxwell equation. For a non-Abelian group G and $\phi = \Omega$ equation (1) coincides with that introduced by Yang and Mills. If P is the bundle of linear frames of M endowed with a linear connection and $\theta = (\theta^\mu)$ denotes the soldering form, then the choice $\phi = (\theta^\mu \wedge \theta^\nu)$, $\mu, \nu = 1, \dots, n$, leads to the Cartan equation of a relativistic theory of gravitation with spin j and torsion $D\theta$ [1 - 3].

In view of the occurrence of the Hodge map in the fundamental equation (1) it is interesting to consider the dependence of $*$ on g and, in particular, its behaviour under deformations of the metric. This results in a formula for the commutator of $*$ with Lie derivation relative to a vector field. The formula has already been used to prove a theorem on the existence of optical (isotropic) Yang-Mills configurations associated with shear-free congruences of null geodesics [4]. The paper is concluded with a section on the «optical geometry» underlying the local structure of such congruences [5 - 7].

2. THE HODGE MAP AND ITS DEFORMATIONS

Let V be an n -dimensional real vector space with a preferred orientation. The group $GL(n, \mathbb{R})$ acts transitively in the manifold $F(V)$ of all linear frames in V ; similarly, $GL^+(n, \mathbb{R})$ acts transitively in the open submanifold $F^+(V) \subset F(V)$ of frames with the preferred orientation. A scalar product in V is defined as a symmetric bilinear map $g : V \times V \rightarrow \mathbb{R}$ which is nonsingular: if $g(u, v) = 0$ for all $u \in V$, then $v = 0$. Let $S(V) \subset V^* \otimes V^*$ be the set (in fact, manifold) of all scalar products in V . If $e = (e_\mu)$, $\mu = 1, \dots, n$, is a frame and $g \in S(V)$ then

the formula

$$g_{\mu\nu}(e) = g(e_\mu, e_\nu)$$

defines the functions $g_{\mu\nu} : S(V) \rightarrow \mathbb{R}$ and

$$g_{\mu\nu}(ea) = g_{\rho\sigma}(e) a_\mu^\rho a_\nu^\sigma,$$

where $a = (a_\sigma^\rho) \in GL(n, \mathbb{R})$. If $\gamma(e) = \det(g_{\mu\nu}(e))$, then

$$\gamma(ea) = \gamma(e) (\det a)^2$$

and the sign of $\gamma(e)$ is an invariant.

The *Grassmann algebra* of forms over V is denoted by

$$\Lambda V^* = \bigoplus_{k=0}^n \Lambda^k V^*,$$

where $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ is the dual of V . For any frame $(e_\mu) \in F(V)$ its dual $(e^\mu) \in F(V^*)$ is determined by

$$\langle e_\mu, e^\nu \rangle = \delta_\mu^\nu,$$

where angular brackets denote the evaluation map $V \times V^* \rightarrow \mathbb{R}$.

Let $(e_\mu) \in F^+(V)$ and (e^μ) be its dual. The volume form

$$\text{vol}(g) = |\gamma(e)|^{1/2} e^1 \wedge e^2 \wedge \dots \wedge e^n$$

depends on $g \in S(V)$, but not on the frame, provided it is of preferred orientation.

A convenient abuse of notation consists in using the same letter

$$(2) \quad g : V \rightarrow V^*$$

for the linear map defined by

$$\langle u, g(v) \rangle = g(u, v)$$

as for the scalar product $g \in S(V)$ itself.

For any k -form α its *Hodge dual* $\sigma(g)\alpha$ is defined as the $(n-k)$ -form given by its value on the vectors $u_{k+1}, \dots, u_n \in V$ as follows:

$$(3) \quad \text{vol}(g) \cdot \sigma(g)\alpha(u_{k+1}, \dots, u_n) = \alpha \wedge g(u_{k+1}) \wedge \dots \wedge g(u_n).$$

When g is fixed once for all, then one usually writes $*\alpha$ instead of $\sigma(g)\alpha$. The latter, more elaborate notation is used here in order to study the dependence of the Hodge map on g . Clearly, $\sigma(g)$ can be extended to a linear map

$$\sigma(g) : \Lambda V^* \rightarrow \Lambda V^*$$

