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THEORETICAL PHYSICS

Discontinuities of Field Derivatives and Radiation in Covariant Theories

by

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The purpose of this paper is to investigate possible forms of discontinuities of field derivatives, and their connexion with the problem of radiation and geometrical optics. In sec. 1, we briefly recall the Cauchy problem and the notion of characteristic surfaces for systems of partial differential equations. Section 2 is devoted to the general theory of discontinuities of the second derivatives of field potentials. This theory is then applied to the electromagnetic (sec. 3) and gravitational (sec. 4) fields. A close relation between the jump conditions in a linear theory and the "geometrical optics" is demonstrated in sec. 5. Covariant notation is used throughout, being essentially the same as in [1]; $\psi_A(x^\nu)$ denotes a tensor field (potentials); Greek indices run from 0 to 3, block Roman – from 1 to N; $\varphi_{[\alpha\beta]} \equiv \varphi_{\alpha\beta} - \varphi_{\beta\alpha}$; repetition of indices implies summation; the ordinary derivative is denoted by a comma ($\partial\psi_A/\partial x^\nu = \psi_{A,\nu}$).

The covariant field equations are supposed to be derivable from a variational principle: $\delta\int d_4x\sqrt{-g}\,L(g_{\mu\nu}, g_{\mu\nu,\varrho}, \psi_A, \psi_{A,\nu}) = 0$, where $g_{\mu\nu}$ is the metric tensor of the Riemannian space-time and $g = \det(g_{\mu\nu})$. Then the field equations are

$$\sqrt{-g}\,L^A = \partial\sqrt{-g}\,L/\partial\psi_A - (\partial\sqrt{-g}\,L/\partial\psi_{A,\nu})_{,\nu} = 0. \tag{1}$$

1. Since the second derivatives of the ψ's enter L^A linearly, we can write (1) in the form:

$$L^A = L^{AB\varrho\sigma}\psi_{B,\varrho\sigma} + \ldots = 0, \tag{2}$$

where $L^{AB\varrho\sigma} = -\frac{1}{2}(\partial^2 L/\partial\psi_{A,\varrho}\partial\psi_{B,\sigma} + \partial^2 L/\partial\psi_{A,\sigma}\partial\psi_{B,\varrho})$ and the dots stand for terms not containing $\psi_{A,\mu\nu}$. Let $\varphi(x^\nu) = 0$ be the equation of a hypersurface S, on which the Cauchy data are to be prescribed. This means that we give on S (in a consistent manner, [8]) the values of ψ_A and $\psi_{A,\nu}$. Introducing a new co-ordinate system $x^{\nu'}=x^{\nu'}(x^\alpha)$, such that $x^{0'}=\varphi(x^\alpha)$, we obtain from (2):

$$L^A = L^{AB\varrho\sigma}\varphi_{,\varrho}\varphi_{,\sigma}\psi_{B,0'0'} + \text{terms not involving } \psi_{B,0,0'}.$$

The possibility of evaluating the derivatives $\psi_{A,0'0'}$ from (3) depends on the rank R of the N-th order matrix $\Lambda^{AB}=L^{AB\varrho\sigma}\varphi_{,\varrho}\varphi_{,\sigma}$. Denoting by M the maximum of this rank in a fixed point of space, $M=\max_{\varphi,\alpha} R(\Lambda^{AB})$, we have always $M \leqslant N$.

If $M=N$ and if S is such that $R(\Lambda^{AB})=M$ on S, then the Cauchy problem formulated on S has a unique solution in the neighbourhood of S. Example: the scalar wave-equation $(\sqrt{-g}\, g^{\mu\nu}\psi_{,\nu})_{,\mu}/\sqrt{-g}+\varkappa^2\psi=0$ and S such that $g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\neq 0$.

If $M<N$, the field equations (1) are underdetermined, and for no S have a unique solution. Additional conditions must be added in order to ensure uniqueness. Examples: Maxwell's and Einstein's equations.

A hypersurface S on which $R(\Lambda^{AB})<M\leqslant N$ is called *characteristic*. The initial value problem cannot be correctly formulated on a characteristic hypersurface.

2. The existence of solutions with discontinuous derivatives is known to be closely related to the hyperbolic character of field equations. A discontinuous solution represents a disturbance of the field propagating with a finite velocity and can be interpreted as radiation.

Our field equations (1) being of the second order, we assume that the ψ's and their first derivatives are continuous throughout and that $\psi_{A,\mu\nu}$ are continuous everywhere except on hypersurface $S(\varphi=0)$. Denoting by $\Delta\psi_{A,\mu\nu}$ the jump of $\psi_{A,\mu\nu}$ across S, we have, from purely geometrical considerations [2],

$$\Delta\psi_{A,\mu\nu}=\eta_A\varphi_{,\mu}\varphi_{,\nu}; \tag{3}$$

where η_A is a tensor field defined on S.

The field equations being valid everywhere, we obtain from (2):

$$L^{AB\varrho\sigma}\Delta\psi_{B,\varrho\sigma}=0 \quad \text{on } S,$$

or

$$\Lambda^{AB}\eta_B=0 \quad (A=1,\ldots,N). \tag{4}$$

The jumps of $\psi_{A,\mu\nu}$ can appear only on such S for which

$$\det(\Lambda^{AB})=0. \tag{5}$$

Thus $\psi_{A,\mu\nu}$ can be discontinuous 1° on any surface when (1) are underdetermined; 2° on a characteristic surface. Examples given below show that only the latter jumps have a physical meaning.

3. In the *electromagnetic* theory the ψ's must be replaced by a vector-potential A_α. The field equations are $L^\alpha=(\sqrt{-g}\,f^{\alpha\beta})_{,\beta}/\sqrt{-g}=0$, where $f_{\alpha\beta}\equiv A_{[\beta,\alpha]}$. $L^{AB\varrho\sigma}$ becomes $L^{\alpha\beta\varrho\sigma}=\frac{1}{2}(g^{\alpha[\varrho}g^{\beta]\sigma}+g^{\alpha[\sigma}g^{\beta]\varrho})$ and the matrix Λ^{AB}:

$$\Lambda^{\alpha\beta}=\varphi^{,\alpha}\varphi^{,\beta}-g^{\alpha\beta}\varphi_{,\nu}\varphi^{,\nu}. \tag{6}$$

We get for the rank of (6):

$$R(\Lambda^{\alpha\beta})=\begin{cases}3=M<N=4 & \text{for} \quad \varphi_{,\nu}\varphi^{,\nu}\neq 0, \quad \text{(a)}\\ 1 & \text{for} \quad \varphi_{,\nu}\varphi^{,\nu}=0. \quad \text{(b)}\end{cases}$$

The Maxwell equations are underdetermined, and their characteristic surfaces coincide with null surfaces [4]. Writing b_α for η_A we have $\Delta A_{\alpha,\mu\nu}=b_\alpha\varphi_{,\mu}\varphi_{,\nu}$.

In case (a), equation (4) leads to

$$(\varphi^{,\alpha}\varphi^{,\beta}-g^{\alpha\beta}\varphi_{,\nu}\varphi^{,\nu})b_\beta=0. \tag{7}$$

The general solution of (7) is $b_\alpha=b\varphi_{,\alpha}$ (b = scalar function); then $\Delta A_{\alpha,\mu\nu}==b\varphi_{,\alpha}\varphi_{,\mu}\varphi_{,\nu}$. But such jumps have no physical significance:

$$\Delta f_{\alpha\beta,\gamma}=\Delta(A_{\beta,\alpha\gamma}-A_{\alpha,\beta\gamma})=0. \tag{8}$$

In case (b), we have $\varphi_{,\nu}\varphi^{,\nu}=0$, and (4) leads to

$$b^\alpha\varphi_{,\alpha}=0. \tag{9}$$

The corresponding jumps in field strength derivatives are

$$\Delta f_{\alpha\beta,\gamma}=b_{[\beta}\varphi_{,\alpha]}\varphi_{,\gamma}. \tag{10}$$

Introducing electric $\vec{E}$ and magnetic $\vec{H}$ field strengths, we can write (9), (10) in vector notation:

$$\Delta\vec{E}_{,\gamma}=\varphi_{,\gamma}\vec{e}, \qquad \Delta\vec{H}_{,\gamma}=\varphi_{,\gamma}\vec{n}\times\vec{e}, \qquad \vec{n}\cdot\vec{e}=0,$$

where $\varphi_{,0}\vec{n}=\operatorname{grad}\varphi$, $e_k=\varphi_{,0}(b_k-n_kb_0)$. The jumps $\Delta\vec{E}_{,\gamma}$ and $\Delta\vec{H}_{,\gamma}$ can be called "weak" in contradistinction to "strong" discontinuities of $\vec{E}$ and $\vec{H}$ (i. e., of the *first* derivatives of the potential). Strong discontinuities of the electromagnetic field were examined by Rubinowicz [3].

4. In Einstein's general relativity, the potentials ψ_A of the *gravitational* field are identified with the metric tensor $g_{\alpha\beta}$. Field equations *in vacuo* are $R_{\alpha\beta}=0$, where $R_{\alpha\beta}=g^{\mu\nu}R_{\alpha\mu\beta\nu}$, and $R_{\alpha\beta\gamma\delta}$ denotes the Riemann curvature tensor of the space-time.

The Cauchy problem and jump conditions for the gravitational field have been investigated by several authors [4]–[7]; we give a detailed and co-variant expression for the jumps of the curvature tensor.

According to Lichnerowicz [4], we assume that there exists a co-ordinate system in which $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma}$ are continuous and $g_{\alpha\beta,\gamma\delta}$ are piecewise continuous. Co-ordinate transformations $x^{\nu} \to x^{\nu'}$ are assumed to be of class C^3. Denoting the jump of $g_{\alpha\beta,\gamma\delta}$ across S by $\Delta g_{\alpha\beta,\gamma\delta}$, we note that this is a *tensor* field on S. Indeed, in virtue of $\Delta g_{\alpha\beta}=0$, $\Delta g_{\alpha\beta,\gamma}=0$, $\Delta \partial x^{\nu}/\partial x^{\nu'}=0$, etc., we have ($A^{\nu}_{\gamma'} = \partial x^{\nu}/\partial x^{\gamma'}$):

$$\Delta g_{\alpha'\beta',\gamma'\delta'} = \Delta[((g_{\alpha\beta}A^{\alpha}_{\alpha'}A^{\beta}_{\beta'})_{,\gamma}A^{\gamma}_{\gamma'})_{,\delta}A^{\delta}_{\delta'}] = A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}A^{\gamma}_{\gamma'}A^{\delta}_{\delta'}\Delta g_{\alpha\beta,\gamma\delta}.$$

It follows from (3) that

$$\Delta g_{\alpha\beta,\gamma\delta} = h_{\alpha\beta}\varphi_{,\gamma}\varphi_{,\delta} \qquad (h_{\alpha\beta}=h_{\beta\alpha} \text{ is a tensor}). \tag{11}$$

Latin indices $A, B, \ldots$ must be identified with the 10 (symmetric) pairs $(\alpha\beta), (\gamma\delta), \ldots$ The symbol $L^{AB\varrho\sigma}$ becomes

$$L^{(\alpha\beta)(\gamma\delta)\varrho\sigma} = \tfrac{1}{8}(g^{\alpha[\gamma}g^{\sigma][\delta}g^{\varrho]\beta} + g^{\alpha[\delta}g^{\sigma][\gamma}g^{\varrho]\beta} + g^{\alpha[\gamma}g^{\varrho][\delta}g^{\sigma]\beta} + g^{\alpha[\delta}g^{\varrho][\gamma}g^{\sigma]\beta}).$$

The 0th order matrix $\Lambda^{AB} = \Lambda^{(\alpha\beta)(\gamma\delta)} = L^{(\alpha\beta)(\gamma\delta)\varrho\sigma}\varphi_{,\varrho}\varphi_{,\sigma}$ has the rank [5]:

$$R(\Lambda^{(\alpha\beta)(\gamma\delta)}) = \begin{cases} 6 = M < N = 10 & \text{for } \varphi_{,\nu}\varphi^{,\nu} \neq 0, \quad \text{(a)} \\ 4 & \text{for } \varphi_{,\nu}\varphi^{,\nu} = 0. \quad \text{(b)} \end{cases}$$

The Einstein equations are underdetermined (general covariance); to them must be added 4 conditions fixing the co-ordinate system.

In case (a), equation (4) implies

$$(\delta^{\mu}_{\alpha}\varphi_{,\beta}\varphi^{,\nu} + \delta^{\mu}_{\beta}\varphi_{,\alpha}\varphi^{,\nu} - \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\varphi_{,\varrho}\varphi^{,\varrho} - g^{\mu\nu}\varphi_{,\alpha}\varphi_{,\beta})h_{\mu\nu} = 0. \tag{12}$$

The general solution of (12) can be easily found:

$$h_{\alpha\beta} = h_{\alpha}\varphi_{,\beta} + h_{\beta}\varphi_{,\alpha} \qquad (h_{\alpha} - \text{arbitrary vector field}),$$

but the corresponding jumps of the curvature tensor vanish:

$$\Delta R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(\Delta g_{\alpha\delta,\beta\gamma} + \Delta g_{\beta\gamma,\alpha\delta} - \Delta g_{\alpha\gamma,\beta\delta} - \Delta g_{\beta\delta,\alpha\gamma}) = 0.$$

Equation (4) leads, in case (b), to

$$h_{\alpha}{}^{\nu}\varphi_{,\nu}\varphi_{,\beta} + h_{\beta}{}^{\nu}\varphi_{,\nu}\varphi_{,\alpha} - h_{\nu}{}^{\nu}\varphi_{,\alpha}\varphi_{,\beta} = 0. \tag{13}$$

Among the 10 equations (13), only 4 are independent. Thus, in an arbitrary co-ordinate system, $h_{\alpha\beta}$ has 6 components algebraically independent. The corresponding discontinuities of the Riemann tensor are:

$$\Delta R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\varphi_{,[\alpha}h_{\beta][\gamma}\varphi_{,\delta]}. \tag{14}$$

Let us summarise: the curvature tensor $R_{\alpha\beta\gamma\delta}$ of empty space-time can be discontinuous only on null surfaces ($\varphi_{,\nu}\varphi^{,\nu}=0$). The jump of $R_{\alpha\beta\gamma\delta}$ is defined at each point by six numbers and has the form (14). The analogy with electrodynamics allows us to consider pure gravitational fields with discontinuous Riemann tensor as representing gravitational radiation [7].

5. Let us restrict ourselves to linear field theories. Solutions of the form $\psi_A = \eta_A e^{i\omega\varphi(x)}$, where ω is a great number, represent, in the electromagnetic case, light (optical) waves. Extending the notion of *geometrical optics* to any linear field theory, we can obtain an "eikonal" equation as follows. Introducting $\psi_A = \eta_A e^{i\omega\varphi}$ in (2), and disregarding terms not containing ω^2 we obtain:

$$-\omega^2 L^{AB\varrho\sigma}\eta_B\varphi_{,\varrho}\varphi_{,\sigma} = 0,$$

i. e., an equation identical in form with (4). Thus (5) is again a necessary and sufficient condition for the existence of a non-vanishing amplitude η_A. The condition

$$R(\Lambda^{AB}) < M$$

can be called, by analogy with electrodynamics, an *eikonal equation.* These few remarks again reveal the connexion between propagation of discontinuities and wave phenomena *).

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*) *Note added in proof.* From the condition $R(\Lambda^{AB}) < M$ defining the characteristic surfaces, we can also obtain the velocity (or velocities) v of propagation of the disturbances. For Maxwell's theory this is trivial: $g^{\mu\varrho}\varphi_{,\mu}\varphi_{,\nu}=0$ implies $v=c$. But if we take, e. g., a non-linear electrodynamics, then we get, apart $v=c$, several different velocities, depending on the field strengths.