MONOGRAPHS AND TEXTBOOKS
IN PHYSICAL SCIENCE

ANDRZEJ TRAUTMAN

DIFFERENTIAL GEOMETRY FOR PHYSICISTS

Stony Brook Lectures

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I

INTRODUCTION

During the academic year 1976-77, at the invitation of Professor Chen Ning Yang, I gave a series of lectures on differential geometry at the Institute for Theoretical Physics of the State University of New York at Stony Brook, N.Y. I also prepared a set of informal notes which were distributed to the audience. The text reproduced below is based on the notes. It is very informal and sketchy.

I thank Professors B. Veit, G. Marmo and R. de Ritis for a critical reading of the manuscript.

Notation. Standard, set-theoretical notation is used in these notes. For example, the notation for *maps* is:

$$f : X \to Y \quad \text{or} \quad X \xrightarrow{f} Y,$$

$$X \ni x \mapsto f(x) = \langle x, f \rangle \in Y.$$

f is *injective* if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, *surjective* if $f(X) = Y$, *bijective* if it is injective and surjective.

Composition of maps $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f : X \to Z$. The following sets of numbers often occur:

$\mathbb{N} = \{0, 1, 2, \dots\}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rationals
$\mathbb{R}$	reals
$\mathbb{C}$	complex numbers
$\mathbb{H}$	quaternions

An *equivalence relation* R in a set X is a subset $R \subset X \times X$ such that $(x, x) \in R$ for any $x \in X$, if $(x, y) \in R$ then $(y, x) \in R$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

The quotient (factor-set) of X by R is defined as:

$$X/R = \{A \subset X \mid \text{if } x, y \in A \text{ then } (x, y) \in R; \text{ if } x \in A \text{ and } (x, y) \in R \text{ then } y \in A\}.$$

If $x \in A$ then $A = [x]$ is the class of elements equivalent to x. The map $X \ni x \mapsto [x] \in X/R$ is a canonical surjection. The symbol $\exists$ means: 'there exists' and $\forall$: 'for any'. The symbol $\Leftrightarrow$ or 'iff' stands for 'if and only if'.

A *group* G *acts in* X *on the right* if there is a map:

$$X \times G \ni (x, a) \mapsto \delta_a(x) = xa \in X$$

such that $\delta_1 = \mathrm{id}_X$ and $\delta_a \circ \delta_b = \delta_{ba}$ where $a, b \in G$ and 1 is the unit element of G. The action of G in X defines an equivalence relation R by $(x, y) \in R \Leftrightarrow \exists a \in G$ such that $y = \delta_a(x)$. The equivalence classes in X are called *orbits* of G in X and the quotient is denoted by X/G. The action of G in X is said to be *transitive* if $X/G = \{X\}$, i.e. if $G \ni a \mapsto \delta_a(x) \in X$ is surjective; it is said to be *free* if this map is injective for any $x \in X$.

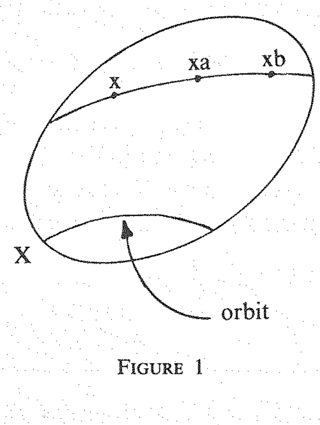


FIGURE 1

II

ALGEBRA

Vector spaces

Given a real vector space V, one can add its elements and multiply them by real numbers. A *frame* is defined as a maximal set of linearly independent vectors; a vector space is finite-dimensional if it has a finite frame; one shows that, in this case, all frames have the same number of elements, say n, which is called the *dimension* of V.

Let

$$e = (e_i) = (e_1, \dots, e_n), \quad \text{where } e_i \in V$$

be a frame in V. If $a = (a^i_j) \in GL(n, \mathbb{R})$, then:

$$e' = ea \quad \text{where } e'_i = e_j \, a^j_i$$

is another frame and the map $(e, a) \mapsto ea$ defines an action of $GL(n, \mathbb{R})$ in the *set* F(V) *of all frames* which is free and transitive.

If $u \in V$ then $u = u^i e_i$ and the numbers $u^1, \dots, u^n$ are the *components* of u with respect to e. The components of u with respect to e′ are $u'^i = a^{-1i}{}_j u^j$.

If U, V are vector spaces, then so is their *direct sum* $U \oplus V$ i.e. the Cartesian product $U \times V$ with addition and multiplication defined by:

$$(u, v) + (u', v') = (u + u', v + v'), \; a(u, v) = (au, av); \; a \in \mathbb{R}.$$

If $U_1, \dots, U_k$, V are vector spaces then so is the set of all multilinear maps:

$$\mathcal{L}(U_1, \dots, U_k; V) = \{f : U_1 \times \dots \times U_k \to V \mid f \text{ multilinear}\}$$

If $U_1 = \dots = U_k = U$ then one writes $\mathcal{L}^k(U; V)$ instead of $\mathcal{L}(U_1, \dots, U_k; V)$.

In particular, $V^* = \mathcal{L}(V; \mathbb{R})$ is the *dual* space of V or the space of (one-) *forms* over V. If $e = (e_i)$ is a frame in V then the *dual frame* $e^* = (e^i)$ in V^* is defined by:

$$e^i(e_j) = \delta^i{}_j.$$

If $u \in V$ then:

$$u^i = e^i(u), \quad i = 1, \dots, n.$$

The vector spaces under consideration being finite-dimensional, $\dim V^* = \dim V$, and the double dual V^{**} *is isomorphic in a natural* (frame-independent) *manner to* V. The isomorphism $\kappa : V \to V^{**}$ is given by:

$$\langle \alpha, \kappa(u) \rangle = \langle u, \alpha \rangle \quad \text{where } \alpha \in V^*.$$

The vector space:

$$V_1^* \otimes \dots \otimes V_k^* = \mathcal{L}(V_1, \dots, V_k; \mathbb{R})$$

is called the *tensor product* of $V_1^*, \dots, V_k^*$. The elements of:

$$(1) \qquad \overset{k}{\otimes} V \otimes \overset{\ell}{\otimes} V^* = \underbrace{V \otimes \dots \otimes V}_{k \text{ factors}} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{\ell \text{ factors}}$$

are called *tensors* of type (k, ℓ) (sometimes: tensors with k contravariant and ℓ covariant indices).

Consider

$$U^* \otimes V^* = \mathcal{L}(U, V; \mathbb{R})$$

and define a bilinear map:

$$k : U^* \times V^* \to U^* \otimes V^*$$

by

$$k(\alpha, \beta) = \alpha \otimes \beta$$

where

$$(\alpha \otimes \beta)(u, v) = \alpha(u)\,\beta(v) \quad \text{for} \quad u \in U,\ v \in V.$$

If (e^i) is a frame in U^* and (f^α) is a frame in V^* then $(e^i \otimes f^\alpha)$ is a frame in $U^* \otimes V^*$; this shows $\dim(U \otimes V) = \dim U \dim V$.

Clearly $(e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e^{j_1} \otimes \ldots \otimes e^{j_\ell})$ is a frame in (1) and a tensor of type (k, ℓ) can be written as:

$$A = A^{i_1 \ldots i_k}_{j_1 \ldots j_\ell}\, e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e^{j_1} \otimes \ldots \otimes e^{j_\ell}.$$

Affine spaces

Let V be a vector space. The set E is an affine space with V as the space of translations if V acts freely and transitively in E. The action of V in E is denoted additively; thus, if $v \in V$ then $v + p \in E$ denotes the point of E obtained from $p \in E$ by *translating* it along v. In other words, V acts as the *group of translations* on E and

$$v + p = p \Leftrightarrow v = 0, \qquad u + (v + p) = (u + v) + p,$$

any $u, v \in V$, $p \in E$.

If E_1, E_2 are affine spaces and V_1, V_2 are their groups of translations then:

$$f : E_1 \to E_2$$

is called an *affine map* if there is a linear map

$$s(f) : V_1 \to V_2$$

such that

$$f(v + p) = s(f)(v) + f(p)$$

for any $v \in V_1$, and $p \in E_1$. A bijective affine map $f\colon E \to E$ is called an *affine transformation* (affine automorphism of E). The set of all affine transformations of E is a group, called the *affine group* GA(E) of E. The map $s : GA(E) \to GL(V)$ is a morphism of groups. A translation may be considered as an affine transformation: if $v \in V$ then $t(v) \in GA(E)$ is given by $t(v)(p) = v + p$. Since $s(t(v)) = id_V$, we have an *exact sequence* of groups:

$$V \xrightarrow{t} GA(E) \xrightarrow{s} GL(V).$$

(Exact means $\operatorname{Im} t = \operatorname{Ker} s$). In particular, one can take $E = V$. In this case, GA(V) may be identified with the (semi-direct) product of groups $GL(V) \times V$: if $a \in GL(v)$ and $b \in V$ then there is the affine transformation $(a, b) \in GA(V)$ given by:

$$(a, b)\, v = a(v) + b.$$

The group GA(V) may be also embedded in $GL(V \times \mathbb{R})$ by

$$(a, b) \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

For any $n \in \mathbb{N}$, there is the *affine number space* $\mathbb{R}^n$ ($E = V = \mathbb{R}^n$). An *affine isomorphism* $f : \mathbb{R}^n \to E$ defines an affine frame (o, e) in E. Namely, $o = f(0)$ and $e_i = s(f)(\varepsilon_i)$ where

$$\varepsilon^i = (0, 0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n$$

and $x^i(p) = f^{-1}(p)^i$ is the ith affine coordinate of p with respect to the affine frame (o, e):

$$p = x^i(p) e_i + o.$$

Forms

Let σ_k be the group of all bijections

$$\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$$

and let sgnσ denote the signature of the permutation σ, sgn$\sigma = \pm 1$. We say that the tensor $f \in \overset{k}{\otimes} V^*$ is skew- or is a k-form over V if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}\sigma\, f(v_1, \dots, v_k),$$

for any $v_i \in V$ and $\sigma \in \sigma_k$.

The set

$$\Lambda^k V^* = \{f \in \overset{k}{\otimes} V^* \mid f \text{ skew}\}$$

is the (vector) space of k-forms and one defines a k-linear map

$$V^* \times \dots \times V^* \ni (\alpha^1, \dots, \alpha^k) \mapsto \alpha^1 \wedge \dots \wedge \alpha^k \in \Lambda^k V^*$$

by

(2) $\quad (\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det \| \alpha^i(v_j) \|$ for $v_i \in V$.

Any form given by formula (2) is called *decomposable* (sometimes: simple). There are non-decomposable forms: if the forms e^1, e^2, e^3, e^4 are linearly independent (therefore $n \geq 4$) then the 2-form $e^1 \wedge e^2 + e^3 \wedge e^4$ is not simple.

The wedge product can be expressed in terms of the tensor product, for example:

$$a \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha, \qquad \alpha, \beta \in V^*$$

From the properties of det one infers that

$\alpha^1 \wedge \dots \wedge \alpha^k \neq 0 \Leftrightarrow$ the set $(\alpha^i)_{i=1,\dots,k}$ is linearly independent.

$(e^{i_1} \wedge \dots \wedge e^{i_k})$ where $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $i_1 < i_2 < \dots < i_k$

is a frame in $\Lambda^k V^*$, thus

$$\dim \Lambda^k V^* = \binom{n}{k}.$$

The *wedge* (exterior) product $\wedge$ may be generalized: if $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^\ell V^*$ then $\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$, where

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) =$$
$$= \frac{1}{k!\ \ell!} \sum_{\sigma \in \sigma_{k+\ell}} \operatorname{sgn}\sigma\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)} \ldots, v_{\sigma(k+\ell)}).$$

The wedge product so defined is *associative.*

The direct sum $\Lambda V^* = \bigoplus_{k=0}^{n} \Lambda^k V^*$ where $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ can be made into an algebra (the *exterior*, or Grassmann, algebra over V) by defining:

$$\omega \wedge \pi = \sum_{i=0}^{n} (\omega_i \wedge \pi_{n-i})$$

where $\omega = (\omega_i)$, $\pi = (\pi_i)$, $i = 0, \ldots, n$ and $\omega_0 \wedge \alpha = \omega_0 \alpha$ (ordinary multiplication by the number $\omega_0 \in \mathbb{R}$).

The definition of ΛV^* implies:

$$\dim \Lambda V^* = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Geometrical interpretation of decomposable (simple) forms. Note that if $\bar{e}^i = A^i_j e^j$ $(i,j = 1, \ldots, k)$ then

$$\bar{e}^1 \wedge \ldots \wedge \bar{e}^k = \det A\, e^1 \wedge \ldots \wedge e^k$$

Consider an $(n-k)$-dimensional subspace U of the n-dimensional space V and a *frame* e *adapted* to U, i.e. such that $e_{k+1}, \ldots, e_n \in U$.

Let (e^i) be the dual frame, then

$$u \in U \Leftrightarrow u^i = e^i(u) = 0 \quad \text{for} \quad i = 1, \ldots, k \Leftrightarrow i(u)\,\alpha = 0$$

where

$$\alpha = e^1 \wedge \ldots \wedge e^k$$

and

$$i(u) : \Lambda^k V^* \to \Lambda^{k-1} V^*$$

is the *interior product* by u, defined by

(3) $$(i(u)\beta)(v_1,\dots,v_{k-1}) = \beta(u, v_1,\dots,v_{k-1})$$

(Note: $\deg \alpha = \operatorname{codim} U$; under a change of adapted frame, the form α acquires a numerical factor).

If $u, v \in V$ and $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^\ell V^*$ then

$$i(u)\, i(v) + i(v)\, i(u) = 0$$

and

$$i(u)(\alpha \wedge \beta) = (i(u)\alpha) \wedge \beta + (-1)^k \alpha \wedge i(u)\beta.$$

The last formula expresses an important property of i(u): it is an odd derivation of the (graded) Grassmann algebra. More generally, the vector space:

$$A = \bigoplus A_i, \qquad i \in \mathbb{Z}$$

is a $\mathbb{Z}$-*graded algebra* if there is given a bilinear map $A \times A \to A$ called the *product* in A, which is coherent with the grading in the sense that:

$$A_i \cdot A_j \subset A_{i+j}.$$

If $a_i \in A_i$ then a_i is said to be homogeneous of degree i.

The algebra is graded commutative (respectively: anticommutative) if, for any $a_i \in A_i$ and $a_j \in A_j$ one has

$$a_j a_i = \pm (-1)^{ij} a_i a_j$$

where the upper (lower) sign refers to the commutative (anticommutative) case. Clearly, $A = \Lambda V^*$ is a $\mathbb{Z}$-graded commutative (and associative) algebra. In this case, A_i is trivial for $i < 0$ and $i > n$. A derivation of degree k is an endomorphism D of A such that

$$D(A_i) \subset A_{i+k}$$

and, for any $a_i \in A$ and $b \in A$ there holds the Leibniz rule

$$D(a_i b) = (D a_i)\, b + (-1)^{ik} a_i\, Db.$$

Sometimes derivations of odd degree are called *antiderivations*.

A *graded Lie algebra* is a $\mathbb{Z}$-graded anticommutative algebra B such that, for any $a \in B_k$ the map

$$Ad_a : B \to B$$

defined by

$$(4) \qquad Ad_a(b) = ab, \qquad b \in B$$

is a derivation of B of degree k. It is customary to define the product in Lie algebras (graded or not) by a bracket so that instead of (4) one writes

$$Ad_a(b) = [a, b].$$

The condition that Ad_a be a derivation of degree k becomes

$$(5) \qquad [a, [b, c]] = [[a, b], c] + (-1)^{ik} [b, [a, c]]$$

where

$$a \in B_k, \; b \in B_i \quad \text{and} \quad c \in B.$$

Formula (5) generalizes the Jacobi condition of the theory of (ungraded) Lie algebras.

If A is any graded algebra, then the vector space

$$B = \operatorname{Der} A = \bigoplus \operatorname{Der}_k A, \qquad k \in \mathbb{Z}$$

of all derivations of A has the structure of a graded Lie algebra provided that one defines the bracket of a derivation D_1 of degree k with a derivation D_2 of degree ℓ by the formula:

$$(6) \qquad [D_1, D_2] = D_1 \circ D_2 - (-1)^{k\ell} D_2 \circ D_1$$

For example, if V is n-dimensional, then

$$\mathrm{Der}\, \Lambda V^* = \bigoplus_{k=-1}^{n-1} \mathrm{Der}_k\, \Lambda V^*$$

and

$$c \in \mathrm{Der}_1\, \Lambda V^*$$

defines in V the structure of an (ordinary) Lie algebra if and only if

$$[c, c] = 0$$

(cf. L. Corwin, Y. Ne'eman and S. Sternberg, Graded Lie algebras in mathematics and physics, Rev. Mod. Phys. *47* (1975), 573-603).

The formula

$$[i(u),\ i(v)] = 0$$

expresses simply the fact that there are no derivations of ΛV^* of degree lower than -1.

Orientation of V is a 'new' element, which must be 'put in by hand'. Two frames, e and $e' = ea$ define the same orientation iff $\det a > 0$. Therefore, an orientation may be defined as an element of $F(V)/GL^+(n, \mathbb{R})$; alternatively, as a half-line of $\Lambda^n V^*$.

An *oriented volume* element is any $\eta \in \Lambda^n V^*$, $\eta \neq 0$. If $u_1, \ldots, u_n \in V$, then $\eta\,(u_1, \ldots, u_n) =$ volume of the parallelepiped spanned by the vectors u_i. A frame e is said to be *unimodular* if $\eta = e^1 \wedge \ldots \wedge e^n$. Clearly, the set of all unimodular frames defines an oriented volume element; therefore, giving an oriented volume in V is equivalent to distinguishing an element (orbit) of $F(V)/SL(n, \mathbb{R})$ where $SL(n, \mathbb{R})$ is the group of $n \times n$ matrices of determinant one.

A *Euclidean* (pseudoeuclidean, Lorentz, Minkowski) *vector space* V has a *scalar product* (metric), i.e. a tensor:

$$g \in \mathfrak{L}^2(V; \mathbb{R}) = V^* \otimes V^*$$

which is symmetric, $g(u,v) = g(v,u)$, and non-degenerate, i.e. such that

$$\forall u \quad g(u,v) = 0 \Rightarrow v = 0.$$

For any v, the map $u \mapsto g(u, v)$ is linear, therefore there exists $\tilde{g}(v) \in V^*$ such that

$$g(u, v) = \langle u, \tilde{g}(v) \rangle.$$

Since g is non-degenerate, the linear map

$$\tilde{g} : V \to V^*$$

is an isomorphism. If (e^i) is a frame in V^*, then $g = g_{ij}\, e^i \otimes e^j$ where

$$g_{ij} = g(e_i, e_j)$$

and

$$\tilde{g}(e_i) = g_{ij}\, e^j.$$

Symmetry of $g \Leftrightarrow g_{ij} = g_{ji}$, non-degeneracy $\Leftrightarrow \det \| g_{ij} \| \neq 0$.

A classical theorem of linear algebra says that one can find an *orthonormal* frame (o_i) such that

$$g = o^1 \otimes o^1 + \dots + o^k \otimes o^k - o^{k+1} \otimes o^{k+1} - \dots - o^n \otimes o^n.$$

If (o_i) is orthonormal then so is $o' = oa$ where $a \in O(k, n-k)$ is an element of the orthogonal group [of type $(k, n-k)$].

Conversely, an orbit of $O(k, n-k)$ in F(V) determines a metric of signature $\underbrace{+ \dots +}_{k} \quad \underbrace{- \dots -}_{n-k}$.

An oriented Euclidean space has a 'natural' *volume element* $\eta = o^1 \wedge \ldots \wedge o^n$ where (o_i) is orthonormal and 'well' oriented. If $e = oa$ is any well oriented frame ($\det a > 0$) then

$$\eta = o^1 \wedge \ldots \wedge o^n = \det a \, e^1 \wedge \ldots \wedge e^n = \sqrt{|\det g_{ij}(e)|}\, e^1 \wedge \ldots \wedge e^n$$

because

$$g_{ij}(e) = g_{k\ell}(o)\, a^k_i\, a^\ell_j \Rightarrow |\det g_{ij}(e)| = (\det a)^2$$

One often writes

$$\eta = \frac{1}{n!}\,\eta_{i_1 \ldots i_n}\, e^{i_1} \wedge \ldots \wedge e^{i_n},$$

where

$$\eta_{i_1 \ldots i_n} = \eta\,(e_{i_1}, \ldots, e_{i_n}).$$

The preceding considerations lend themselves to an interesting generalization. Let G be a closed Lie subgroup of $GL(n, \mathbb{R})$. A G-*structure* on V is an element of $F(V)/G$, i.e. an orbit of G in $F(V)$.

Examples: an orientation, an oriented volume element, and a scalar product are $GL^{\pm}$, SL-, and O-structures respectively.

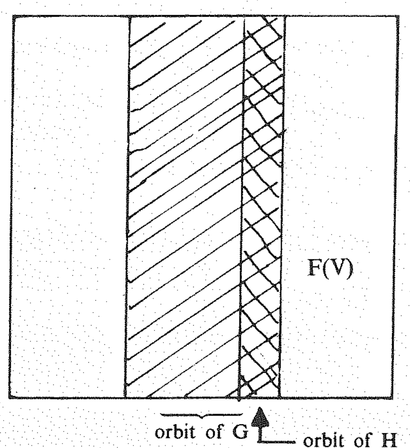


FIGURE 2

If $H \subset G$ the H-structure is finer (richer, stronger) than the G-structure; an H-structure determines a G-structure. The $\{I\}$-structure is the strongest (a preferred frame in V), the $GL(n, \mathbb{R})$-structure is the weakest (no additional element whatsoever).

Some of the important subgroups of $GL(n, \mathbb{R})$, and the corresponding G-structures on an $n = 2m$-dimensional real space, are enumerated below.

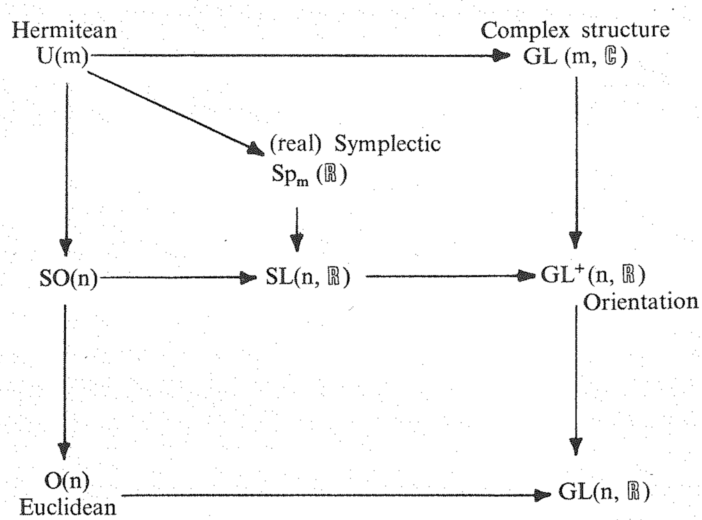


FIGURE 3

The injection $k : GL(m, \mathbb{C}) \to GL^+(n, \mathbb{R})$ is given by:

$$k(A + iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ (A, B real } m \times \text{ matrices).}$$

Let $j = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$, where I is the unit $m \times m$ matrix, then

$a \in GL(m, \mathbb{C}) \Leftrightarrow aj = ja$ and $\det a \neq 0$;
$a \in Sp_m(\mathbb{R}) \Leftrightarrow {}^t aja = j$, where ${}^t a$ denotes the matrix transpose of a;

$a \in U(m) \Leftrightarrow {}^t aa = I$ and $aj = ja$;
$a \in O(n) \Leftrightarrow {}^t aa = I_n$; $a \in SO(n) \Leftrightarrow a \in O(n)$ and $\det a > 0$.

Exercise: describe geometrically the G-structure on an n-dimensional vectore space V for:

$$G = \left\{ \begin{pmatrix} A & B \\ O & C \end{pmatrix} \in GL(n, \mathbb{R}) \mid A \in GL(k, \mathbb{R}), \right.$$

$$\left. C \in GL(n-k, \mathbb{R}),\ B = \text{any } k \times (n-k) \text{ matrix} \right\}.$$

Geometric objects of type ρ over an n-dimensional vector space V. Consider a representation of $GL(n, \mathbb{R})$ in $\mathbb{R}^m$, i.e. a homomorphism of groups

$$\rho : GL(n, \mathbb{R}) \to GL(m, \mathbb{R})$$
$$\rho(ab) = \rho(a)\,\rho(b), \qquad \rho(1) = 1.$$

The group $GL(n, \mathbb{R})$ acts in $F(V) \times \mathbb{R}^m$ on the right,

$$(e, q)a = (ea, \rho(a^{-1})q)$$

where $e \in F(V)$, $q \in \mathbb{R}^m$ and $a \in GL(n, \mathbb{R})$. The quotient

$$\rho(V) = F(V) \times \mathbb{R}^m / GL(n, \mathbb{R})$$

is the *space of objects of type* ρ. Let

$$k : F(V) \times \mathbb{R}^m \to \rho(V)$$

be the canonical map; by definition

(7) $k(e, q) = k(e', q') \Leftrightarrow \ \exists\ a \in GL(n, \mathbb{R})\ e' = ea$ and $q' = \rho(a^{-1})q$.

The partial map $k_e : \mathbb{R}^m \to \rho(V)$ defined by $k_e(q) = k(e, q)$ is bijective and may be used to make $\rho(V)$ into a vector space by putting

$$k_e(q_1) + k_e(q_2) = k_e(q_1 + q_2), \qquad \lambda k_e(q) = k_e(\lambda q).$$

By (7), this definition is correct: addition does not depend on e.

To any object $u \in \rho(V)$ there corresponds a map

$$\bar{u} : F(V) \to \mathbb{R}^m \tag{8}$$

defined by

$$\bar{u}(e) = k_e^{-1}(u).$$

It follows from (7) that

$$\bar{u}(ea) = \rho(a^{-1})\, \bar{u}(e). \tag{9}$$

Conversely, any map (8) which satisfies the '*transformation law*' (9) gives rise to an object of type ρ, $u = k(e, \bar{u}(e))$. The bar over u is usually omitted; this abuse of notation is often convenient and only occasionally confusing.

EXAMPLES

1. The representation ρ_w in $\mathbb{R}$, $\rho_w(a) = (\det a)^w$, corresponds to scalar densities of weight $w \in \mathbb{Z}$.
2. The one-dimensional representation $\pi : GL(n, \mathbb{R}) \to GL(1, \mathbb{R})$, $\pi(a) = \operatorname{sgn} \det a$ corresponds to 'pseudoscalars'.
3. The identity representation $\mathrm{id} : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $\mathrm{id}\,(a) = a$, corresponds to V.
4. The adjoint representation ad of $GL(n, \mathbb{R})$ in $\mathfrak{L}(\mathbb{R}^n)$, $\mathrm{ad}(a)\,(c) = aca^{-1}$, $c \in \mathfrak{L}(\mathbb{R}^n)$, corresponds to the space $\mathfrak{L}(V) = V \otimes V^*$ of 'mixed' tensors.
5. If ρ is a representation, then so is the contragredient map $\check{\rho}$, $\check{\rho}(a) = {}^t\rho(a^{-1})$, (t denotes the transpose). E.g., $\check{\mathrm{id}}$ corresponds to V^*.
6. If $\rho_i\,(i = 1, 2)$ are two representations, then so are $\rho_1 \oplus \rho_2$ and $\rho_1 \otimes \rho_2$, where

$$(\rho_1 \oplus \rho_2)\,(a)\,(q_1, q_2) = (\rho_1(a)q_1, \rho_2(a)q_2),$$
$$(\rho_1 \otimes \rho_2)\,(a)\,(q_1 \otimes q_2) = (\rho_1(a)q_1 \otimes \rho_2(a)q_2).$$

Duality.

Consider a vector space V with a metric tensor g and an orientation. This gives rise to an n-form η which is the oriented volume element in V.

One can now construct an isomorphism (the Hodge map)

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^*$$

by putting

$$(10) \qquad {}^*\alpha(v_{k+1},\dots, v_n)\,\eta = \alpha \wedge \tilde{g}(v_{k+1}) \wedge \dots \wedge \tilde{g}(v_n),$$
$$\text{for any } v_i \in V, \ i = k+1,\dots, n.$$

The Hodge map transforms decomposable forms into decomposable forms. To get a geometric interpretation of ${}^*\alpha$, consider $i(u)^*\alpha$:

$$\begin{aligned} \eta \cdot (i(u)^*\alpha)\,(v_{k+2},\dots, v_n) &= {}^*\alpha(u, v_{k+2},\dots, v_n)\,\eta \\ &= \alpha \wedge \tilde{g}(u) \wedge \tilde{g}(v_{k+2}) \wedge \dots \wedge \tilde{g}(v_n) \\ &= \eta \cdot {}^*(\alpha \wedge \tilde{g}(u))\,(v_{k+2},\dots, v_n) \end{aligned}$$

thus

$$(11) \qquad i(u)^*\alpha = {}^*(\alpha \wedge \tilde{g}(u))$$

$$(12) \qquad u \in U \Leftrightarrow i(u)\alpha = 0$$

$$(13) \qquad u \in U^\perp \Leftrightarrow i(u)^*\alpha = 0.$$

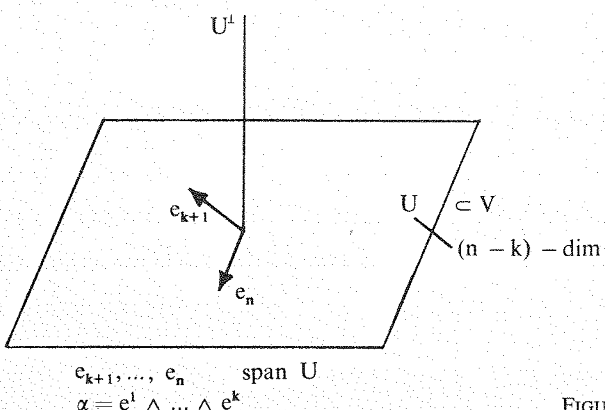


FIGURE 4

Proof of the last statement: $i(u)^*\alpha = 0 \Leftrightarrow \alpha \wedge \tilde{g}(u) = 0 \Leftrightarrow \tilde{g}(u)$ is a linear combination of $e^1,\ldots, e^k$, but $u \in U^\perp \Leftrightarrow g(u, e_i) = 0$ for $i = k+1,\ldots, n \Leftrightarrow \langle e_i, \tilde{g}(u)\rangle = 0$, and the last condition is equivalent to the statement that $\tilde{g}(u)$ is a linear combination of $e^1,\ldots, e^k$.

Since $^{**}\alpha$ corresponds to $U^{\perp\perp} = U$ one has $^{**}\alpha \parallel \alpha$; in fact $^{**}\alpha = \pm\, \alpha$. More precisely, $^{**}\alpha = (-1)^{k(n+1)}\, \mathrm{sgn}\ \det g\ \alpha$ if $\alpha \in \Lambda^k V^*$.

Another important property:

(14) if $\alpha, \beta \in \Lambda^k V^*$ then $^*\alpha \wedge \beta = {}^*\beta \wedge \alpha$.

It follows from the definition of * that:

$$^*\eta = 1\,.$$

To get the expression of $^*\alpha$ in terms of the components of α consider the volume element:

$$\eta = \frac{1}{n!}\,\eta_{i_1\ldots i_n}\, e^{i_1} \wedge \ldots \wedge e^{i_n} = \eta_{1\ldots n}\, e^1 \wedge \ldots \wedge e^n$$

where

$$\eta_{1\ldots n} = \left|\det g\right|^{1/2}$$

and define

$$\eta^{i_1\ldots i_n} = (\mathrm{sgn}\ \det g)\, g^{i_1 j_1} \ldots g^{i_n j_n}\, \eta_{j_1\ldots j_n}$$

so that

$$\eta^{1\ldots n} = \frac{1}{\eta_{1\ldots n}}$$

and

(15) $$e^{i_1} \wedge \ldots \wedge e^{i_n} = \eta^{i_1\ldots i_n}\, \eta$$

• Any k-form α can be written as $\alpha = \dfrac{1}{k!} \alpha_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}$

where

$$\alpha_{i_1 \dots i_k} = \alpha(e_{i_1}, \dots, e_{i_k})$$

and similarly for $*\alpha$. From the definition of the dual form,

$$\eta \cdot *\alpha_{i_{k+1} \dots i_n} = \eta \cdot *\alpha(e_{i_{k+1}}, \dots, e_{i_n}) = \alpha \wedge \tilde{g}(e_{i_{k+1}}) \wedge \dots \wedge \tilde{g}(e_{i_n})$$
$$= \frac{1}{k!} \alpha_{j_1 \dots j_k} g_{i_{k+1} j_{k+1}} \dots g_{i_n j_n} e^{j_1} \wedge \dots \wedge e^{j_n} ;$$

thus

$$(16) \qquad *\alpha_{i_{k+1} \dots i_n} = \frac{1}{k!} \alpha_{j_1 \dots j_k} \eta^{j_1 \dots j_n} g_{i_{k+1} j_{k+1}} \dots g_{i_n j_n}.$$

Example 1. In Euclidean space $\mathbb{R}^3$ with standard metric g, consider an orthonormal frame (e_i),

$$g(e_i, e_j) = \delta_{ij}, \quad \tilde{g}(e_i) = e^i, \quad \eta = e^1 \wedge e^2 \wedge e^3,$$

then

$$*1 = e^1 \wedge e^2 \wedge e^3,\ *e^1 = e^2 \wedge e^3,\ *(e^2 \wedge e^3) = e^1,\ *(e^1 \wedge e^2 \wedge e^3) = 1,$$

so that

$*(\alpha \wedge \beta)$ corresponds to $\vec{\alpha} \times \vec{\beta}$, $*(\alpha \wedge \beta \wedge \gamma)$ corresponds to $\vec{\alpha} \cdot (\vec{\beta} \times \vec{\gamma})$,

$*(*\alpha \wedge \beta)$ corresponds to $\vec{\alpha} \cdot \vec{\beta}$, $i(u)(\alpha \wedge \beta) = *(*(\alpha \wedge \beta) \wedge \tilde{g}(u))$ corresponds to $\vec{u} \times (\vec{\alpha} \times \vec{\beta})$, etc.

In $\mathbb{R}^3$, for any α, $**\alpha = \alpha$.

Example 2. Consider Minkowski space with metric of signature $+ - - -$.

Let (e_0, e_1, e_2, e_3) be an orthonormal frame, $\eta = e^0 \wedge e^1 \wedge e^2 \wedge e^3$, then

$$*1 = -\eta$$

$$*e^0 = -e^1 \wedge e^2 \wedge e^3, \quad *e^1 = -e^0 \wedge e^2 \wedge e^3, \text{ etc.}$$

$$*(e^0 \wedge e^1) = e^2 \wedge e^3, \quad *(e^2 \wedge e^3) = -e^0 \wedge e^1, \text{ etc.}$$

$$*(e^0 \wedge e^1 \wedge e^2) = -e^3, \quad *(e^1 \wedge e^2 \wedge e^3) = -e^0, \text{ etc.}$$

$$*(e^0 \wedge e^1 \wedge e^2 \wedge e^3) = 1$$

$$**\alpha = (-1)^{k+1}\alpha \quad \text{for} \quad \alpha \in \Lambda^k V^*.$$

The electromagnetic field may be represented by the 2-forms

$$f = E_x\, e^0 \wedge e^1 + \ldots - B_x\, e^2 \wedge e^3 - \ldots$$
$$*f = B_x\, e^0 \wedge e^1 + \ldots + E_x\, e^2 \wedge e^3 + \ldots$$

One has

$$f \wedge f = -2\,\vec{E} \cdot \vec{B}\,\eta,$$

$$*f \wedge f = (\vec{E}^2 - \vec{B}^2)\,\eta,$$

and $f \wedge f = 0 \Leftrightarrow f$ is simple.

Put $F = f - i * f$ and $\vec{F} = \vec{E} - i\vec{B}$, then

$$*F = iF, \qquad F \wedge F = -2i\,\vec{F}^2\,\eta.$$

An electromagnetic field is said to be null (optical, isotropic) iff

$$F \wedge F = 0 \Leftrightarrow \vec{E} \cdot \vec{B} = 0 = \vec{E}^2 - \vec{B}^2.$$

This implies

$$f \wedge f = 0 \Rightarrow f = \alpha \wedge x$$
$$*f \wedge f = 0 \Rightarrow *f = \beta \wedge x.$$

Let $W \subset V + iV$ be the complex 2-dimensional plane corresponding to the simple complex 2-form $F = (\alpha - i\beta) \wedge x$. Since $*F \parallel F$ we have $W^\perp = W$; therefore

$$(\alpha - i\beta)^2 = x^2 = x \cdot (\alpha - i\beta) = 0$$

or

$$x^2 = 0 = x \cdot \alpha = x \cdot \beta, \quad \alpha^2 = \beta^2, \quad \alpha \cdot \beta = 0.$$

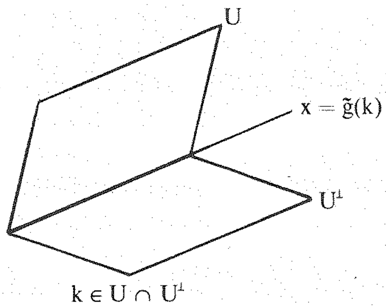


FIGURE 5

For an arbitrary frame e in Minkowski space, define

$$\eta_{ijk} = e^l \eta_{ijkl}, \quad \eta_{ij} = \frac{1}{2} e^k \wedge \eta_{ijk}, \quad \eta_i = \frac{1}{3} e^j \wedge \eta_{ij},$$

then

$$\eta = \frac{1}{4} e^i \wedge \eta_i,$$

and the following formulae are sometimes useful:

$$\begin{aligned} e^l \wedge \eta_{ijk} &= \delta^l_k \, \eta_{ij} + \delta^l_i \, \eta_{jk} + \delta^l_j \, \eta_{ki} \\ e^k \wedge \eta_{ij} &= \delta^k_j \, \eta_i - \delta^k_i \, \eta_j \\ e^j \wedge \eta_i &= \delta^j_i \, \eta \end{aligned}$$

also

$$*\eta_{ijkl} = \tilde{g}(e_i) \wedge \tilde{g}(e_j) \wedge \tilde{g}(e_k) \wedge \tilde{g}(e_l)$$

$$
\begin{aligned}
{}^*\eta_{ijk} &= -\tilde{g}(e_i) \wedge \tilde{g}(e_j) \wedge \tilde{g}(e_k) \\
{}^*\eta_{ij} &= \tilde{g}(e_i) \wedge \tilde{g}(e_j) \\
{}^*\eta_i &= -\tilde{g}(e_i) \\
{}^*\eta &= 1 .
\end{aligned}
$$

Exercise. Given an arbitrary vector space with a metric and orientation, extend the definition of the interior product (3) to $i(u) : \Lambda V^* \to \Lambda V^*$, define $j(u) : \Lambda V^* \to \Lambda V^*$ by $j(u)\alpha = \tilde{g}(u) \wedge \alpha$ and prove

$$
\begin{aligned}
&i(u)\, i(v) + i(v)\, i(u) = 0, \\
&j(u)\, j(v) + j(v)\, j(u) = 0, \\
&i(u)\, j(v) + j(v)\, i(u) = g(u, v), \quad \text{where } u, v \in V.
\end{aligned}
$$

Moreover, if one defines a scalar product on ΛV^* by:

$$(\alpha, \beta)\, \eta = \sum_{k=0}^{n} \alpha_k \wedge {}^*\beta_k, \quad \text{where } \alpha = (\alpha_k),\ \beta = (\beta_k) \in \Lambda V^*,$$

then

$$(i(u)\alpha, \beta) = (\alpha, j(u)\beta).$$

Remark: a natural extension of $i(u)$ to $\Lambda^0 V^* = \mathbb{R}$ is given by $i(u) 1 = 0$.

III

DIFFERENTIAL MANIFOLDS

A *chart* on a topological space M is a homeomorphism $x : U \to V$ of an open subset U of M onto an open subset V of $\mathbb{R}^n$. U is called the *domain* of the chart, n is its *dimension* and the real-valued functions x^i, $x = (x^i)$, $i = 1, \ldots, n$, are the *local coordinates* defined by the chart. An *atlas* on M is a collection of charts whose domains cover M. If all the charts of the atlas are n-dimensional, then n is said to be the dimension of the atlas.

An n-*dimensional manifold* is a Hausdorff space which has a countable basis for its topology and admits an n-dimensional atlas. Consider two charts x_1 and x_2 on an n-dimensional manifold M, $x_i : U_i \to V_i$; they are said to be C^∞-*compatible* if the composite maps:

$$x_2 \circ x_1^{-1} : x_1(U_1 \cap U_2) \to x_2(U_1 \cap U_2) \subset \mathbb{R}^n$$

and

$$x_1 \circ x_2^{-1} : x_2(U_1 \cap U_2) \to x_1(U_1 \cap U_2) \subset \mathbb{R}^n$$

are of class C^∞ (i.e., if they have all partial derivatives). An *atlas* is of *class* C^∞ if all pairs of its charts are C^∞-compatible.

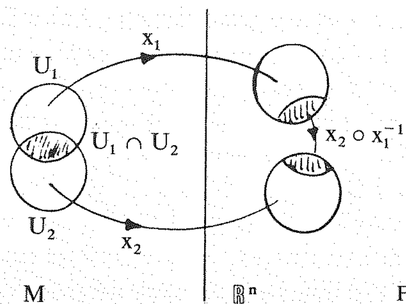


FIGURE 6

A *differential manifold of class* C^∞ is a manifold with a maximal atlas of class C^∞. Any atlas of class C^∞ is contained in an unique maximal atlas; therefore, to define the structure of a differential manifold it is enough to specify an atlas of class C^∞.

If M, N are differential manifolds of class C^∞ then a map $f : M \to N$ is said to be differentiable (smooth) if, for any charts x on M and y on N, the composite map $y \circ f \circ x^{-1}$ is differentiable wherever defined.

A *diffeomorphism* is a homeomorphism $f : M \to N$ such that both f and f^{-1} are smooth.

From now on we consider only differentiable manifolds of class C^∞ and smooth maps.

Examples of smooth maps: if $N = \mathbb{R}$ then $f : M \to \mathbb{R}$ is a (real valued) *function* on M. The function f is smooth iff its *expression by local coordinates* $f \circ x^{-1}$ is smooth for all charts. If $M = \mathbb{R}$, then $f : \mathbb{R} \to N$ is a *curve* in N.

Examples of manifolds

1. $\mathbb{R}^n$ with its 'natural' differential structure, defined by the global chart $x : \mathbb{R}^n \to \mathbb{R}^n$, where $x^i(t^1, \ldots, t^n) = t^i$.
2. An n-dimensional *affine space* is defined to be a set M on which an n-dimensional space V acts freely and transitively. If $p_0 \in M$ and $e \in F(V)$ then the bijective map $x : M \to \mathbb{R}^n$ defined by

$$x^i(p)\, e_i + p_0 = p$$

   induces on M a manifold topology and a differentiable structure; they are both independent of p_0 and e. (Explanation: if $u \in V$ and $p \in M$, then $u + p$ is the result of action of u on p: it is the point of M translated by the vector u).
3. The n-*sphere*

$$\mathbb{S}_n = \{q \in \mathbb{R}^{n+1} \mid q_1^2 + q_2^2 + \ldots + q_{n+1}^2 = 1\}$$

   has an atlas consisting of two charts defined by the stereographic projection from two distinct points of $\mathbb{S}_n$.

4. The (real) *projective space* $\mathbb{RP}_n$ is defined as the quotient of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation

$$R = \{(s, t) \mid s = \lambda t, \ 0 \neq \lambda \in \mathbb{R}\},$$

where

$$t = (t^k), \ s = (s^k) \in \mathbb{R}^{n+1} - \{0\}.$$

For any α ($\alpha = 1,\ldots,n+1$) one defines a chart $x_\alpha : U_\alpha \to \mathbb{R}^n$ by $U_\alpha = \{k(t) \in \mathbb{RP}_n \mid t^\alpha \neq 0\}$ where $k : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}_n$ is the canonical map, $x^i_\alpha(k(t)) = t^i/t^\alpha$, $i \neq \alpha$. Exercise: check compatibility of these charts.
5. The sets $\mathbb{C}^n$ and $\mathbb{H}^n$ have a natural structure of differentiable manifolds of dimension 2n and 4n, respectively.
6. Exercise. Define the complex and quaternionic projective spaces and describe their differential structure.
7. If N is an open subset of an n-dimensional differential manifold M, then N has a natural structure of an n-dimensional differential manifold. Example: $GL(n, \mathbb{R}) \subset \mathfrak{L}(\mathbb{R}^n) \approx \mathbb{R}^{n^2}$ is an n^2-dimensional differential manifold.

Problem. Is it possible to have *distinct* differential structures on the same topological manifold? Yes; consider $\mathbb{R}$ and the following two global charts $x_1, x_2 : \mathbb{R} \to \mathbb{R}$, where $x_1(t) = t$ (the 'natural' chart) and $x_2(t) = t^3$. These two charts are *not* compatible (exercise: explain why), therefore they define distinct differential structures on $\mathbb{R}$. However, these structures are *equivalent* in the following sense: there is a diffeomorphism f of $\mathbb{R}$ with one structure onto $\mathbb{R}$ with the other (namely, $f(t) = t^{1/3}$). J. Milnor [Ann. Math. *64* (1956), 339] has shown that $\mathbb{S}_7$ has *non equivalent* differential structures. More surprisingly, it has been shown recently by S. K. Donaldson that $\mathbb{R}^4$ admits differential structures inequivalent to the standard one (*cf.* Proceedings of the 1982 International Congress of Mathematicians, Warsaw, 1983).

The set $C^\infty(M)$ of all differentiable functions on M forms an associative and commutative algebra (functions can be multiplied by numbers, by each other and added). A smooth map

$h : M \to N$ induces a homomorphism of algebras

$$h^* : C^\circ(N) \to C^\circ(M)$$

defined by

$$h^*(g) = g \circ h \qquad \text{for } g \in C^\circ(N).$$

Note that if

$$M \xrightarrow{h_1} N \xrightarrow{h_2} P \quad \text{then} \quad (h_2 \circ h_1)^* = h_1^* \circ h_2^*.$$

A *vector field* u on M is a linear map $u: C^\circ(M) \to C^\circ(M)$ such that

$$u(fg) = fu(g) + gu(f)$$

for any $f,g \in C^\circ(M)$; in other words, u is a derivation of the algebra $C^\circ(M)$. If u and v are vector fields on M, then so is their commutator (bracket) $[u, v]$ defined in the usual way:

$$[u,v]\,(f) = u(v(f)) - v(u(f)).$$

The bracket is skew, $[u,v] + [v,u] = 0$ and satisfies the Jacobi identity

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0.$$

Therefore, the set V(M) of all vector fields on M is a Lie algebra. If $p \in M$ and $u \in V(M)$ then there is a linear map

$$u_p : C^\circ(M) \to \mathbb{R}$$

such that

(L) $$u_p(fg) = f(p)\, u_p(g) + g(p)\, u_p(f)$$

defined by

$$u_p(f) = u(f)\,(p).$$

Any linear map from $C^\infty(M)$ to $\mathbb{R}$ satisfying the Leibniz rule (L) is called a *tangent vector* to M at p; the tangent vector defined by the last formula is induced by the vector field u.

Any curve $k : \mathbb{R} \to M$ through p, $k(0) = p$, defines a tangent vector to M at p by

$$f \mapsto \frac{d}{dt} f \circ k(t) \Big|_{t=0}$$

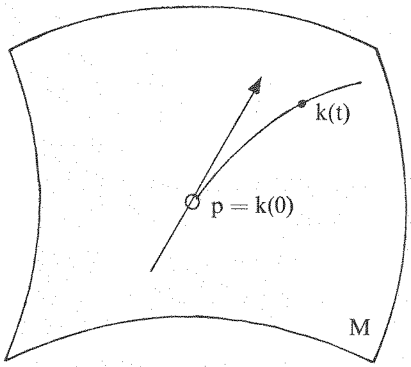


FIGURE 7

An alternative and equivalent definition of a tangent vector to M at p is:

$$\left.\begin{array}{l}\text{tangent vector}\\ \text{to M at p}\end{array}\right\} = \left\{\begin{array}{l}\text{an equivalence class of curves through p,}\\ \text{two curves } k_1 \text{ and } k_2 \text{ being considered}\\ \text{as equivalent iff, for any } f \in C^\infty(M)\\ \dfrac{d}{dt} f \circ k_1(t) \Big|_{t=0} = \dfrac{d}{dt} f \circ k_2(t) \Big|_{t=0}.\end{array}\right.$$

Coordinate lines and coordinate vectors on M are defined by a chart $x : U \to V \subset \mathbb{R}^n$ as follows. Let $p \in U$, $x^i(p) = t_0^i$, then

$$k_i(t) = x^{-1}(t_0^1, \ldots, t_0^{i-1}, t, t_0^{i+1}, \ldots, t_0^n)$$

defines the ith *coordinate line* through p.

Note:

$$x^j(k_i(t)) = \begin{cases} t_0^j & \text{for } j \neq i \\ t & \text{for } j = i. \end{cases}$$

The ith coordinate vector at p is given by its action on $f \in C^\infty(M)$ as follows:

$$e_i(f) = \frac{d}{dt} f \circ k_i(t) \Big|_{t = t_0^i}$$

$$= \frac{\partial}{\partial t^i} f \circ x^{-1}(t^1, \ldots, t^n) \Big|_{\text{all } t^i = t_0^i}.$$

Because of the last formula, the vector e_i is often denoted by $\partial/\partial x^i$ [or $(\partial/\partial x^i)_p$ if one wishes to emphasize the point to which it is attached]. Thus a chart $x : U \to V$ defines at any point $p \in U$ a collection of n vectors

$$e_1 = \frac{\partial}{\partial x^1},\ e_2 = \frac{\partial}{\partial x^2},\ \ldots,\ e_n = \frac{\partial}{\partial x^n}.$$

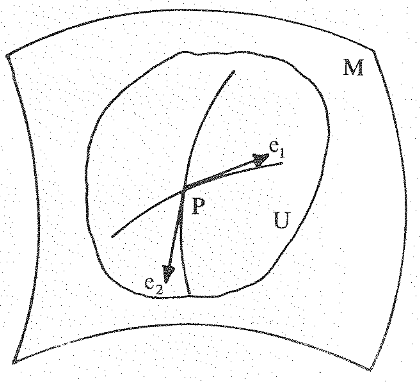


FIGURE 8

One can prove the following

Theorem. The vectors (e_i) form a frame of the vector space T_pM tangent to M at p; therefore $\dim T_pM = \dim M$.

The functions x^i defining a chart may be extended to functions defined over all M; in general it is necessary to restrict them first to a compact set $V \subset U$; but, given $p \in U$, one can always extend the functions x^i to M in such a way that the extensions coincide with x^i in a neighbourhood of p. Therefore, it makes sense to evaluate the vectors $e_i \in T_pM$ on x^j:

$$e_i(x^j) = \frac{\partial}{\partial t^i} x^j \circ x^{-1}(t^1, \ldots, t^n) \bigg|_{t^i = t^i_0} = \delta^j_i$$

or

$$\frac{\partial}{\partial x^i}(x^j) = \delta^j_i .$$

If $u \in T_pM$ and $u^i \underset{\text{def}}{=} u(x^i)$ then $u = u^i \frac{\partial}{\partial x^i}$; the numbers u^i are thus the components of the vector u with respect to the coordinate frame (e_i).

Tensor fields, fields of k-forms, etc. may be introduced according to the following scheme. Consider the dual of the tangent space,

$$T^*_p M \underset{\text{def}}{=} (T_pM)^*;$$

form

$$\overbrace{T_pM \otimes \ldots \otimes T_pM}^{k} \otimes \overbrace{T^*_p M \otimes \ldots \otimes T^*_p M}^{l} = T^{(k,l)}_p M ,$$

and define a tensor field A of type (k,l) to be a map

$$M \ni p \mapsto A_p \in T^{(k,l)}_p M$$

which is smooth in a sense explained on the example of a *field of* k-*forms* ω,

$$M \ni p \mapsto \omega_p \in \Lambda^k T^*_p M .$$

The field ω is smooth if, for any $v_1, \ldots, v_k \in V(M)$, the function

$$M \ni p \mapsto \omega_p(v_{1p}, v_{2p}, \ldots, v_{kp}) \in \mathbb{R}$$

is smooth. Let $C^k(M)$ be the vector space of (smooth) fields of k-forms. If $\omega \in C^k(M)$ and $\pi \in C^l(M)$, then $\omega \wedge \pi \in C^{k+l}(M)$ is defined by

$$(\omega \wedge \pi)_p = \omega_p \wedge \pi_p$$

and this 'point-wise' method of extending algebraic operations from the tangent spaces to fields may be used to define:

$i(u)\alpha$ for $u \in V(M)$ and $\alpha \in C^k(M)$,
$\tilde{g}(u)$, where g is a metric tensor field, $u \in V(M)$,
$*\alpha$, etc.

Exterior derivative. Let $f \in C^0(M)$; one defines $df \in C^1(M)$ to be such that, for any $u \in V(M)$,

$$\langle u, df \rangle = u(f). \tag{1}$$

Clearly, if x is a local coordinate system, then

$$\langle e_i, dx^j \rangle = \delta_i^j.$$

The forms $(dx^i)_p$ constitute the dual frame relative to $\left(\frac{\partial}{\partial x^i}\right)_p$ and

$$df = \frac{\partial}{\partial x^i}(f)\, dx^i \quad \text{or simply} \quad \frac{\partial f}{\partial x^i}\, dx^i. \tag{1 bis}$$

The linear map $d : C^0(M) \to C^1(M)$ is extended to a unique linear map

$$d : C^k(M) \to C^{k+1}(M),\ k = 0, 1, \ldots, n = \dim M \quad \text{with } C^{n+1}(M) = \{0\} \tag{2}$$

called the *exterior derivative*; its characteristic properties are

(3) linearity

(4) d on $C^0(M)$ is given by (1)

(5) $d^2 = 0$

(6) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in C^k(M)$.

Conditions (2), (3) and (6) can be summarized by saying that d is a derivation of degree 1 of the Cartan algebra $\bigoplus_k C^k(M)$ of differential forms.

For example, if

$$\alpha = \frac{1}{k!} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \text{ then}$$

$$d\alpha = \frac{1}{k!} d\alpha_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \frac{1}{k!} \alpha_{i_1 \dots i_k, j}\, dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where

$$\alpha_{i_1 \dots i_k, j} \underset{\text{def}}{=} \frac{\partial}{\partial x^j} \alpha_{i_1 \dots i_k} \qquad \text{(compare (1bis));}$$

thus

$$(d\alpha)_{i_1 \dots i_k i_{k+1}} = (-1)^k (k+1)\, \alpha_{[i_1 \dots i_k, i_{k+1}]}$$

with the square brackets denoting antisymmetrisation over the indices enclosed.

Remark. Nothing compels us to use coordinate frames on a manifold; one can – and often does – use an arbitrary *field* of frames

$$M \ni U \ni p \mapsto e_p \in F(T_p M)$$

and the field of dual frames (*coframes*)

$$M \ni U \ni p \mapsto e_p^* \in F(T_p^* M)$$

with

$$\langle e_j, e^i \rangle = \delta_j^i .$$

Note that locally

$$e^i = dx^i \Leftrightarrow de^i = 0 .$$

Indeed the implication $\Rightarrow$ is true by (5), and the implication $\Leftarrow$ is true locally by a theorem known as the Poincaré Lemma. Sometimes one says that $de^i = 0$ characterizes *holonomic frames*.

EXAMPLES

1. *Vector analysis* in $\mathbb{R}^3$ with Euclidean metric and orientation:

$$\begin{array}{lll} \text{if } f \in C^0(\mathbb{R}^3), & \text{then} & df = (\text{grad } f)_i \, dx^i \\ \text{if } \alpha \in C^1(\mathbb{R}^3), & \text{then} & {}^*d\alpha = (\text{curl } \alpha)_i \, dx^i \\ & & {}^*d{}^*\alpha = \text{div } \alpha . \end{array}$$

Therefore, both $\text{div curl} = 0$ and $\text{curl grad} = 0$ are consequences of $d^2 = 0$. Formulae of the type

$$\text{div}(\alpha \times \beta) = \beta \text{ curl } \alpha - \alpha \text{ curl } \beta$$

may be derived from the Leibniz rule for d.

2. *Maxwell's equations* in Minkowski space.

Let $A = A_i \, dx^i$ be the 1-form of the electromagnetic potential, $A_0 = \varphi$, $A_1 = -A_x$ etc. then

$$f = dA$$

is the 2-form of the electromagnetic field. The Lorentz condition is

$$d^*A = 0,$$

whereas Maxwell's equations are

$$df = 0 \quad \text{and} \quad d{*}f = -4\pi {*}j,$$

where

$$j = j_i dx^i$$

is the 1-form of current, conserved as a consequence of $d^2 = 0$:

$$d{*}j = 0.$$

3. *Lienard-Wiechert potentials* and field of a unit charge in Minkowski space $\mathbb{R}^4$. Consider the motion of a unit charge, refer everything to an orthonormal frame. Given the world-line $z^i(s)$ of the charge, define function σ on $\mathbb{R}^4$ by

$$g_{ij}\rho^i(x)\rho^j(x) = 0, \ \rho^0 \geq 0,$$

where

$$\rho^i(x) = x^i - z^i \ (\sigma(x)).$$

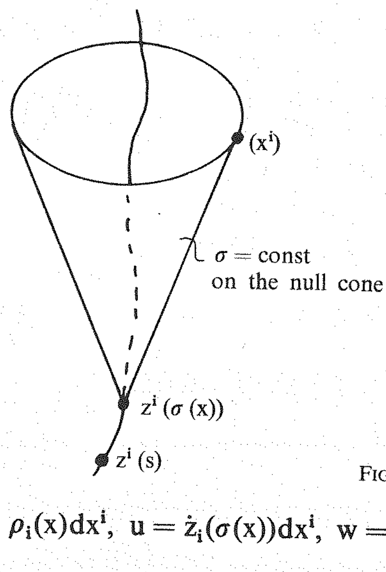


FIGURE 9

Define $\rho = \rho_i(x)dx^i$, $u = \dot{z}_i(\sigma(x))dx^i$, $w = \ddot{z}_i(\sigma(x))dx^i$

$$R = u \cdot \rho \ (= g^{ij} \, \dot{z}_i \, \rho_j);$$

show $\rho = Rd\sigma$, $du = d\sigma \wedge w$, $dR = (w \cdot \rho)d\sigma + u - d\sigma$, and differentiate $A = u/R$ to obtain the field

$$f = \frac{d\sigma \wedge u}{R^2} + \frac{d\sigma \wedge (w - (w \cdot d\sigma)u)}{R}$$

(Note that $(d\sigma)^2 = 0$ and $d\sigma \perp w - (w \cdot d\sigma)u$).
In some respects, '*forms are better than vectors*': they can be

differentiated by means of d;
integrated;
transported (pulled back) by smooth maps.

If $h : M \to N$ is smooth then there is a linear map

$$h^* : C^k(N) \to C^k(M)$$

defined by

$$(h^*\alpha)_p\,(u_1,\ldots,u_k) = \alpha_{h(p)}\,(T_p h(u_1),\ldots,T_p h(u_k)), \tag{7}$$

where

$$\alpha \in C^k(N);\;\; p \in M;\;\; u_1,\ldots,u_k \in T_pM,$$

and

$$T_p h : T_pM \to T_{h(p)}N$$

is a linear map, called the tangent (*derived*) *map* of h at p, given by

$$\langle g, T_p h(u)\rangle = \langle g \circ h, u\rangle,\;\; g \in C^0(N),\;\; u \in T_pM.$$

In terms of local coordinates x on M and y on N, $T_p h$ is given by the Jacobian matrix:

$$T_p h\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial(y^\alpha \circ h)}{\partial x^i}\,\frac{\partial}{\partial y^\alpha}\,.$$

The direct sum $C(M) = \bigoplus_{k=0}^{n} C^k(M)$ can be made into an algebra relative to $\wedge$; this is the *Cartan algebra* of differential forms on M. The pull-back map h^* can be extended to

$$h^* : C(N) \to C(M).$$

Moreover, $(h_1 \circ h_2)^* = h_2{}^* \circ h_1^*$, $id_M^* = id_{C(M)}$; one says that C and $h \mapsto h^*$ define a (contravariant) functor from the category of differential manifolds to the category of associative algebras with (anti)derivation.

Indeed, h^* is not only linear but also

$$h^*(\alpha \wedge \beta) = h^*\alpha \wedge h^*\beta; \tag{8}$$

$$h^* \circ d = d \circ h^*. \tag{9}$$

Many of the potentials and fields occurring in physics are (Lie algebra – or spinor-valued) differential forms.

Vector fields occur in a natural manner in connection with problems of *invariance* and *transformation groups*.

A *diffeomorphism* $h : M \to N$ can be used to transport ('drag along', 'Lie transport') any tensor field from N to M. It is enough to define the transport for vector fields; any tensor field can be represented as a sum of products of vector fields and forms. Denoting by V(M) the set of all vector fields on M, we define

$$h^* : V(N) \to V(M)$$

by

$$(h^*v)(f) = v(f \circ h^{-1}) \circ h, \tag{10}$$

where

$$v \in V(N) \quad \text{and} \quad f \in C^0(M).$$

A *one-parameter group of transformations* of M is given by a smooth map

$$M \times \mathbb{R} \ni (p, t) \mapsto \varphi_t(p) \in M$$

such that

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \text{and} \quad \varphi_0 = \mathrm{id}_M.$$

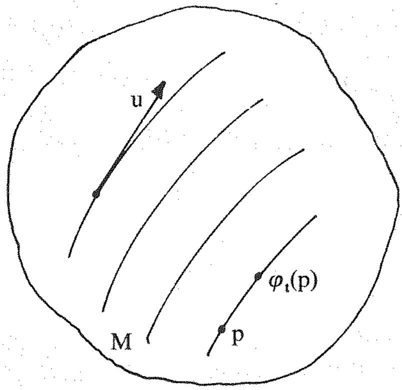


FIGURE 10

Clearly,

$$\varphi_t^{-1} = \varphi_{-t};$$

therefore, φ_t is a diffeomorphism of M onto itself. The group (φ_t) induces a vector field u on M,

$$u(f) = \frac{d}{dt} f \circ \varphi_t \Big|_{t=0}, \quad f \in C^0(M).$$

Conversely, any $u \in V(M)$ *generates* a (local) one-parameter group of (local) transformations of M. Given $u \in V(M)$, one defines the *Lie derivative* of a tensor field A on M to be

$$L_u A = \frac{d}{dt} \varphi_t^* A \Big|_{t=0}.$$

Now,

$$\frac{d}{dt}\varphi_t^* A = \frac{d}{ds}\varphi_{t+s}^* A\Big|_{s=0} = \frac{d}{ds}\varphi_t^*\varphi_s^* A\Big|_{s=0} =$$

$$= \varphi_t^* \frac{d}{ds}\varphi_s^* A\Big|_{s=0}$$

gives

$$\frac{d}{dt}\varphi_t^* A = \varphi_t^* L_u A\,;$$

therefore there holds the

Theorem. A is invariant under (φ_t),

$$\text{i.e., } \varphi_t^* A = A \text{ for any } t \in \mathbb{R} \Leftrightarrow L_u A = 0.$$

Properties of the Lie derivative useful in computations.

Consider the *tensor algebra* $\mathcal{T}(M)$ over M: its elements are collections of arbitrary tensor fields on M; multiplication is defined by the tensor product $\otimes$. Then, for any $u \in V(M)$:

(11) $$L_u : \mathcal{T}(M) \to \mathcal{T}(M)$$

is linear and type-preserving;

(12) $$L_u(A \otimes B) = (L_u A) \otimes B + A \otimes L_u B;$$

(L_u is a derivation of degree 0).

(13) if $f \in C^0(M)$, then $L_u f = u(f)$;

(14) if $v \in V(M)$, then $L_u v = [u, v]$.

Proof: let (φ_t) be generated by u, then from equation (10)

$$(\varphi_t^* v)(f) = v(f \circ \varphi_t^{-1}) \circ \varphi_t,$$

and

$$\frac{d}{dt}(\varphi_t^* v)(f)\Big|_{t=0} = \frac{d}{dt} v(f \circ \varphi_t^{-1})\Big|_{t=0} + \frac{d}{dt} v(f) \circ \varphi_t \Big|_{t=0}$$

$$= -\frac{d}{dt} v(f \circ \varphi_t)\Big|_{t=0} + u(v(f))$$

$$= [u, v](f).$$

Moreover,

(15) the map $u \mapsto L_u$ is a Lie algebra homomorphism,

$$L_{[u, v]} = [L_u, L_v].$$

The Lie derivative restricted to the *Cartan algebra* C(M) enjoys the following properties:

(16) $L_u \circ d = d \circ L_u$;

(17) $L_u(\alpha \wedge \beta) = (L_u \alpha) \wedge \beta + \alpha \wedge L_u \beta$;

(18) $L_u = d \circ i(u) + i(u) \circ d$;

(19) $[i(u), L_v] = i([u, v])$.

Formulae such as (15), (16), (18) and (19) can be proved by noting that a commutator of a derivation and an antiderivation is an antiderivation (similarly, the anticommutator of two antiderivations is a derivation; cfr. the paragraph on graded algebras in the Chapter on algebra), and taking into account that C(M) is generated by $C^0(M) \cup C^1(M)$ so that any (anti)derivation vanishing on functions and 1-forms vanishes on all of C(M).

Examples and exercises.

1. If $x = (x^i)$ is a local coordinate system, then, by equations (13) and (16),

$$L_u dx^i = dL_u x^i = d(u(x^i)) = du^i.$$

2. If $\alpha = \alpha_i dx^i \in C^1(M)$, then

$$L_u\alpha = (L_u\alpha_i)dx^i + \alpha_i L_u dx^i = u(\alpha_i)dx^i + \alpha_i du^i = \\ = (\alpha_{i,j}u^j + \alpha_j u^j_{,i})\, dx^i,$$

where one uses (17) and $u = u^i \dfrac{\partial}{\partial x^i}$, $du^i = \partial u^i/\partial x^j dx^j$.

3. Show

$$L_u g = (g_{ij,k}u^k + g_{kj}u^k_{,i} + g_{ik}u^k_{,j})\, dx^i \otimes dx^j \tag{20}$$

where

$$g = g_{ij}dx^i \otimes dx^j \text{ is a metric tensor field.}$$

4. If $\alpha \in C^1(M)$ and $u, v \in V(M)$, then

$$d\alpha(u, v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u, v]). \tag{21}$$

Proof: substitute (18) into (19) to get

$$i(u)di(v) + i(u)i(v)d - di(v)i(u) - i(v)di(u) = i([u,v]).$$

Evaluate both sides of the last equation on α and use

$$i(v)i(u)\alpha = 0 \text{ for } \alpha \in C^1(M),\ i(u)i(v) = -i(v)i(u), \\ d\alpha(u,v) = i(v)i(u)d\alpha,\ u(\alpha(v)) = i(u)di(v)\alpha.$$

Similarly, if $\alpha \in C^2(M)$ and $u, v, w \in V(M)$, then

$$d\alpha(u,v,w) = u(\alpha(v,w)) + v(\alpha(w,u)) + w(\alpha(u,v)) + \\ + \alpha(u,[v,w]) + \alpha(v,[w,u]) + \alpha(w,[u,v]). \tag{22}$$

5. Prove the following: if $h: M \to N$ is a diffeomorphism, then

1° $h^*[u,v] = [h^*u, h^*v]$;
2° if (φ_t) is a one-parameter group of transformations on N, generated by $u \in V(N)$, then $(h^{-1} \circ \varphi_t \circ h)$ is a one-parameter group on M, generated by $h^*u \in V(M)$.

6. Symmetries of plane electromagnetic waves.

Consider a plane, linearly polarized wave moving in the z direction,

$$f = a(u)du \wedge dx, \quad {}^*f = a(u)du \wedge dy,$$

where $\sqrt{2}\,u = t - z$ and a is an arbitrary function. If $\sqrt{2}\,v = t + z$ then $g = 2dudv - dx^2 - dy^2$ (here dx^2 is short for $dx \otimes dx$, etc.).

Symmetries are generated by vector fields ξ, solutions of

$$L_\xi g = 0 \quad \text{and} \quad L_\xi f = 0.$$

For $a \neq 0$, these equations admit 5 linearly independent solutions, e.g.

$$\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial v} + u\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y}.$$

The three vector fields

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial v} + u\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y}$$

generate the 'little group' of the null vector $\partial/\partial v$.

7. Divergence of a vector field.

Let η be a volume n-form on M. E.g., if M has a metric $g = g_{ij}dx^i \otimes dx^j$ and $\gamma = |\det g_{ij}|^{1/2}$, then η may be taken to be $\gamma dx^1 \wedge \ldots \wedge dx^n$.

For any vector field u on M define its divergence by

$$L_u\eta = (\text{div } u)\,\eta.$$

Using equation (18) one obtains

$$L_u\eta = d(i(u)\eta),$$

and

$$\begin{aligned} L_u\eta &= d(\gamma u^1 \wedge dx^2 \wedge \ldots \wedge dx^n - \gamma dx^1 \wedge u^2 \wedge \ldots \wedge dx^n + \ldots) \\ &= (\gamma u^1)_{,1} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n + \\ &+ (\gamma u^2)_{,2} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n + \ldots = \frac{1}{\gamma}(\gamma u^k)_{,k}\,\eta; \end{aligned}$$

thus

$$\operatorname{div} u = \frac{1}{\gamma}(\gamma u^k)_{,k}.$$

8. Behaviour of integrals under infinitesimal transformations.
Let λ be an n-form on an n-dimensional oriented manifold M and let $x : U \to \mathbb{R}^n$ be a local coordinate system which agrees with the orientation of M. Write

$$\lambda = \lambda_{12\dots n}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

$$\int_U \lambda = \int_{x(U)} \lambda_{12\dots n} \circ x^{-1} \cdot d\xi^1 d\xi^2 \dots d\xi^n,$$

where the integral on the right is the 'ordinary' (Riemann) integral over $\xi \in x(U) \subset \mathbb{R}^n$.

If $x' : U \to \mathbb{R}^n$ is another coordinate map of the same orientation as x and

$$\lambda = \lambda'_{12\dots n}\, dx'^1 \wedge dx'^2 \wedge \dots \wedge dx'^n,$$

then

$$\lambda_{12\dots n} = \lambda'_{12\dots n} \det\left(\frac{\partial x'^i}{\partial x^j}\right),$$

so that the definition of $\int_U \lambda$ does not depend on the choice of the map x.

If $h : M \to M$ is an orientation preserving diffeomorphism, then, by considering the maps $x : h(U) \to \mathbb{R}^n$ and $x \circ h : U \to \mathbb{R}^n$ one proves:

(23) $$\int_{h(U)} \lambda = \int_U h^* \lambda.$$

Let (φ_t) be a one-parameter group of transformations of M generated by $u \in V(M)$, then

$$\frac{d}{dt}\int_{\varphi_t(U)} \lambda = \frac{d}{dt}\int_U \varphi_t^* \lambda = \int_U \varphi_t^* L_u \lambda = \int_{\varphi_t(U)} L_u \lambda.$$

If $\lambda = \eta L$, where L is a Lagrangian, then

$$L_u \lambda = (L \operatorname{div} u + L_u L)\eta .$$

9. Canonical formalism of classical mechanics.

A symplectic (phase) space consists of an even-dimensional manifold P with a two-form β which is *closed*, $d\beta = 0$, and *non-singular*: for any $p \in P$, the map

$$T_p P \ni u_p \mapsto i(u_p)\,\beta \in T_p^* P$$

is invertible. In local coordinates,

$$\beta = dp_i \wedge dq^i .$$

A diffeomorphism $h: P \to P$ is called a *canonical transformation* iff $h^* \beta = \beta$.

Let $u \in V(P)$ generate a one-parameter group (φ_t) of canonical transformations. Then

$$0 = L_u \beta = (d \circ i(u) + i(u) \circ d)\,\beta = d(i(u)\beta).$$

Therefore, locally

$$i(u)\,\beta = dU$$

where $U \in C^0(P)$ is the generating function of (φ_t), defined up to $U \mapsto U + \text{const}$. If $u_1, u_2 \in V(P)$ generate canonical transformations then so does $[u_1, u_2]$. Let $i(u_i)\,\beta = dU_i$, $i = 1, 2$ and $i([u_1, u_2])\,\beta = dU$. Using equation (19) we obtain

$$dU = i([u_1, u_2])\,\beta = [i(u_1), L_{u_2}]\,\beta = -L_{u_2} i(u_1)\,\beta =$$
$$= L_{u_2} dU_1 = -d\,(L_{u_2} U_1) = d(L_{u_1} U_2).$$

Define the *Poisson bracket* by

$$\{U_1, U_2\} = L_{u_1} U_2$$

to obtain

$$U = \{U_1, U_2\} + \text{const}.$$

If $H \in C^0(P)$ is a *Hamiltonian function* and (ψ_t) is the corresponding group of transformations of P (= motion of the classical system) then $U \in C^0(P)$ is a *constant of the motion* iff

$$\psi_t^* U = U \Leftrightarrow L_v U = 0 \Leftrightarrow \{H, U\} = 0,$$

where v is the Hamiltonian vector field: $i(v)\beta = dH$.

If $\dim P = 2n$, then $n!\,\eta = \beta \wedge \beta \wedge \ldots \wedge \beta$ (n times) is a volume form on P and different versions of the *Liouville theorem* read

$$\psi_t^* \eta = \eta, \qquad \operatorname{div} v = 0, \qquad \int_Q \eta = \int_{\psi_t(Q)} \eta \qquad (Q \subset P)$$

Exercise. Consider local coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n)$ on P such that $\beta = dp_i \wedge dq^i$ (such coordinates always exist locally by a theorem due to Darboux). If $i(u)\beta = dU$ then write

$$dU = \frac{\partial U}{\partial p_i} dp_i + \frac{\partial U}{\partial q^i} dq^i$$

and show

$$u = \frac{\partial U}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial U}{\partial p_i} \frac{\partial}{\partial q^i};$$

$$\{U_1, U_2\} = \frac{\partial U_1}{\partial q^i} \frac{\partial U_2}{\partial p_i} - \frac{\partial U_1}{\partial p_i} \frac{\partial U_2}{\partial q^i};$$

$$\eta = dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \wedge \ldots \wedge dp_n \wedge dq^n.$$

IV

LIE GROUPS

A Lie group is a group G, which is also a differential manifold, and such that the map

$$G \times G \ni (a, b) \mapsto a^{-1}b \in G$$

is smooth.

It follows from the definition that, for any $a \in G$, the *left translation*

$$\gamma_a : G \to G, \qquad \gamma_a(b) = ab$$

and the *right translation*

$$\delta_a : G \to G, \qquad \delta_a(b) = ba$$

are diffeomorphisms; the same is true of the internal automorphism of G,

$$\mathrm{ad}_a : G \to G, \qquad \mathrm{ad}_a(b) = aba^{-1}.$$

A vector field $A \in V(G)$ is left-invariant, if for any $a \in G$

$$\gamma_a^* A = A.$$

Since

$$h^*[A, B] = [h^*A, h^*B]$$

for any diffeomorphism h, the set $G' \subset V(G)$ of all left-invariant vector fields on G is closed under the bracket; it forms the *Lie* *algebra* of G. Any $A \in G'$ is defined by its value at the unit element of G:

$$A_a = T_e \gamma_a(A_e).$$

Therefore, G' may be identified with T_eG, and $\dim G' = \dim G$. If (e_i) is a frame in G', then

$$[e_i, e_j] = c_{ij}^k e_k,$$

where c_{ij}^k are the *structure constants* of G. From the properties of the bracket they are skew in (i, j) and satisfy the Jacobi identity.

One-parameter groups of transformations of G generated by elements of G' may be described as follows. Let (φ_t) be a group generated by $A \in G'$; since A is left-invariant, by example 5.2° at the end of chapter III,

$$(\circledast) \qquad \gamma_a \circ \varphi_t = \varphi_t \circ \gamma_a \quad \text{for any} \quad a \in G.$$

Define the exponential map

$$\exp : G' \to G$$

by

$$\exp A = \varphi_1(e).$$

Since sA generates the group φ_{st}, we have $\exp tA = \varphi_t(e)$ and $(\circledast)$ gives:

$$\varphi_t = \delta_{\exp tA}.$$

One shows that exp restricted to a sufficiently small neighbourhood of 0 in G' is a diffeomorphism.

By putting $a = \varphi_s(e)$ in $(\circledast)$ and evaluating both sides of this equation on e, one obtains

$$\varphi_t(e) \cdot \varphi_s(e) = \varphi_{t+s}(e) ;$$

thus

$$\exp tA \cdot \exp sA = \exp (t + s) A.$$

Examples of Lie groups

1. Let V be a real or complex vector space. The group GL(V) of all linear transformations of V is a Lie group. Moreover, GL(V)′ may be identified with $\mathfrak{L}(V)$:

$$[A, B] \text{ in } GL(V)' \text{ corresponds to } AB - BA \text{ in } \mathfrak{L}(V);$$

$$\exp A \text{ in } GL(V)' \text{ corresponds to } I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots \text{ in } \mathfrak{L}(V).$$

2. Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Consider the vector space K^n, where the multiplication by quaternions is taken on the right, i.e. if $q = (q^i) \in K^n$, $\lambda \in K$, then $q\lambda = (q^i\lambda) \in K^n$. Put $\bar{\lambda} = \lambda$ for $K = \mathbb{R}$ and $\bar{\lambda} =$ conjugate of λ for $K = \mathbb{C}$ or $\mathbb{H}$, consider the quadratic form in K^n:

$$Q(q) = \bar{q}^1 q^1 + \dots + \bar{q}^k q^k - \bar{q}^{k+1} q^{k+1} - \dots - \bar{q}^{k+\ell} q^{k+\ell}$$

where $n = k + \ell$. The group G defined by

$$G = \{a \in GL(K^n) \mid Q(a(q)) = Q(q), \text{ for any } q \in K^n\}$$

is

$$G = \begin{cases} O(k,\ell) & \text{for } K = \mathbb{R}; \quad O(n) = O(n, 0) \\ U(k,\ell) & \text{for } K = \mathbb{C}; \quad U(n) = U(n, 0) \\ Sp(k,\ell) & \text{for } K = \mathbb{H}; \quad Sp(n) = Sp(n, 0). \end{cases}$$

The groups O(n), U(n) and Sp(n) are compact.

Note that the quaternionic symplectic group Sp(n) is different from the real symplectic group $Sp_n(\mathbb{R})$ defined in Chapter II.

According to that Chapter, the group U(n) is isomorphic to $O(2n) \cap GL(n, \mathbb{C})$ and to $O(2n) \cap Sp_n$; the isomorphism is obtained by representing the complex matrix

$$a = A + iB \in U(n)$$

by the real $2n \times 2n$ matrix

$$k(a) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and noting that

$$a^{+}a = 1 \Leftrightarrow {}^{t}k(a)\,k(a) = 1\,.$$

In order to obtain similar isomorphisms for Sp(n), consider the representation ℓ of quaternions by complex numbers given by

$$\mathbb{H} \ni \lambda = t + ix + jy + kz \mapsto$$
$$\ell(\lambda) = (t + iz,\ y + ix) = (\xi, \eta) \in \mathbb{C}^2$$

then

$$\ell(\bar{\lambda}) = (\bar{\xi}, -\eta) \text{ and } (\xi_1, \eta_1)(\xi_2, \eta_2) =$$
$$= (\xi_1\xi_2 - \bar{\eta}_1\eta_2,\ \eta_1\xi_2 + \bar{\xi}_1\eta_2).$$

Reminder of basic notions on *quaternions*: any $\lambda \in \mathbb{H}$ is written as $\lambda = t + ix + jy + kz$, where $t, x, y, z \in \mathbb{R}$ and i, j, k are the unit quaternions; they satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$ and cyclic; $\bar{\lambda} = t - ix - jk - kz$ so that $\bar{\lambda}\lambda = t^2 + x^2 + y^2 + z^2$ and any $\lambda \neq 0$ is invertible: $\lambda^{-1} = \bar{\lambda}/\bar{\lambda}\lambda$. There is an obvious representation of quaternion units by Pauli matrices. Namely

$$i \leftrightarrow \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \text{ etc.}$$

Let $a \in GL(\mathbb{H}^n)$, then

$$\ell(a) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$$

is the corresponding representation of a by a pair of complex matrices $A,\ B \in \mathfrak{L}(\mathbb{C}^n)$.

Moreover,

$$a^{+}a = 1 \Leftrightarrow \ell(a) \in U(2n)\,,$$

where $^{+}$ is transpose of the quaternion conjugate matrix, and

$${}^{t}\ell(a)\,j\,\ell(a) = j, \quad \text{where} \quad j = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Therefore

$$\mathrm{Sp(n)} \quad \text{is isomorphic to} \quad \mathrm{U(2n)} \cap \mathrm{Sp_{2n}}(\mathbb{C}),$$

where

$$\mathrm{Sp_m}(\mathbb{C}) = \{f \in \mathrm{GL}(m, \mathbb{C}) \mid {}^t f j f = j\}.$$

3. Cartan's classification of compact non-commutative simple Lie groups (simple means all normal subgroups are discrete or = G itself):

$$\begin{array}{lll} \mathrm{SO(n)} & \text{for} & n = 3 \text{ and } \geq 5; \\ \mathrm{SU(n)} & \text{for} & n \geq 2; \\ \mathrm{Sp\ (n)} & \text{for} & n \geq 1. \end{array}$$

There are 5 'exceptional' groups of dimension 14, 52, 78, 133 and 248.

There are isomorphisms and local isomorphisms (= Lie algebra isomorphisms) among these groups. For example, the map ℓ described above establishes an isomorphism

$$\mathrm{Sp(1)} = \mathrm{SU(2)}.$$

Namely,

$$\lambda = t + ix + jy + kz \in \mathrm{Sp(1)} \Leftrightarrow \bar{\lambda}\lambda = 1 \Leftrightarrow$$

$$\Leftrightarrow \det \begin{pmatrix} t + iz, & -y + ix \\ y + ix, & t - iz \end{pmatrix} = 1;$$

$$\Leftrightarrow \begin{pmatrix} t + iz, & -y + ix \\ y + ix, & t - iz \end{pmatrix} \in \mathrm{SU(2)}.$$

There are also isomorphisms of Lie algebras:

$$\begin{array}{l} \mathrm{Sp(1)'} = \mathrm{SO(3)'}; \\ \mathrm{Sp(1)'} \times \mathrm{Sp(1)'} = \mathrm{SO(4)'}; \\ \mathrm{Sp(2)'} = \mathrm{SO(5)'}; \\ \mathrm{SU(4)'} = \mathrm{SO(6)'}. \end{array}$$

Exercise. Prove

$$\dim SO(n) = \frac{1}{2} n(n-1), \quad \dim SU(n) = n^2 - 1,$$
$$\dim Sp(n) = (2n+1)n.$$

Morphisms of Lie groups; derived morphisms of Lie algebras. Let $h : G \to H$ be a morphism of Lie groups, i.e., a smooth homomorphism. One defines the *derived morphism* of Lie algebras

$$h' : G' \to H'$$

as follows. If $A \in G'$, then $\delta_{\exp tA}$ is the one-parameter group of transformations of G, generated by A; its characteristic property is that it is left-invariant:

$$\delta_{\exp tA} \circ \gamma_a = \gamma_a \circ \delta_{\exp tA} \text{ for any } a \in G.$$

The transformed one-parameter group on H given by $\delta_{h(\exp tA)}$ is also left-invariant; therefore, the vector field h'_A on H, generating $\delta_{h(\exp tA)}$ is left invariant, $h'_A \in H'$. An alternative and equivalent definition is to put $h'_A(f) \circ h = A(f \circ h)$ for any $f \in C^0(H)$. Clearly, $h' : G' \to H'$ is linear and one shows that it is a Lie algebra homomorphism,

(1) $$h'_{[A,B]} = [h'_A, h'_B].$$

Moreover, since h'_A generates $\delta_{h(\exp tA)}$, we have

$$\delta_{\exp t h'_A} = \delta_{h(\exp tA)},$$

thus

(2) $$\exp h'_A = h(\exp A).$$

The adjoint representation

For any $a \in G$, the map $ad_a : G \to G$, $ad_a(b) = aba^{-1}$ is an automorphism of G; the derived morphism

$$ad'_a : G' \to G'$$

is an automorphism of the Lie algebra G′ (clearly, $(\mathrm{ad}'_a)^{-1} = \mathrm{ad}'_{a^{-1}}$). Therefore, if we put $\mathrm{Ad}_a = \mathrm{ad}'_a$, then

$$\mathrm{Ad} : G \to GL(G')$$

is a morphism of Lie groups, called the *adjoint representation* of G in G′.

The derived morphism

$$\mathrm{Ad}' : G' \to GL(G')' \approx \mathfrak{L}(G')$$

is the adjoint representation of the Lie algebra. To compute Ad′, consider G embedded in GL(V) for some vector space V. Then (2) may be written as

$$e^{th'_A} = h(e^{tA}), \qquad h : G \to G \subset \mathfrak{L}(V);$$

thus

$$h'_A = \frac{d}{dt} h(e^{tA}) \bigg|_{t=0}$$

Apply this to Ad:

$$\mathrm{Ad}'_A(B) = \frac{d}{dt} \mathrm{Ad}_{e^{tA}}(B) \bigg|_{t=0} = \frac{d}{dt} e^{tA} B e^{-tA} \bigg|_{t=0} = [A, B].$$

Exercise. Check that condition (1) for h = Ad is the same thing as the Jacobi identity for the Lie bracket.

The *canonical form* on a Lie group G is defined as a Lie algebra valued 1-form, $\tilde{\omega}_G : TG \to G'$, such that, for any $u \in T_aG$, $\tilde{\omega}_G(u)$ is that element of G′ (i.e. a left invariant vector field on G) which coincides with u at $a \in G$:

$$\tilde{\omega}_G(u)_a = u \quad \text{for} \quad u \in T_aG.$$

Here $TG = \bigcup_{a \in G} T_aG$ and we denote by $Th : TG \to TH$ the map given by $Th \mid T_aG = T_ah$.

If $h : G \to H$ is a morphism of Lie groups, then

$$h' \circ \tilde{\omega}_G = \tilde{\omega}_H \circ Th. \tag{3}$$

Moreover,

$$\gamma_a^* \tilde{\omega}_G = \tilde{\omega}_G \quad \text{for any } a \in G, \tag{4}$$

and (3) applied to $h = ad_a$ gives

$$ad_a^* \tilde{\omega} = \tilde{\omega} \circ Tad_a = ad_a' \circ \tilde{\omega} = Ad_a \tilde{\omega},$$

thus

$$\delta_a^* \tilde{\omega}_G = Ad_{a^{-1}} \circ \tilde{\omega}_G. \tag{5}$$

Let α, β be differential forms defined on a manifold M, with values in G′. Let (e_i) be a frame in G′ and put $\alpha = \alpha^i e_i$, $\beta = \beta^i e_i$, where α^i and β^i are $\mathbb{R}$-valued forms on M. Define a G′-valued form $[\alpha, \beta]$ on M by

$$[\alpha, \beta] = \alpha^i \wedge \beta^j [e_i, e_j].$$

For example, if α and β are G′-valued 1-forms, then $[\alpha, \beta]$ is a G′-valued 2-form and

$$[\alpha, \beta](u, v) = [\alpha(u), \beta(v)] - [\alpha(v), \beta(u)].$$

Note that $[\alpha, \alpha] = 0$ if α is an even form, but $[\alpha, \alpha]$ may be different from 0 if α is odd. If α is odd, then $[[\alpha, \alpha], \alpha] = 0$. The space $G' \otimes C(M)$ is a graded Lie algebra.

Consider equation (21) of Chapter III and note $\tilde{\omega}(u) = u$ for $u \in G'$ to prove the Maurer-Cartan equation:

$$d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0. \tag{6}$$

In terms of the frame (e_i), $\tilde{\omega} = e_i \otimes e^i$, and (6) becomes:

$$de^i + \frac{1}{2} c^i_{jk}\, e^j \wedge e^k = 0. \tag{7}$$

The 1-forms (e^i) constitute a frame dual with respect to (e_i), and $\tilde{\omega}(u) = u$ for $u \in G'$, is equivalent to

$$e^i(e_j) = \delta^i_j.$$

Alternatively, one can define $\tilde{\omega} \wedge \tilde{\omega}$ by

$$\tilde{\omega} \wedge \tilde{\omega}(u, v) = [\tilde{\omega}(u), \tilde{\omega}(v)]$$

and write the Maurer-Cartan equation as

$$d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = 0. \tag{8}$$

Invariant metrics on Lie groups

A metric g on G is *left invariant* if $\gamma_a^* g = g$ for any $a \in G$; it is *right invariant* if $\delta_a^* g = g$; it is *biinvariant* if it is both left and right invariant.

Let

$$k : G' \times G' \to \mathbb{R}$$

be a scalar product in the Lie algebra of G, then g defined on G by $g(u, v) = k(\tilde{\omega}(u), \tilde{\omega}(v))$ (symbolically, $g = k \circ \tilde{\omega}$) is left invariant: indeed,

$$\gamma_a^* g = g \circ T\gamma_a = k \circ \tilde{\omega} \circ T\gamma_a = k \circ \gamma_a^* \tilde{\omega} = k \circ \tilde{\omega} = g \text{ by (5).}$$

Let us check when g is right invariant:

$$\delta_a^* g = g \circ T\delta_a = k \circ \tilde{\omega} \circ T\delta_a = k \circ \mathrm{Ad}_{a^{-1}} \circ \tilde{\omega},$$

where formula (5) has been used. Therefore, $k \circ \tilde{\omega}$ is biinvariant $\Leftrightarrow k \circ \mathrm{Ad}_a = k$ for any $a \in G$. For example, if G is Abelian, then g is biinvariant for any choice of k.

For an arbitrary Lie group G one defines its *Killing form* by

$$K(A, B) = \mathrm{Tr}(\mathrm{Ad}'_A \circ \mathrm{Ad}'_B).$$

Recall that, for any $f_1, f_2 \in \mathcal{L}(V)$ and $a \in GL(V)$, $Tr(f_1 \circ f_2) = Tr(f_2 \circ f_1)$ and $Tr(a \circ f_1 \circ a^{-1}) = Tr f_1$. Therefore, K is symmetric, $K(A, B) = K(B, A)$, and invariant, $K \circ Ad_a = K$. By definition, a group G is *semi-simple* iff its Killing form is non degenerate.

In that case $g = K \circ \tilde{\omega}$ is a (pseudo)Riemannian, biinvariant metric on G. Consider a frame (e_i) in G', $[e_i, e_j] = c^k_{ij} e_k$, write $g = g_{ij}\ e^i \otimes e^j$, thus:

$$g_{ij} = g(e_i, e_j) = K(e_i, e_j) = Tr(Ad'_{e_i} \circ Ad'_{e_j}).$$

Then

$$Ad'_{e_i} \circ Ad'_{e_j}(e_k) = [e_i, [e_j, e_k]] = c^\ell_{jk}\, c^m_{i\ell}\, e_m.$$

Thus

$$g_{ij} = c^k_{i\ell}\, c^\ell_{jk}. \tag{9}$$

One can prove that if G is compact then (g_{ij}) is negative definite.

The property of $g = k \circ \tilde{\omega}$ to be biinvariant is easily expressed in terms of the structure constants. Clearly, $k \circ Ad_a = k$ implies $k \circ Ad_{\exp tA} = k$, where $A \in G'$ and $t \in \mathbb{R}$. By differentiating the last equation with respect to t at $t = 0$ one obtains:

$$(*) \quad k([A, B], C) + k(B, [A, C]) = 0 \text{ for any } A, B, C \in G'.$$

Putting

$$k_{ij} = k(e_i, e_j) \quad \text{and} \quad c_{ijk} = k_{i\ell} c^\ell_{jk},$$

one obtains from (*):

$$c_{jki} + c_{ikj} = 0,$$

so that invariance of k is equivalent to complete antisymmetry of c_{ijk}.

SPACES

We define a *space* as a differential manifold M on which a Lie group G acts as a group of transformations, i.e. there is a smooth map

$$\psi : M \times G \to M$$

such that

$$\psi_a \circ \psi_b = \psi_{ba}, \text{ and } \psi_e = \mathrm{id}_M, \tag{10}$$

where $\psi_a(p) = \psi(p, a)$, $p \in M$; $a, b \in G$, and $e \in G$ is the unit element.

One often writes pa instead of $\psi_a(p)$, and one says that condition (10) means that G acts in M on the right. To emphasize the role of the group G one says that M is a G-*space*.

For any $p \in M$, the subset of M

$$pG = \{\psi_a(p) \mid a \in G\}$$

is the *orbit* of p, and

$$G_p = \{a \in G \mid \psi_a(p) = p\}$$

is a closed subgroup of G, called the *isotropy group* of $p \in M$.

If $pG = M$ then we have a *homogeneous space* (the action of G on M is transitive).

If, for any $p \in M$, $G_p = \{e\}$, then G is said to *act freely* on M and M is a *principal* space.

Let M be a homogeneous space and $o \in M$. Consider the quotient (right coset space):

$$G/G_o = \{G_o a \mid a \in G\}.$$

G/G_o is also a space: the group G acts on G/G_o by right translations : $G/G_o \times G \to G/G_o$, $\delta(G_o b, a) = G_o ba$ for $a, b \in G$.

Clearly, G/G_o is homogeneous under this action. One shows that G/G_o has a natural structure of a differential manifold, δ is smooth and the bijection

$$G/G_o \ni G_o a \mapsto \psi_a(o) \in M$$

is a diffeomorphism h which is equivariant with respect to the action of G in the sense that the diagram

$$\begin{array}{ccc} G/G_o \times G & \xrightarrow{\delta} & G/G_o \\ {\scriptstyle h \times \mathrm{id}_G} \downarrow & & \downarrow {\scriptstyle h} \\ M \times G & \xrightarrow{\psi} & M \end{array}$$

commutes. If o' is another point of M, then $o' = \psi_a(o)$ for some $a \in G$; therefore $G_{o'} = a^{-1} G_o a$ and the spaces G/G_o and $G/G_{o'}$ are diffeomorphic: conjugate closed subgroups of G lead to the same homogeneous spaces; all homogeneous G-spaces may be obtained by considering quotients G/H, where $H \subset G$ is closed.

The assumption that $H \subset G$ be closed is essential; e.g. consider the torus $G = U(1) \times U(1)$ and its subgroup H_λ consisting of pairs of complex numbers $(e^{2\pi i t}, e^{2\pi i \lambda t})$, where $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is fixed. If λ is rational, then H_λ is isomorphic to $U(1)$ and closed in G; the quotient G/H_λ is isomorphic to $U(1)$. But if λ is irrational, H_λ is isomorphic to $\mathbb{R}$ and 'fills the torus': $H_\lambda \neq \overline{H_\lambda} = G$; the quotient G/H_λ is a 'pathological' space and is not a manifold in any natural sense.

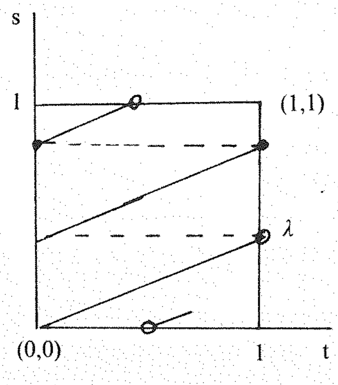


FIGURE 11

Returning to the regular case, consider a space M; the map $\psi : M \times G \to M$ defines a homomorphism of Lie algebras

$$\psi' : G' \to V(M) \tag{11}$$

given by

$\psi'_A =$ vector field on M induced by the one-parameter group $\psi_{\exp tA}$ of transformations of M.

If $a \in G$, then $\psi_{\exp t \operatorname{Ad}_a A}$ induces $\psi'_{\operatorname{Ad}_a A}$. From formula (2) one has

$$\exp t \operatorname{Ad}_a A = \operatorname{ad}_a \exp t A.$$

Therefore

$$\psi_{\exp t \operatorname{Ad}_a A} = \psi_{a^{-1}} \circ \psi_{\exp tA} \circ \psi_a \text{ induces } \psi_a^* \psi'_A,$$

and

$$\psi'_{\operatorname{Ad}_a A} = \psi_a^* \psi'_A. \tag{12}$$

If the action of G in M is regular * then one can form the quotient manifold M/G: its elements (points) are orbits of G in M.

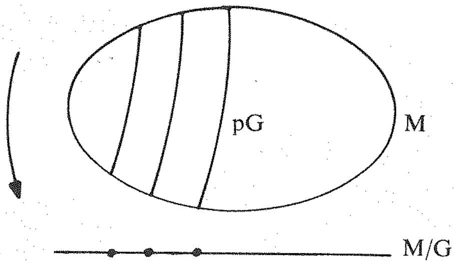


FIGURE 12

* To define a regular action of G in M consider $R \subset M \times M$ corresponding to the equivalence relation given by the group: $(x, y) \in R \Leftrightarrow \exists a \in G \; y = \psi_a(x)$; let $pr_2 : R \to M$ be the (second) projection, $pr_2(x, y) = y$. The action is regular if R is a submanifold of $M \times M$ and the tangent map to pr_2 is everywhere surjective. These assumptions are necessary to dispose of the pathologies such as the one concerning the embedding of $\mathbb{R}$ in the torus. For further details, see J. P. Serre, Lie Algebras and Lie Groups, Benjamin, New York 1965.

If the action of G is free then the orbits are diffeomorphic to G and $\dim M/G + \dim G = \dim M$.

Example. *Hopf spaces* associated with $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Consider the vector space K^{n+1}; in the case of quaternions define multiplication by numbers on the right.

Consider the positive-definite quadratic form Q on K^{n+1}:

$$Q(q) = \bar{q}^0 q^0 + \bar{q}^1 q^1 + \ldots + \bar{q}^n q^n.$$

Remark. Most of the following considerations apply to the more general, indefinite quadratic forms defined at the beginning of this Chapter.

Let

$$S_n(K) = \{q \in K^{n+1} \mid Q(q) = 1\};$$

then

$$S_n(K) = \begin{cases} \mathbb{S}_n & \text{for } K = \mathbb{R} \\ \mathbb{S}_{2n+1} & K = \mathbb{C} \\ \mathbb{S}_{4n+3} & K = \mathbb{H}, \end{cases}$$

and if

$$G_{n+1}(K) = \{a \in GL(K^{n+1}) \mid Q(aq) = Q(q)\}$$

then

$$G_{n+1}(K) = \begin{cases} O(n+1) & \text{for } K = \mathbb{R} \\ U(n+1) & K = \mathbb{C} \\ Sp(n+1) & K = \mathbb{H}. \end{cases}$$

Theorem. $S_n(K)$ is a homogeneous $G_{n+1}(K)$-space. (Reference: N. Steenrod, Topology of Fibre Bundles, Princeton University Press, Princeton 1965). The proof is based on the 'Schmidt orthogonalization procedure': take the canonical frame in K^{n+1}:

$$\begin{aligned} e_0 &= (1,0,\ldots,0) \\ e_1 &= (0,1,\ldots,0) \\ &\ldots\ldots\ldots\ldots \\ e_n &= (0,0,\ldots,1) \end{aligned}$$

and an arbitrary $z_0 \in S_n(K)$; complete z_0 to a Q-orthonormal frame $z_0, z_1, \ldots, z_n$; there is then an element a of $G_{n+1}(K)$ sending e into z, $z_i = a(e_i)$, thus $z_0 = a(e_0)$; this proves transitivity of the action of $G_{n+1}(K)$ on $S_n(K)$. Moreover, the isotropy group of e_0 consists of all elements of $G_{n+1}(K)$ of the form:

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad \text{where } b \in G_n(K).$$

Therefore, if we put $\pi(a) = a(e_0)$, then π and the canonical map $G_{n+1}(K) \to G_{n+1}(K)/G_n(K)$ identifies $S_n(K)$ with $G_{n+1}(K)/G_n(K)$. Consider next the *projective space* (cf. example 4 at the beginning of Chapter III):

$$K\mathbb{P}_n = K^{n+1} - \{0\} \Big/ K - \{0\}$$

where $K^{n+1} - \{0\}$ is considered as a space relative to the multiplicative group $K - \{0\}$: elements of $K\mathbb{P}_n$ are 'directions' in K^{n+1}; depending on whether $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, these directions are one, two or four dimensional in the real sense.

The sphere $S_n(K)$ may be also regarded as a $G_1(K) \subset K - \{0\}$ space: if $\lambda \in G_1(K)$, i.e. if $\bar{\lambda}\lambda = 1$, then $Q(q) = 1 \Rightarrow Q(q\lambda) = 1$. Let $p(q) \in K\mathbb{P}_n$ be the direction containing $q \in S_n(K)$; the map

$$p : S_n(K) \to K\mathbb{P}_n$$

is surjective ('the sphere meets all directions passing through

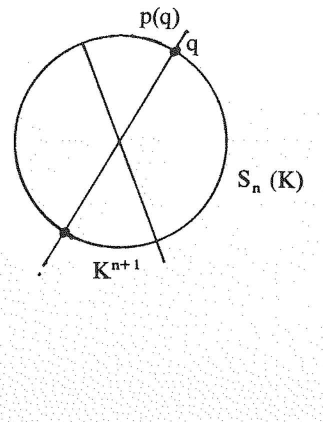


FIGURE 13

its origin'), and

$$p(q) = p(q') \Leftrightarrow \exists_{\lambda \in G_1(K)} \qquad q' = q\lambda.$$

Therefore, $K\mathbb{P}_n$ is the quotient manifold of $S_n(K)$ by $G_1(K)$. We have a sequence of maps (projections onto quotients)

$$(13) \qquad G_{n+1}(K) \xrightarrow{\pi} G_{n+1}(K)/G_n(K) = S_n(K) \xrightarrow{p} K\mathbb{P}_n.$$

Clearly,

$$G_1(K) = \begin{cases} O(1) = \mathbb{Z}_2 & \text{for} \quad K = \mathbb{R} \\ U(1) & \text{for} \quad K = \mathbb{C} \\ Sp(1) \approx SU(2) & \text{for} \quad K = \mathbb{H}. \end{cases}$$

The last two groups are structure (gauge) groups of electromagnetism and the Yang-Mills theory, respectively. The Hopf spaces are closely related to the 'topologically non-trivial' solutions of Maxwell ($K = \mathbb{C}$) and Yang-Mills equations. For example, the Hopf map $S_3 \to S_2 (K = \mathbb{C}, n = 1)$ carriers a geometry corresponding to the Dirac pole of lowest strength $g = 1/2e$, whereas $S_7 \to S_4 (K = \mathbb{H},\ n = 1)$ is associated with the BPST 'instanton' solution of the Yang-Mills equations, as will be shown in detail in Chapter VII.

Example 1. Some details on the classical groups $G_n(K)$, $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Write a^+ for the conjugate transpose of $a \in G_n(K)$; in the real case $a^+ = {}^t a$. The defining equation of $G_n(K) = G$

$$Q(aq) = Q(q)$$

is equivalent to $a^+ a = 1$.

Writing $a = \exp tA$, $A \in \mathfrak{L}(K^n)$, we obtain

$$(14) \qquad A \in G' \Leftrightarrow A^+ + A = 0.$$

The matrix elements a^i_j of $a \in G$ are K-valued functions on the differential manifold G; therefore da^i_j are 1-forms on G, and we write

$$da = (da^i_j)\,.$$

Consider a^+da; this 1-form is G′-valued because

$$(da^+)a + a^+da = 0 \quad \text{implies} \quad (a^+da)^+ + a^+da = 0\,.$$

Moreover, a^+da is left-invariant on G: if $b \in G$, then

$$(ba)^+ d(ba) = a^+b^+bda = a^+da\,,$$

and

$$\langle A,\ a^+da\rangle_{a=1} = A\,.$$

Therefore,

$$\tilde{\omega} = a^+da$$

is the canonical form on $G_n(K)$; we may check the Maurer-Cartan equation (8) by noting that

$$d\tilde{\omega} = da^+ \wedge da = -a^+da \wedge a^+da = -\tilde{\omega} \wedge \tilde{\omega}\,.$$

Example 2. The group $Sp(1) = SU(2)$ as a manifold is homeomorphic to S_3:

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = a \in SU(2) \Leftrightarrow |\, z_1\,|^2 + |\, z_2\,|^2 = 1\,,$$

where $z_1, z_2 \in \mathbb{C}$. A convenient system of local coordinates on $\mathbb{S}_3$ is given by the Euler angles χ, ϑ, φ:

$$z_1 = e^{\frac{i}{2}(\chi+\varphi)} \cos\frac{\vartheta}{2}, \qquad z_2 = e^{\frac{i}{2}(\chi-\varphi)} \sin\frac{\vartheta}{2}.$$

The canonical form on SU(2) is

$$\tilde{\omega} = \mathrm{a}^{+}\mathrm{da} = \frac{\mathrm{i}}{2}\begin{pmatrix} \zeta & \xi - \mathrm{i}\eta \\ \xi + \mathrm{i}\eta & -\zeta \end{pmatrix},$$

where

$$\xi + \mathrm{i}\eta = \mathrm{e}^{\mathrm{i}\varphi}\,(\mathrm{i}\,\mathrm{d}\vartheta + \sin\vartheta\,\mathrm{d}\chi), \qquad \zeta = \mathrm{d}\varphi + \cos\vartheta\,\mathrm{d}\chi\,,$$

and the Maurer-Cartan equations are $\mathrm{d}\xi = \eta \wedge \zeta$, etc.

V

FIBRE BUNDLES

Heuristic considerations

A map (function)

$$f : M \to N$$

is often represented by its *graph*, defined as the image of M under

$$\tilde{f} : M \to E \underset{\text{def}}{=} M \times N$$

where

$$\tilde{f}(p) = (p, f(p)).$$

The construction of the graph of f requires the introduction,

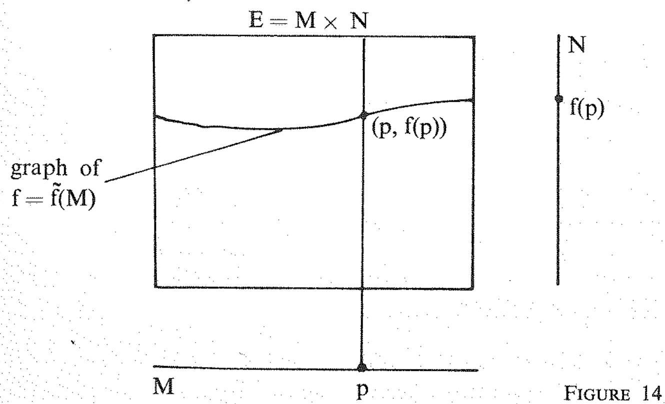


Figure 14

in a trivial manner, of (some of) the ingredients of a fibre bundle:

E	is the *total space* of the bundle (one often says simply that E is the bundle);
M	is its *base*;
$\pi : E \to M$	given by $\pi(p, q) = p$ is the *projection*;
N	is the *typical fibre*;
$E_p = \pi^{-1}(p) \subset E$	is the *fibre* over $p \in M$;
$\tilde{f} : M \to E$	is a (global) section of π, $\pi \circ \tilde{f} = id_M$.

Of course, this is a trivial example which does not justify the introduction of all this new terminology.

Consider next a vector field u on a manifold M and assume first that M admits a global coordinate system x. Relative to this coordinate system,

$$u = u^i(p) \left(\frac{\partial}{\partial x^i}\right)_p, \qquad p \in M$$

and the vector field is given by a map

$$(u^i) : M \to \mathbb{R}^n = N. \tag{1}$$

This is the classical (XIXth Century), coordinate description of vector fields.

But it is deficient for at least three reasons:

1° it is coordinate dependent;

2° vector spaces at different points of the manifold are distinct, whereas they are identified with $\mathbb{R}^n$ in (1);

3° in some important cases, M does not admit global coordinates (e.g. if $M = \mathbb{S}_2$).

There is a way out: first define the tangent vector space T_pM at $p \in M$ (cf. Chapter III), put

$$TM = \bigcup_{p \in M} T_pM.$$

Define an obvious projection $\pi : TM \to M$ by $\pi(u) = p$ if $u \in T_pM$.

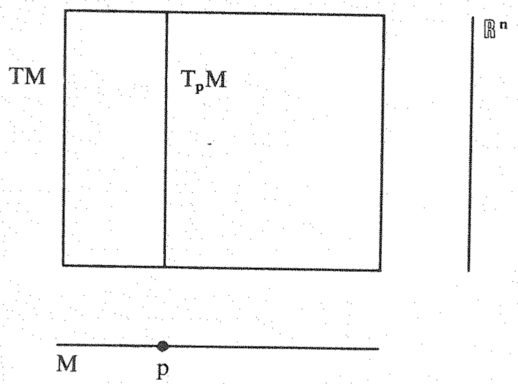


FIGURE 15

A vector field u on M defines a section of π,

$$u : M \to TM; \quad u(p) \in T_pM \Leftrightarrow \pi \circ u = \mathrm{id}_M. \tag{2}$$

But how can we describe in terms of the map (2) the smooth character of u? So far, TM is only a set and u(p) may vary wildly with p. To solve the problem, one makes TM into a differential manifold by constructing an atlas as shown below. Consider a chart $x : U \to \mathbb{R}^n$ on M and define a map

$$\hat{x} : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

as follows. Let $u \in \pi^{-1}(U) \subset TM$, then $u \in T_pM$ for some $p \in M$ and

$$u = u^i \left(\frac{\partial}{\partial x^i}\right)_p.$$

Put

$$\hat{x}(u) = (x^1(p), \ldots, x^n(p), u^1, \ldots, u^n).$$

The map $\hat{x}$ is bijective; moreover, if x and y are compatible charts on M, then $\hat{x}$ and $\hat{y}$ are compatible on TM; in other words, the hat operation lifts an atlas on M to an atlas on TM and makes the latter into an 2n-dimensional differential manifold, called the *tangent bundle* of M. It is now not difficult to check that

$$\left.\begin{array}{l}\text{a vector field on M is smooth}\\ \text{(i.e. defined as in Chapter III)}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}u : M \to TM \text{ is a smooth}\\ \text{section of the tangent bundle}\end{array}\right.$$

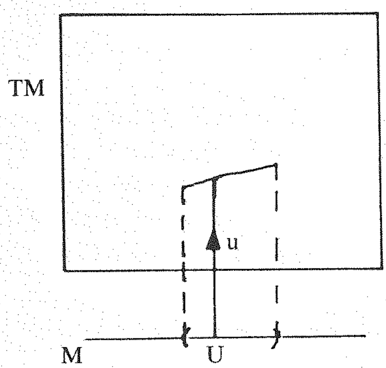


FIGURE 16

A vector field on $U \subset M$ is described by a section $u : U \to TM$, $\pi \circ u = id_u$.

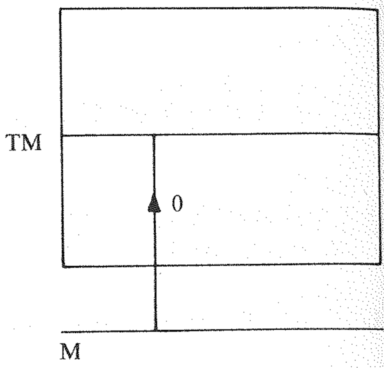


FIGURE 17

The tangent bundle always admits global sections; one of them is the zero section $0 : M \to TM$ (zero vector field on M).

More formally,

$$\pi : E \to M$$

is a fibre bundle with typical fibre N, if π (the *projection*) is a smooth surjective map of differential manifolds and any $p \in M$ has a neighbourhood U and a diffeomorphism

$$h : U \times N \to \pi^{-1}(U),$$

such that $\pi(h(p, y)) = p$ (*local triviality* of the fibre bundle: the portion of E over a sufficiently small $U \subset M$ looks like $U \times N$).

A bundle is *trivial* if there is a diffeomorphism $h\colon M \times N \to E$ such that

$$\pi(h(p, q)) = p$$

for any $p \in M$ and $q \in N$.

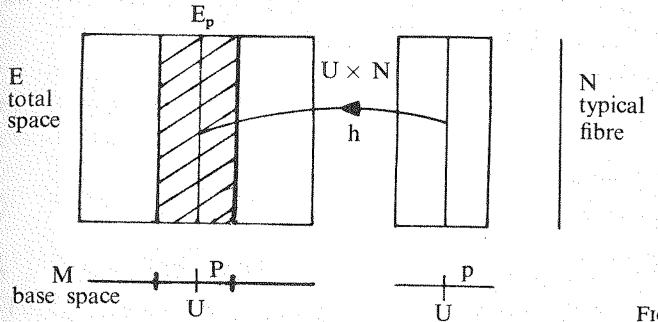


FIGURE 18

Clearly, if M admits a global coordinate system, then TM is a trivial bundle ($\hat{x}$ defined above may be used to construct h), but the converse is not true: $T\mathbb{S}_3 = TSU(2)$ is trivial (why?) though $\mathbb{S}_3$ has no global chart.

OTHER EXAMPLES OF BUNDLES

1. (Important and general). If $G \subset H$, and is a closed Lie subgroup of H, then there is the bundle

$$\pi : H \to H/G = M.$$

2. In particular, for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we have the bundles over spheres

$$G_{n+1}(K) \to G_{n+1}(K)/G_n(K) = \mathbb{S}_{(n+1)\dim K - 1},$$

and also the Hopf bundles over projective spaces.

3. The map

$$\pi_n : U(1) \to U(1), \quad \pi_n(e^{it}) = e^{int}, \quad n = 1, 2, 3, \ldots$$

defines U(1) as a bundle over U(1); the typical fibre $\mathbb{Z}_n$ may be identified with the n-element set (group) of nth order roots of 1. (Such a set may be considered as a 0-dimensional differential manifold).

This bundle is trivial only for $n = 1$.

A related example is the (universal covering) bundle over U(1), $\pi : \mathbb{R} \to U(1)$, $\pi(t) = e^{2\pi i t}$. Here the typical fibre may be identified with $\mathbb{Z}$.

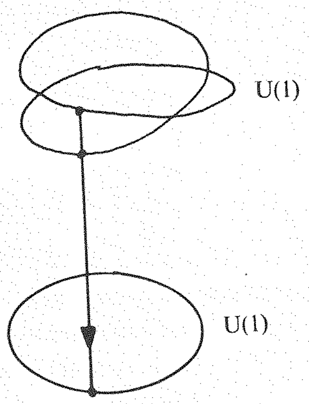


FIGURE 19

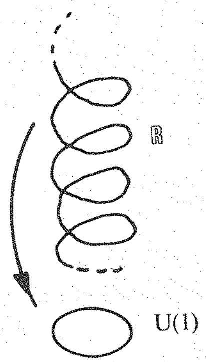


FIGURE 20

4. *Frame bundle*: the set $FM = \bigcup_{p \in M} F(T_pM)$ of all frames of M can be made into a bundle over M; $\pi : FM \to M$ maps a frame e into the point p of M at which the frame is attached; if $x : U \to \mathbb{R}^n$ is a chart, then $\hat{x} : \pi^{-1}(U) \to \mathbb{R}^n \times GL(n, \mathbb{R})$ is defined by $\hat{x}(e) = (x^1(p), \ldots, x^n(p), a^i_j)$ where $a^i_j = \langle e_j, (dx^i)_p \rangle$; $\hat{x}$ is bijective and the hat operation lifts an atlas on M to an atlas on FM.

Fibre bundles occurring in examples 1-4 above all have *groups* as their *typical fibres*; these *groups act freely and transitively on the fibres*, and, in each case, the base manifold may be considered as the quotient of the total space by the action of the group. Such fibre bundles are called *principal*; other bundles may be obtained from principal bundles by a construction paralleling the one of geometric objects of type ρ over a vector space (cf. Chapter II).

Formally, a fibre bundle $\pi : P \to M$ is a *principal bundle* if its typical fibre is a Lie group G which acts freely on P,

$$\psi : P \times G \to P, \quad \psi_a \circ \psi_b = \psi_{ba},$$

the action of G is transitive on the fibres,

$$\pi(e) = \pi(e') \Leftrightarrow \exists_{a \in G}\; e' = \psi_a(e) \text{ for } e, e' \in P,$$

and for any $p \in M$ there is a neighbourhood $U \subset M$ and a diffeomorphism $h : U \times G \to \pi^{-1}(U)$ such that

$$(2) \qquad \pi(h(p, a)) = p \quad \text{and} \quad h(p, ba) = \psi_a(h(p, b))$$

for any $p \in U$, $a, b \in G$.

The group G is the structure group of the bundle $\pi : P \to M$.

Theorem. A principal bundle admits a section iff it is trivial (Exercise: prove the theorem).

Remark: this is not true for other bundles.

EXAMPLES OF PRINCIPAL BUNDLES

1. If G is a closed Lie subgroup of H, then $\pi : H \to H/G$ is a principal bundle with G as the structure group.
2. In particular,

$$(3) \quad \begin{cases} O(n+1) \to \mathbb{S}_n & \text{is a principal bundle with } O(n) \text{ as the structure group;} \\ U(n+1) \to \mathbb{S}_{2n+1} & \text{is a principal bundle with } U(n) \text{ as the structure group;} \\ Sp(n+1) \to \mathbb{S}_{4n+3} & \text{is a principal bundle with } Sp(n) \text{ as the structure group.} \end{cases}$$

Also, one has the Hopf fibrations,

$\mathbb{S}_n \to \mathbb{RP}_n$ is a principal bundle with $\mathbb{Z}_2$ as the structure group;

(4) $\mathbb{S}_{2n+1} \to \mathbb{C}\mathbb{P}_n$ is a principal bundle with U(1) as the structure group;

$\mathbb{S}_{4n+3} \to \mathbb{H}\mathbb{P}_n$ is a principal bundle with Sp(1) = SU(2) as the structure group.

3. $\pi_n : U(1) \to U(1)$ is a principal bundle with

$$\mathbb{Z}_n \approx \{1, e^{\frac{2\pi i}{n}}, e^{2 \cdot \frac{2\pi i}{n}}, \ldots, e^{(n-1)\frac{2\pi i}{n}}\}$$

as the structure group; action of the group is ordinary multiplication.

$\pi : \mathbb{R} \to U(1)$, $\pi(t) = e^{2\pi i t}$ is a principal bundle with $\mathbb{Z}$ as the structure group; action is $\psi_a(t) = t + a$, where $t \in \mathbb{R}$ and $a \in \mathbb{Z}$.

4. FM → M, where M is n-dimensional, has $GL(n, \mathbb{R})$ as the structure group; the action is as usual, $\psi_a(e) = ea = (e_j\, a^j_i)$.

Associated bundles.

Consider now the construction of geometric objects of type ρ described in Chapter II. Given a representation,

$$\rho : GL(n, \mathbb{R}) \to GL(m, \mathbb{R})$$

and an n-dimensional vector space V, one constructs

$$V \rightsquigarrow F(V) \rightsquigarrow \rho(V).$$

Apply this to $V = T_pM$, dim M = n, and 'bundle up' = form $\bigcup_{p \in M}$ to get a new bundle over M:

$$T_pM \rightsquigarrow F(T_pM) \rightsquigarrow \rho(T_pM)$$

$$TM \rightsquigarrow FM \rightsquigarrow \rho M$$

We shall now describe this construction directly, in the general case of a principal bundle $\pi : P \to M$ with group G acting also in N:

$$\rho : G \times N \to N$$
$$\rho_a \circ \rho_b = \rho_{ab}, \ \rho_a(q) = \rho(a, q); \quad a, b \in G, \ q \in N.$$

Form $P \times N$ and define an action of G in $P \times N$ by

$$(e, q)a = (\psi_a(e), \ \rho_{a^{-1}}(q)).$$

Let E be the quotient of $P \times N$ by the action of G, and $k : P \times N \to E$ be the canonical map:

$$k(e,q) = k(e', q') \Leftrightarrow \exists_{a \in G} \ e' = \psi_a(e) \text{ and } q' = \rho_{a^{-1}}(q).$$

There is a natural projection:

$$\pi_E : E \to M \text{ given by } \pi_E(k(e,q)) = \pi(e), \ e \in P,$$

and E can be made into a differential manifold; π_E is smooth. If $h : U \times G \to \pi^{-1}(U)$ defines a local trivialization of $\pi : P \to M$, in the sense of (2), then $\ell : U \times N \to \pi_E^{-1}(U)$ defined by $\ell(p, q) = (h(p, 1), q)$ provides a local trivialization of $\pi_E : E \to M$ (here 1 denotes the unit element of G).

To sum up: given a principal bundle $\pi : P \to M$ with structure group G and a (left) G-space N, one constructs as associated fibre bundle $\pi_E : E \to M$ whose typical fibre is N and $E = (P \times N)/G$.

G is also said to be the structure group of the associated bundle.

Examples of fibre bundles *associated* with principal bundles.

1. Let $\pi : FM \to M$ be the frame bundle of an n-dimensional manifold M and let $\rho : GL(n, \mathbb{R}) \to GL(m, \mathbb{R})$ be a morphism of Lie groups; put $\rho M = E$ and $\pi_\rho = \pi_E$, then $\pi_\rho : \rho M \to M$ is the bundle of geometric objects of type ρ over M. In particular, if

   $\rho = \text{id}$ then ρM may be identified with TM (tangent bundle);

$\rho = \check{\mathrm{id}}$ then ρM mat be identified with T*M (cotangent bundle);

$\rho = \mathrm{ad}$ then ρM may be identified with the bundle of mixed tensors (tensors of valence (1,1));

$\rho = \Lambda^k \check{\mathrm{id}}$ then ρM may be identified with the bundle of k-forms, $\Lambda^k T^*M$.

(5) $(\rho_1 \oplus p_2)M$ is denoted $\rho_1 M \oplus \rho_2 M$ and called the *Whitney sum* of the bundles, and

(6) $(\rho_1 \otimes \rho_2)M$ is denoted $\rho_1 M \otimes \rho_2 M$; it is the *tensor product* of bundles.

All these bundles, obtained from FM by linear representations of GL(n, $\mathbb{R}$), are examples of *vector bundles*: their fibres are vector spaces which, under any local trivialization, are isomorphic to the typical fibre (also a vector space). In general, bundles are named after the properties of their fibres. Clearly,

$$\dim E = \dim M + \dim N.$$

2. Consider $\pi_2 : U(1) \to U(1)$ defined previously, $\pi_2(z) = z^2$, $N = [-1, 1] \subset \mathbb{R}$, and consider the action of the structure group of π_2, $G = \mathbb{Z}_2 \approx (1, -1\}$ defined by ordinary multiplication. The associated bundle E over U(1) is called the *Möbius band*.

Description of *sections of associated bundles*.

Sections of these bundles are important in physics and geometry: tensor fields and wave functions are examples of such sections. There are two equivalent ways of describing sections. From its definition, a section is a map.

$$s : M \to E \quad \text{such that} \quad \pi \circ s = \mathrm{id}_M. \tag{7}$$

Define (cf. Chapter II)

$$\bar{s} : P \to N \quad \text{by} \quad \bar{s}(e) = k_e^{-1}(s(\pi(e))), \tag{8}$$

where

(9) $$k_e : N \to \pi_E^{-1}(\pi(e)), \quad k_e(q) = k(e, q)$$

is a diffeomorphism between N and the fibre of E over $\pi(e)$, defined by e. From the definition of the canonical map $k : P \times N \to E$, it follows that

(10) $$\bar{s}(ea) = \rho_{a^{-1}} \circ \bar{s}(e).$$

This is the 'transformation law' of classical tensor analysis.

Conversely, if $\bar{s} : P \to N$ satisfies (10), then $s(p) = k(e, \bar{s}(e))$, $e \in \pi^{-1}(p)$, defines a section (7).

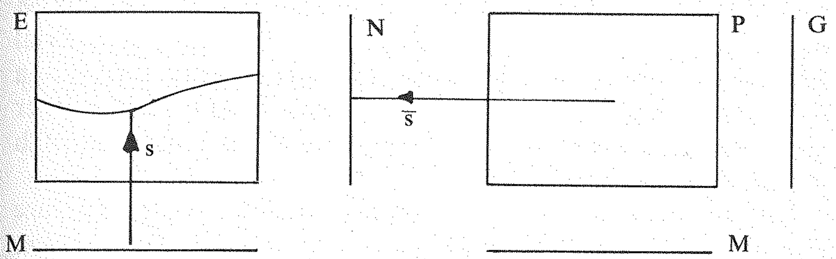


FIGURE 21

Two ways of representing a section of the associated bundle $\pi_E : E \to M$.

Interpretation: $\bar{s}(e)$ are the components of the field s, at the point $\pi(e) \in M$, with respect to the (generalized) frame $e \in P$. Under a change of the frame, $e \mapsto ea$, the components transform according to the representation ρ.

MORPHISMS OF FIBRE BUNDLES

For each 'category' of mathematical objects, there are natural transformations among these objects, which 'agree' with their structure; e.g. for groups, these maps are homomorphisms, for vector spaces they are linear maps, for differential manifolds they are smooth maps. We now define such 'morphisms' for fibre bundles.

Let $\pi_i : E_i \to M_i$ $(i = 1,2)$ be two fibre bundles; to alleviate the language, we refer to π_i as the bundle. A pair (h, f) of smooth maps is called a *morphism* from π_1 to π_2 if the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{h} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes. A bundle $\pi : E \to M$ with typical fibre N is called *trivial* if there is a morphism (h, id_M) of π to $pr_1 : M \times N \to M$, such that h is a diffeomorphism. For example, if the manifold M admits a global coordinate system x, then $\pi : TM \to M$ is trivial because $h : TM \to M \times \mathbb{R}^n$, given by $h(u) = (\pi(u), u(x^i))$, is a diffeomorphism. A morphism of a vector bundle into another such bundle is a *vector bundle morphism* if the restriction of h to any fibre is linear.

Let $\pi_i : P_i \to M_i$ be two principal bundles with structure groups G_i acting in P_i according to $\psi_i : P_i \times G_i \to P_i$. A *morphism of principal bundles* is a triple (h, k, f) of maps $h : P_1 \to P_2$, $k : G_1 \to G_2$, $f : M_1 \to M_2$, where k is a Lie group morphism, and the diagram

$$\begin{array}{ccc} P_1 \times G_1 & \xrightarrow{h \times k} & P_2 \times G_2 \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle \psi_2} \\ P_1 & \xrightarrow{h} & P_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes.

In the case $M_1 = M_2$, one usually restricts f *to be the identity map*. The most interesting cases are those for which h and k are both injective or both surjective. (More precisely, one requires that not only h and k but their tangent maps should enjoy this property; this is to exclude maps such as $x \mapsto x^3$).

If both h and k are injective, then π_1 is a *restriction* of π_2 or π_2 is an *extension of* π_1.

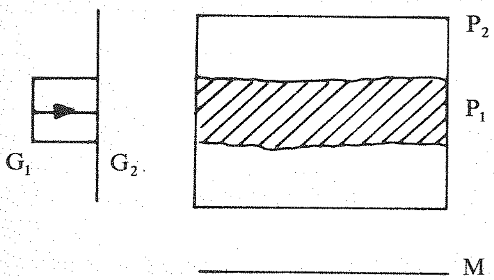


FIGURE 22

If both h and k are surjective, then π_1 is a *prolongation* of π_2 or π_2 is a *reduction* of π_1.

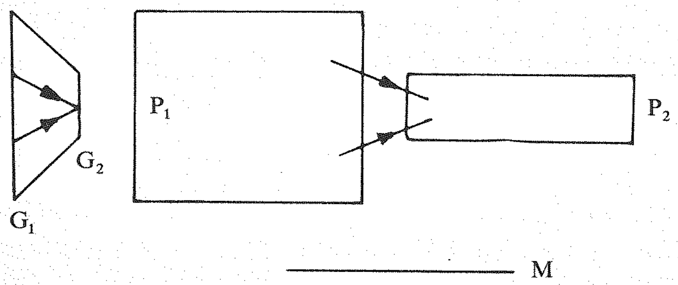


FIGURE 23

These notions are important since they occur frequently in geometry and physics.

Examples.

1. The bundle of *orthonormal frames* of a proper Riemannian manifold (M, g) is a *restriction* of the *bundle* of all *linear frames* $FM = P_2$. In this case $G_1 = O(n)$, $G_2 = GL(n, \mathbb{R})$. More precisely,

to give a proper Riemannian metric g an n-dimensional manifold M	$\Leftrightarrow$	to restrict FM to a subbundle $P_1 \subset FM$, the structure group of P_1 being $O(n)$.

If g is given, one defines P_1 as the set of all orthonormal frames relative to g; conversely, if P_1 is given and $e \in P_1$, then one defines the metric tensor at $\pi(e)$ by

$$g_{\pi(e)} = e^1 \otimes e^1 + \ldots + e^n \otimes e^n.$$

2. The bundle of *affine frames* of M is an *extension* of the bundle of linear frames $P_2 = FM$, corresponding to the injection

$$k : GL(n, \mathbb{R}) \to GA(n, \mathbb{R})$$

of the general linear group into the *general affine group* defined as follows:

$$GA(n, \mathbb{R}) = GL(n, \mathbb{R}) \times \mathbb{R}^n \text{ (semi-direct product)},$$

if $(a, q) \in GA(n, \mathbb{R})$, i.e. if $a = (a^i_j) \in GL(n, \mathbb{R})$ and $q = (q^i) \in \mathbb{R}^n$, then $(a,q)\ (a',q') = (aa', q + aq')$, and $k(a) = (a, 0)$. Alternatively, one represents (a,q) by the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} a & q \\ 0 & 1 \end{pmatrix}$$

The bundle AM of *affine frames* consists of all pairs (e,u), where $e \in F(T_pM)$ and $u \in T_pM$. The action of $GA(n, \mathbb{R})$ on AM is $(e,u)\ (a,q) = (ea, eq + u)$, where $eq = e_i q^i$. In this case, $h : FM \to AM$ is given by $h(e) = (e, 0)$.

3. Consider the *proper Lorentz bundle* $P_2 \subset FM$, i.e. the bundle whose total space P_2 is the set of all orthonormal, time- and space oriented frames of a Lorentz manifold M (= 4-dimensional manifold with metric of signature $+ - - -$); its structure group $G_2 = O_0(1, 3)$ is the connected component of the identity of the Lorentz group O(1,3). Define

$$SL(2, \mathbb{C}) \to O_0(1,3)$$

in the standard manner:

if $X = (t,x,y,z)$, then $\ell(X) = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$ is hermitean and det $\ell(X) = t^2 - x^2 - y^2 - z^2$; these properties are preserved by $\ell(X) \mapsto a\ell(X)a^+$, where $a \in SL(2, \mathbb{C})$; we put

$$\ell(k(a)X) = a\ell(X)a^+$$

and check $k(ab) = k(a)k(b)$; k is surjective, ker $k = \{I, -I\}$.

By definition, a *spin structure* on M is a prolongation π_1 of the proper Lorentz bundle $\pi_2 : P_2 \to M$, associated with k. Cf. J. Milnor, *Spin structures on manifolds*, Enseign. Math. 9 (1963) 198.

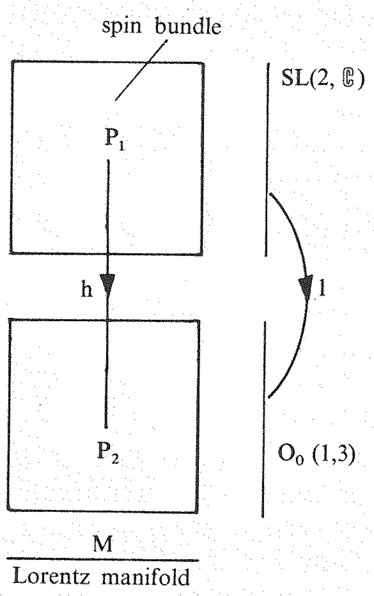


FIGURE 24

4. Let $P_1 = FM$ and consider the bundle P_2 of all *projective frames* on M: an element of P_2 is an equivalence class of linear frames attached to a point of M, the equivalence being defined by

$$e \equiv e' \Leftrightarrow \pi(e) = \pi(e') \text{ and } \underset{0 \neq \lambda \in \mathbb{R}}{\exists} \quad e'_i = \lambda e_i, \; i = 1, \dots, n \qquad e, e' \in FM$$

The structure group G_2 of P_2 is the *general projective group* $PGL(n, \mathbb{R})$, quotient of $GL(n, \mathbb{R})$ by the multiplicative group of the reals. The map $h : FM = P_1 \to P_2$ which sends any e into its equivalence class defines a *reduction* of the bundle of linear frames to the bundle of *projective frames*. This bundle plays a role in conformal geometry.

Warning. In the literature, the word 'reduction' is often used for what we call here a 'restriction'. The names for 'extension' and 'prolongation' are also sometimes interchanged.

5. A generalization of example 1 along the lines indicated in Chapter II leads to the notion of a G-*structure* on a manifold M: by definition, a G-structure on M is a restriction of FM to a subbundle P whose structure group G is a closed Lie subgroup of $GL(n, \mathbb{R})$:

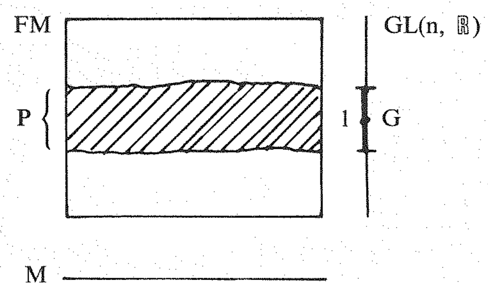


FIGURE 25

In this case, both h and k are the natural injections i.e.

There are *topological obstacles* to introducing G-structures on manifolds; given M and $G \subset GL(n, \mathbb{R})$, a G-structure on M may not exist, and, if it does, it need not be unique.

For example, a GL^+-structure on M is the same thing as an orientation; therefore, it exists on M iff M is orientable and, in this case, there are two of them.

A classical theorem asserts that on any (paracompact) manifold there exists a (proper) Riemannian metric and therefore there exists an O(n)-structure; this is no longer true if O(n) is replaced by $O(k, \ell)$; for example, $\mathbb{S}_2$ does not admit a metric of signature $+ -$.

Canonical form on the bundle of linear frames.

The bundle of linear frames has 'more structure', is 'richer' than an abstract bundle; there is defined on FM a natural, *canonical 1-form*

$$\theta : \mathrm{TFM} \to \mathbb{R}^n$$

(one can say that θ is an $\mathbb{R}^n$-valued 1-form on FM, or that $\theta = (\theta^i)$, $i = 1, \ldots, n$ is a collection of n real-valued 1-forms).

If $u \in T_e\mathrm{FM}$, then $\theta^i(u)$ is the i-th component of the projection

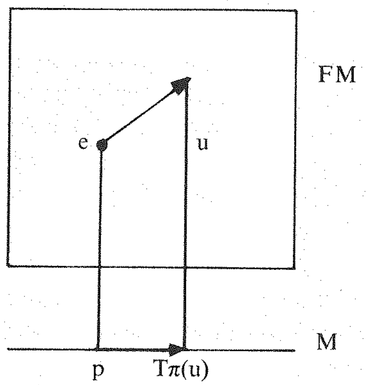


FIGURE 26

of u onto M relative to the frame e,

$$\theta^i_e(u) = \langle T_e\pi(u), e^i \rangle .$$

Consider the behaviour of θ under the action ψ of GL(n, $\mathbb{R}$) on FM:

$$(\psi^*_a\, \theta^i)_e(u) = \theta^i_{ea}(T_e\psi_a(u)) = \langle T_{ea}\pi \circ T_e\psi_a(u), (ea)^i \rangle =$$
$$= \langle T_e\pi(u), (ea)^i \rangle = a^{-1}{}^i_j \langle T_e\pi(u), e^j \rangle = a^{-1}{}^i_j\, \theta^j_e(u),$$

where we have used equation (7) of Chapter III and the 'chain rule':

$$T_{ea}\pi \circ T_e\psi_a = T_e(\pi \circ \psi_a) = T_e\pi .$$

By comparing the first and last term of these equalities, one obtains

(11) $$\psi_a^* \theta = a^{-1} \theta, \quad a \in GL(n, \mathbb{R}).$$

The bundle AM of affine frames, has a similar canonical form θ, and, in addition, a *canonical function* $\rho : AM \to \mathbb{R}^n$ defined by $\rho^i(e, u) = \langle u, e^i \rangle$. It is easy to check that

$$\psi_{(a, q)}^* \rho^i = a^{-1}{}^i_j (\rho^j + q^j),$$

where

$$(a, q) \in GL(n, \mathbb{R}) \times \mathbb{R}^n = GA(n, \mathbb{R}).$$

VI

CONNECTIONS AND GAUGE FIELDS

Connections are needed in order to transport, from one point of a manifold to another, geometrical and 'physical' objects such as vectors, tensors, values of wave functions etc. For geometric objects it is enough to know how the frames are transported (by parallel transfer). By definition, a geometric object is parallelly transported if its components with respect to a parallelly transferred frame are constant.

The notion of a connection – and the associated idea of covariant differentiation – may be defined in many equivalent ways.

Here is a partial list:

(i) the coefficients of a linear connection Γ^i_{jk} on a manifold are defined in local coordinates and are subject to well-known transformation rules (coordinate definition);

(ii) covariant differentiation is defined in terms of a map ∇_u, $u \in V(M)$, which acts linearly on tensor fields, satisfies the Leibniz rule and

$$\nabla_{fu+cv} = f\nabla_u + g\nabla_v,$$

$$\nabla_u(fv) = f\nabla_u v + u(f) \cdot v,$$

$$\nabla_u f = u(f),\ \nabla_u \langle v, \alpha \rangle = \langle \nabla_u v, \alpha \rangle + \langle v, \nabla_u \alpha \rangle,$$

for any f, $g \in C^0(M)$, u, $v \in V(M)$ and $\alpha \in C^1(M)$.

If (e_i) is a field of frames on M, then one defines Γ in terms of ∇:

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$

This is an 'algebraical definition'.

(iii) a definition with a clear *geometric* significance is possible in terms of fibre bundles; its advantages is that it is equally applicable to linear connections and to connections (= potentials) associated with gauge fields. This will now be described at some length.

CONNECTIONS ON A PRINCIPAL BUNDLE

This definition is general, but *it is useful to think* of the bundle of linear frames.

Heuristic considerations. Given a principal bundle $\pi : P \to M$ with group G, a point $p \in M$ and $e \in P$ above $p = \pi(e)$, we wish to be able to transport e from p to neighbouring points, along vectors originating from p, such as $u \in T_pM$:

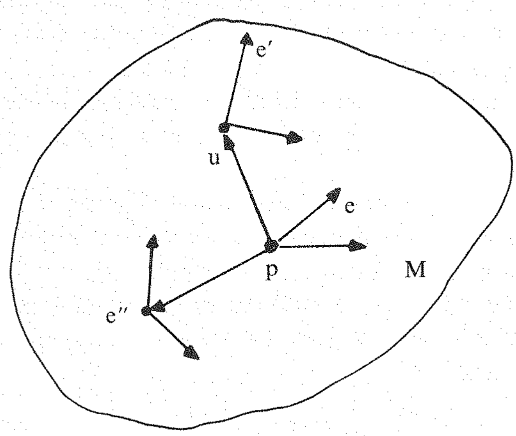


FIGURE 27

The law of parallel transport should:

(a) smoothly depend on p;

(b) allow transfer along all vectors emanating from p;

(c) be linear: if e is transported along u and results in e′, then ea transported along u results in $(ea)' = e'a$.

More *formally*, a connection on a principal bundle $\pi : P \to M$ is a distribution

$$P \ni e \mapsto H_e \subset T_eP$$

of vector spaces H_e which is:

(A) smooth;

(B) such that $T_e\pi : H_e \to T_{\pi(e)}M$ is an isomorphism for any $e \in P$;

(C) invariant under the action of G: $T_e\psi_a(H_e) = H_{ea}$.

Clearly, conditions (A)-(C) correspond to similar conditions (a)-(c). In particular, it follows from (B) that any vector $u \in T_pM$ lifts to a unique vector:

$$\text{lift}_e u \in H_e, \text{ where } \pi(e) = p. \tag{1}$$

The latter vector contains information: (i) about a displacement in M, namely:

$$T_e\pi\,(\text{lift}_e u) = u; \tag{2}$$

(ii) about an infinitesimal change in e under this displacement: $\text{lift}_e u$ points from e to a neighbouring element of P (frame if P = FM).

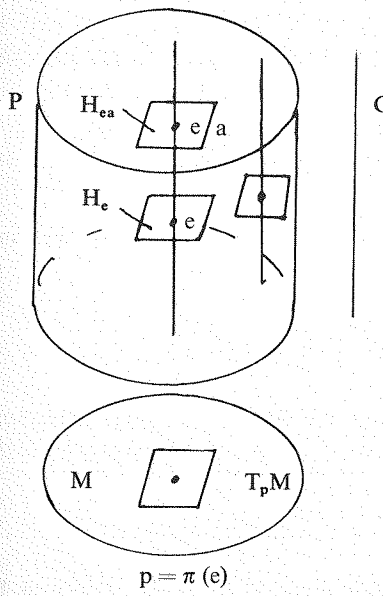


FIGURE 28

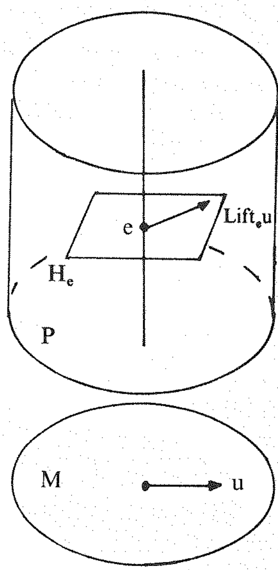


FIGURE 29

Returning for a while to the general concept of a principal bundle, one defines at any $e \in P$ the *vertical space*

$$\mathrm{ver}_e\, P = \{v \in T_eP \mid T_e\pi(v) = 0\};$$

$\mathrm{ver}_e P$ is the vector space of all vectors tangent to the fibre of P through e. The action of G in P defines a linear injection (equation (11) of Chapter IV),

$$\psi' : G' \to V(P).$$

Since G acts 'vertically': $\pi \circ \psi_a = \pi$, the vector field ψ'_A is vertical. Therefore, the map

$$\psi'_e : G' \to \mathrm{ver}_e\, P$$

defined by

$$\psi'_e(A) = \psi'_A(e) = \text{value of the vector field } \psi'_A \text{ at } e \in P$$

is an isomorphism. Equation (2) of Chapter IV implies the transformation law

$$\psi'_{ea} = T\psi_a \circ \psi'_e \circ \mathrm{Ad}_a. \tag{3}$$

Clearly, if a connection is given on $\pi : P \to M$, then one has a *direct sum* decomposition

$$T_eP = H_e \oplus \mathrm{ver}_e\, P, \quad e \in P.$$

A connection on P allows one to define *parallel transport* along a curve $c : [0, 1] \to M$. Given e in the fibre over $c(0) \in M$, one defines the lift $\bar{c} : [0, 1] \to P$ of c to P by

(i) $\bar{c}(0) = e$;
(ii) $\pi \circ \bar{c} = c$;
(iii) $\bar{c}$ is horizontal, i.e. the tangent vector to $\bar{c}$ at t belongs to $H_{\bar{c}(t)}$ for all $t \in [0, 1]$.

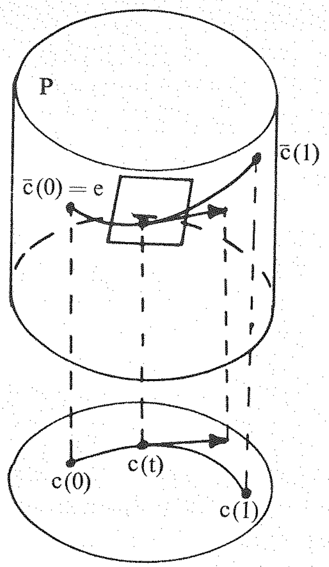


FIGURE 30

One says that $\bar{c}(1)$ is obtained by parallel transport of e along c. In general, even if c is closed, $c(0) = c(1)$, $\bar{c}$ is not: $\bar{c}(1) = \bar{c}(0)a$ where $a \in G$. The set of all a's which can be obtained in this way forms a subgroup of G, called the *holonomy group* of the connection at point c(0).

For example, the bundle $\pi : \mathbb{R} \to U(1)$, $\pi(s) = e^{i2\pi s}$ admits only one connection: H_s = tangent space to $\mathbb{R}$ at s. Let $c : [0, 1] \to U(1)$

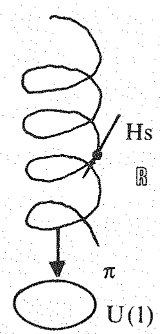


FIGURE 31

be given by $c(t) = e^{i2\pi t}$ and $\bar{c}(0) = 0$, then $\bar{c}(t) = t$, $\bar{c}(1) = 1$, and the holonomy group coincides with the structure group $\mathbb{Z}$ of the bundle. Clearly, if M is (archwise) connected then the holonomy group does not depend on c(0); a proof of this is obtained by inspecting fig. 32. More precisely, the groups corresponding to c(0) and c′(0) are conjugate to each other.

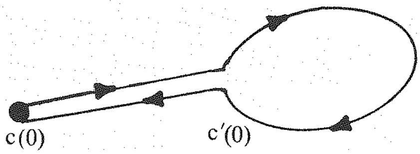


FIGURE 32

If one restricts the loops c to be contractible, then the above construction leads to the *restricted holonomy group*, which is a subgroup of the (full) holonomy group. In the case of the bundle $\pi : \mathbb{R} \to U(1)$ (fig. 31), the restricted holonomy group is trivial (i.e. contains only the unit element).

A connection on P is *completely integrable* if, for any $e_0 \in P$, there exists a submanifold S of P passing through e_0 and such that, for any $e \in S$, $T_eS = H_e$. Clearly, if the connection is completely integrable, then the restricted holonomy group reduces to the identity, but not necessarily so the full holonomy group: an example is provided again by $\pi : \mathbb{R} \to U(1)$, fig. 31.

The notion of complete integrability can be contemplated in the general case of a *differential system* on a manifold P (not necessarily on a bundle), defined as a smooth distribution $P \ni e \mapsto H_e \subset T_eP$ of subspaces of the tangent spaces to P. The common dimension $\dim H_e = n$ is called the dimension of the system. Complete integrability is defined as in the preceding paragraph.

A vector field u on P is said to belong to the differential system H if $u_e \in H_e$ for all $e \in P$. We denote by V(H) the set of all vector fields on P which belong to the differential system H.

Dually, we define

$$\mathfrak{F}(H) = \{\omega \in C^1(P) \mid \omega \mid H_e = 0, \text{ any } e \in P\}.$$

A classical *theorem* due to *Frobenius* asserts that the following properties are equivalent:

(i) the differential system H is completely integrable;
(ii) V(H) is a Lie algebra;
(iii) if $(\omega^\mu)_{\mu = n+1,\ldots,\dim P = p}$ is a frame in the vector space $\mathfrak{F}(H)$, then there exist forms $\alpha^\mu_\nu \in C^1(P)$ such that:

$$d\omega^\mu = \alpha^\mu_\nu \wedge \omega^\nu;$$

(iv) if $\mathfrak{J}(H) \subset C(P)$ is the ideal in the Cartan algebra generated by $\mathfrak{F}(H) \subset C^1(P) \subset C(P)$, then:

$$d\,\mathfrak{J}(H) \subset \mathfrak{J}(H).$$

Sketch of proof (for details, see R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland, Amsterdam 1968):

(i) $\underset{\text{easy}}{\Rightarrow}$ (ii): Let S be a maximal integral manifold, choose coordinates x such that $x^{n+1},\ldots,\ x^p = \text{const}$ on S, then

$$u \in V(H) \Leftrightarrow u = \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i}$$

and check that $u, v \in V(H) \Rightarrow [u, v] \in V(H)$;

(ii) $\underset{\text{hard}}{\Rightarrow}$ (i): choose a frame $(u_i)_{i=1,\ldots,n}$ in V(H): use $[u_i, u_j] \in V(H)$ to show that one can find $v_i = \Sigma a_{ij} u_j$ such that $[v_i, v_j] = 0$, then put

$$v_i = \frac{\partial}{\partial x^i}\ (i = 1,\ldots,\ n);$$

complete the x's to a coordinate system $x^1, \ldots, x^p$, then S is given by $x^{n+1},\ldots,\ x^p = \text{cont}$.

(ii) $\underset{\text{easy}}{\Leftrightarrow}$ (iii): use equation (21) in Chapter III.

(iii) $\underset{\text{easy}}{\Leftrightarrow}$ (iv) consider the definition of an ideal.

Examples of applications.

$p = 2,\ n = 1$: condition (iii) is always trivially satisfied; if $n = p - 1$, then (iii) is equivalent to $\omega \wedge d\omega = 0$ (note that $\mathfrak{F}(H)$ is one-dimensional) and also to $\omega = TdS$ (T is an 'integrating factor' of ω).

Carathéodory's statement of the second law of thermodynamics says that the distribution defined by the 1-form of heat is completely integrable.

$p = 3$, $n = 2$: condition $\omega \wedge d\omega = 0$ may be written as $\vec{\omega} \cdot \operatorname{curl} \vec{\omega} = 0$.

Connection form. Motivated by the Frobenius theorem, we wish to describe the distribution H of the horizontal subspaces on $\pi : P \to M$ by means of a collection of 1-forms ω. At any $e \in P$ we have the direct sum decomposition

$$T_eP = H_e \oplus \mathrm{ver}_e\, P.$$

Therefore, any $u \in T_eP$ may be uniquely decomposed into its horizontal and vertical components

$$u = \mathrm{hor}\, u + \mathrm{ver}\, u, \quad \text{where} \quad \mathrm{hor}\, u \in H_e.$$

Using the isomorphism $\psi'_e : G' \to \mathrm{ver}_e\, P$ we define the *connection form*

$$\omega : TP \to G' \tag{4}$$

by the following

$$\omega(u) = (\psi'_e)^{-1}\,(\mathrm{ver}\, u). \tag{5}$$

It is obvious that

$$u \in H_e \Leftrightarrow \omega(u) = 0, \tag{6}$$

$$\omega(\psi'_e(A)) = A, \tag{7}$$

and it follows from equation (3) that

$$\psi_a^* \,\omega \equiv \omega \circ T\psi_a = \mathrm{Ad}_{a^{-1}} \circ \omega. \tag{8}$$

The last property may be also easily shown to be true by checking it separately on horizontal and vertical vectors: on horizontal vectors both sides of (8) vanish because of (6) and the invariance of H under the action of the group; if $u = \psi'_e(A) \in \mathrm{ver}_e P$ then u is the tangent vector to $t \mapsto \psi_{\exp tA}(e)$ at $t = 0$; the right-hand-side of (8) is then equal to $\mathrm{Ad}_{a^{-1}}(A)$, whereas on the left-hand-side we have $T\psi_a(u) =$ tangent vector to

$$t \mapsto \psi_a \circ \psi_{\exp tA}(e) = \psi_{a^{-1}(\exp tA)a}(ea) = \psi_{\exp t\mathrm{Ad}_{a^{-1}}A}(ea)$$

because of equation (2) in Chapter IV; we use now (7) to establish the equality of both sides of (8).

Conversely, if there is given a G′-valued 1-form ω on P such that (7) and (8) hold, then (6) defines a connection on P (i.e. an invariant horizontal differential system).

According to form (iii) of the Frobenius theorem, the distribution H is completely integrable on P iff, at any $e \in P$:

$$u, v \in H_e \Rightarrow d\omega(u, v) = 0.$$

Alternatively, if we define a G′-values 2-form Ω on P by

$$\Omega(u, v) = d\omega\,(\mathrm{hor}\ u, \mathrm{hor}\ v),$$

then

(9) $$\text{complete integrability of H} \Leftrightarrow \Omega = 0.$$

The *curvature form* Ω has the following properties

(10) it is horizontal: if $u \in \mathrm{ver}_e P$, then $\Omega(u, v) = 0$;

(11) $$\psi_a^* \Omega = \mathrm{Ad}_{a-1} \cdot \Omega\,;$$

(12) $$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \quad \text{or} \quad d\omega + \omega \wedge \omega.$$

Equation (11) follows from (8) and the definition of Ω; equation (12) may be checked by evaluation of both sides on pairs of vectors: hor, hor; hor, ver and ver, ver, and using equation (21) of Chapter III.

The definition of Ω suggests a generalization. Let ρ be a representation of G in a vector space V, i.e. let

$$\rho : \mathrm{G} \rightarrow \mathrm{GL(V)}$$

be a morphism of Lie groups; a k-*form of type* ρ on P is a V-valued k-form α on P such that

$$\psi_a^* \alpha = \rho_{a-1} \circ \alpha .$$

For example, ω is a 1-form of type Ad, Ω is a 2-form of type Ad and θ is a 1-form of type id on FM.

We define hor α by

$$\text{hor } \alpha(u_1,\dots,u_k) = \alpha\,(\text{hor } u_1,\dots, \text{hor } u_k).$$

A form α is *horizontal* if hor $\alpha = \alpha$; e.g. Ω is horizontal; also

$$\text{hor } \omega = 0. \tag{13}$$

If α is a k-form of type ρ then so is hor α. The *exterior covariant derivative* of a k-form of type ρ on a principal bundle with a connection is defined as:

$$\mathrm{D}\alpha = \text{hor } \mathrm{d}\alpha . \tag{14}$$

Clearly, $\mathrm{D}\alpha$ is a horizontal $(k + 1)$-form of type ρ. Since $\mathrm{dd}\omega = 0$ and there holds (13), the curvature form satisfies the *Bianchi identity*

$$\mathrm{D}\Omega = 0. \tag{15}$$

The exterior covariant derivative of a horizontal k-form of type ρ may be evaluated as follows: consider the derived homomorphism of Lie algebras.

$$\rho' : \mathrm{G}' \rightarrow \mathfrak{L}(\mathrm{V})$$

and its composition with ω:

$$\rho'(\omega) : TP \to \mathfrak{L}(V),$$

then

$$(16) \qquad D\alpha = d\alpha + \rho'(\omega) \wedge \alpha,$$

where the symbol $\rho'(\omega) \wedge \alpha$ implies exterior product of forms *and* an evaluation of $\rho'(\omega)$, considered as an element of $\mathfrak{L}(V)$, on α, considered as an element of V. In other words, if one refers α to a frame (e_A) in V and ω to a frame (e_μ) in G', then

$$(17) \qquad D\alpha^A = d\alpha^A + \rho^A_{B\mu}\, \omega^\mu \wedge \alpha^B.$$

We may now use (16) to evaluate (15); since Ad′ is given by $Ad'_A(B) = [A, B]$, we have

$$(18) \qquad D\Omega = d\Omega + [\omega, \Omega].$$

From (17) and the definition of Ω there follows the formula for D^2

$$(19) \qquad D^2\alpha = \rho'(\Omega) \wedge \alpha.$$

Indeed,

$$\begin{aligned} D^2\alpha &= \text{hor } d\, D\alpha = \text{hor } d(\rho'(\omega) \wedge \alpha) = \\ &= \text{hor } [\rho'(d\omega) \wedge \alpha - \rho'(\omega) \wedge d\alpha] = \text{hor } [\rho'(d\omega) \wedge \alpha] \\ &= \rho'(\Omega) \wedge \alpha. \end{aligned}$$

If α and β are horizontal forms on P of degree k and ℓ respectively, then $\alpha \wedge \beta$ is horizontal of degree $k + \ell$ and

$$(20) \qquad D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^k \alpha \wedge D\beta.$$

According to Chapter V, if α is a 0-form of type ρ on P, then it defines a section of the associated bundle $P \times V/G$ and $D\alpha$

represents the *covariant derivative* of that section. Therefore, D generalizes the notions of both covariant and exterior derivatives: if ρ is the trivial representation, $\rho(a) = 1$, then $D = d$.

LINEAR CONNECTIONS

A *linear connection* is a connection on the bundle FM of linear frames of a manifold M. If the manifold is n-dimensional, then $G = GL(n, \mathbb{R})$, $G' = \mathfrak{L}(\mathbb{R}^n)$ and $\omega = (\omega^i_j)$ is a collection of n^2 1-forms. Equations (8), (6), (17), (18) and (19) become, respectively,

(21) $$\omega^i_j \circ T\psi_a = a^{-1}{}^i_k \omega^k_\ell a^\ell_j;$$

(22) $$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j;$$

(23) $$D\alpha^A = d\alpha^A + \rho^{Ai}_{Bj} \omega^j_i \wedge \alpha^B;$$

(24) $$D\Omega^i_j = d\Omega^i_j + \omega^i_k \wedge \Omega^k_j - \Omega^i_k \wedge \omega^k_j;$$

(25) $$D^2\alpha^A = \rho^{Ai}_{Bj} \Omega^j_i \wedge \alpha^B.$$

Since the canonical form $\theta = (\theta^i)$ is horizontal of type id, we have the following expression for the *torsion* 2-form:

(26) $$\Theta^i \underset{\text{def}}{=} D\theta^i = d\theta^i + \omega^i_j \wedge \theta^j.$$

Since both Ω^i_j and Θ^i are horizontal and $\theta^i(u) = 0 \Leftrightarrow u$ is vertical, the curvature and torsion forms may be represented as

(27) $$\Omega^i_j = \frac{1}{2} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell;$$

(28) $$\Theta^i = \frac{1}{2} Q^i_{k\ell}\, \theta^k \wedge \theta^\ell.$$

$(R^i_{jk\ell})$ and $(Q^i_{k\ell})$ are called, respectively, the curvature and the torsion tensors of the connection. They are defined here as functions on FM.

It follows from (21) and $\psi^*_a \theta^i = a^{-1}{}^i_j \theta^j$ that these tensors 'transform' as one would expect, i.e.

$$R^i_{jk\ell}(ea) = a^{-1i}{}_m a^n_j a^p_k a^q_\ell R^m_{npq}(e),$$

and similarly for Q. In addition to the *Bianchi identity for the curvature form*

$$D\Omega^i_j = 0, \tag{29}$$

there is a *Bianchi identity for torsion* which is easily obtained from (25) for $\alpha = \theta$:

$$D\Theta^i = \Omega^i_j \wedge \theta^j. \tag{30}$$

• If $\alpha = (\alpha^A)$ is a V-valued function (= 0-form) of type ρ (*tensor of type* ρ), then its covariant derivative may be written as

$$D\alpha^A = \theta^i \nabla_i \alpha^A \quad \text{or} \quad \theta^i \alpha^A{}_{;i}. \tag{31}$$

(Since $D\alpha^A$ is a horizontal 1-form it may be represented in this manner). The *components of the covariant derivative* $\nabla_i\alpha^A$ correspond to the horizontal part of $d\alpha^A$:

$$d\alpha^A = \theta^i \nabla_i \alpha^A - \rho^{Ai}_{Bj} \omega^j_i \alpha^B. \tag{32}$$

Note that at each point e of FM the set (ω^i_j, θ^k) of $n^2 + n$ 1-forms constitutes a frame in $T^*_e FM$: the manifold FM admits teleparallelism. In other words, for any (paracompact) manifold M, the bundle F(FM) over FM is trivial.

If we now apply (25) to (31) and use (26)-(28), then we obtain

$$\nabla_i \nabla_j \alpha^A - \nabla_j \nabla_i \alpha^A = \rho^{Ak}_{B\ell} R^\ell{}_{kij} \alpha^B - Q^k_{ij} \nabla_k \alpha^A. \tag{33}$$

Exercise. Use (27) and (28) in (29) and (30) to get an 'explicit' form of the Bianchi identities:

$$R^i_{j[k\ell;m]} + R^i_{jn[m} Q^n_{k\ell]} = 0; \tag{29'}$$

$$Q^i_{[jk;\ell]} = R^i_{[jk\ell]}. \tag{30'}$$

A *metric tensor* (field) g on a manifold may be equivalently defined:

(a) as a section of the bundle of symmetric tensors, associated with FM;

(b) as a set of $\frac{n(n+1)}{2}$ functions $g_{ij} : FM \to \mathbb{R}$ such that

$$g_{ij}(ea) = a_i^k\, a_j^\ell\, g_{k\ell}(e), \qquad a \in GL(n, \mathbb{R}),$$

where $g_{ij}(e)$ is the scalar product of e_i and e_j;

(c) as a restriction of the bundle FM to O(n),

$$P(M, g) = \{e \in FM \mid g_{ij}(e) = \delta_{ij}\}.$$

The bundle $P(M,g) \subset FM$ of orthonormal frames (for $n = 4$ called also 'tetrads' or 'Vierbeins') defines g: if $e = (e_i) \in P(M, g)$ is given, then at $\pi(e)$,

$$g = e^1 \otimes e^1 + \ldots + e^n \otimes e^n.$$

Clearly, for an indefinite (say, Lorentz) metric, O(n) should be replaced by $O(k,\ell)$ (say, O(1, 3)).

A linear connection H on FM is said to be *compatible* with g if, at any $e \in P(M,g)$:

$$H_e \subset T_e P(M,g).$$

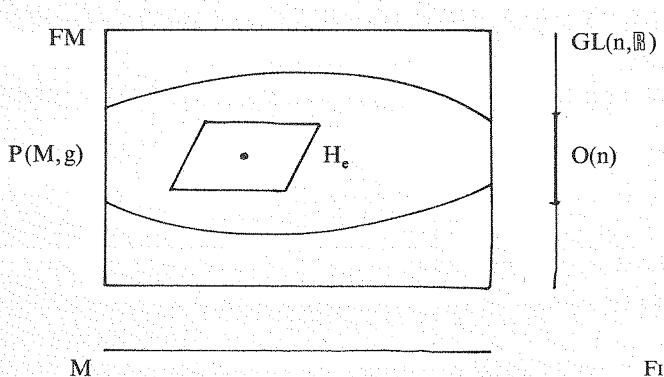


FIGURE 33

According to the definition of H,

$$\text{H and g are compatible} \Leftrightarrow \text{length is preserved by parallel transport} \Leftrightarrow Dg_{ij} = 0.$$

(This is so because $H_e \subset T_eP(M,g)$ implies that the lift of a curve through $e \in P(M,g)$ is contained in $P(M,g) \Rightarrow$ an orthonormal frame remains orthonormal under parallel transport).

Another formulation of compatibility of a connection H and of a metric g: compatibility of H and $g \Leftrightarrow \omega$ is $O(n)'$-valued on $P(M,g)$.

According to equation (16),

$$(34) \qquad Dg_{ij} = dg_{ij} - \omega_{ij} - \omega_{ji}, \quad \text{where} \quad \omega_{ij} = g_{ik}\,\omega^k_j.$$

Therefore

$$(35) \qquad Dg_{ij} = 0 \Leftrightarrow \omega_{ij} + \omega_{ji} = 0 \quad \underline{\text{on } P(M,g).}$$

It follows from equation (19) that

$$(36) \qquad Dg_{ij} = 0 \Rightarrow \Omega_{ij} + \Omega_{ji} = 0.$$

Transformation law of the 'coefficients' of a connection form pulled back from P to M by a (local) section.

For any connection defined by ω on P over M, and any local section $f: U \to P$ one defines:

$$\Gamma = f^*\omega.$$

The Γ's are G′-valued 1-forms on U; depending on the details of P, G and f, they are called 'Christoffel symbols' and 'Ricci rotation coefficients' (in geometry) or 'potentials of a gauge field' (in physics). It is important to recognize how they change under a transformation of the section.

Let $\psi: P \times G \to P$ be the action of the group G on P; in agreement with previous chapters we put

$$\psi_a(e) = \psi(e, a) = \psi_e(a) = ea \quad \text{for any} \quad e \in P,\ a \in G.$$

The property of the connection form expressed by (7) is equivalent to

$$\omega \circ T\psi_e = \tilde{\omega}, \tag{37}$$

where $\tilde{\omega}$ is the canonical form on G. Let

$$S : U \to G$$

and let $f' : U \to P$ be the section obtained from f by transforming it by means of S, i.e.

$$f'(p) = \psi(f(p),\ S(p)), \qquad p \in U.$$

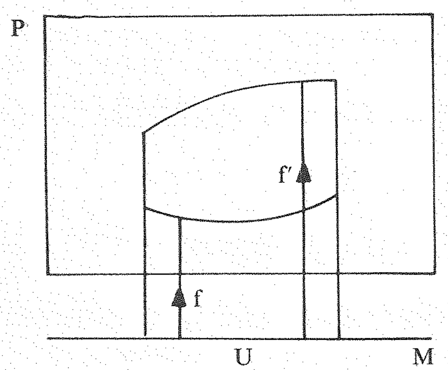


FIGURE 34

We ask for the relation between Γ and $\Gamma' = f'^*\omega$. The rule for differentiation of composite functions,

$$\frac{d}{dt} F(x(t),\ y(t)) = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}$$

leads to the

Lemma. If $h : X \times Y \to Z$ is smooth, $\xi \in T_x X$, $\eta \in T_y Y$, then

$$T_{(x,y)}\, h(\xi,\eta) = T_y\, h_x(\eta) + T_x h_y(\xi),$$

where

$T_{(x,y)}X \times Y$ is identified with $T_xX \oplus T_yY \ni (\xi,\eta)$,
$h_x : Y \to Z$,
$h_y : X \to Z$,
and $h_x(y) = h(x,y) = h_y(x)$.

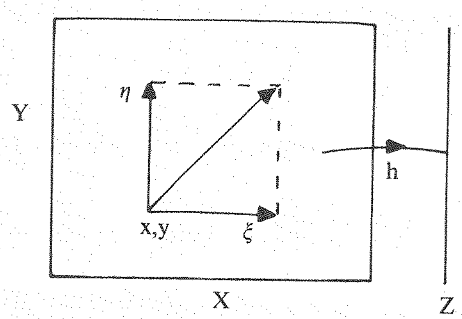


FIGURE 35

Therefore

$$\Gamma' = f'^*\omega = \omega \circ Tf' = \omega \circ T\psi(f, S) = \omega \circ (T(\psi_f \circ S) + T(\psi_S \circ f)),$$

$\omega \circ T\psi_f = \tilde{\omega}$ according to (37),
$\omega \circ T\psi_S = Ad_{S^{-1}} \circ \omega$ according to (8);

thus

$$\Gamma' = Ad_{S^{-1}} \circ \Gamma + \tilde{\omega} \circ TS. \tag{38}$$

If α is a *horizontal* k-form on P of type ρ, $A = f^*\alpha$, $A' = f'^*\alpha$, then

$$A' = \rho_{S^{-1}} \circ A. \tag{39}$$

This is so because α, as a horizontal form, is annihilated by any vertical vector such as $T\psi_f \circ TS(v)$. The inhomogeneous transformation law (38) of Γ is due to the fact that ω is not a horizontal form. (For this reason, horizontal forms are often called *tensorial forms*). For example

$R = f^*\Omega$ transforms according to $R' = Ad_{S^{-1}} \circ R$ $(\rho = Ad)$,
$f^i = f^*\theta^i$ transforms according to $f'^i = S^{-1}{}^i_j f^j$ $(\rho = id)$,

and similarly for the torsion form.

If $G \subset GL(W)$ for some vector space W – as is always the case for classical groups – then G' is identified with a subspace of $\mathfrak{L}(W)$, $\tilde{\omega} = a^{-1}da$ (cf. Chapter IV) where the a's are now understood as coordinates on G. In this case (38) becomes:

$$\Gamma' = S^{-1}\Gamma S + S^{-1}dS, \tag{40}$$

and

$$R' = S^{-1}RS. \tag{41}$$

Exercises. Derive (41) from (40) by using $R = d\Gamma + \Gamma \wedge \Gamma$.

2. Define $DA = f^* D\alpha$, where α is a horizontal k-form of type ρ, then, from (16), derive

$$DA = dA + \rho'(\Gamma) \wedge A,$$

and prove

$$DA' = \rho'_{S^{-1}} DA.$$

3. In particular,

$$DR = dR + \Gamma \wedge R - R \wedge \Gamma.$$

Check $DR = 0$ from $R = d\Gamma + \Gamma \wedge \Gamma$.

The coefficients Γ *of a linear connection compatible with a metric tensor* g may be determined as follows. Let $f = U \to FM$ be a field of frames on an-dimensional manifold M, $g_{ij} : FM \to \mathbb{R}$ its metric tensor, and

$$\omega = (\omega^i_j) : TFM \to \mathfrak{L}(\mathbb{R}^n)$$

a linear connection compatible with g. We denote, as before,

$$f^*\omega^i_j = \Gamma^i_{jk} f^k,$$

and have

$$f^* \theta^i = f^i \qquad \text{(check from the definition of } \theta\text{)},$$

but we also *abuse the notation* by writing

$$g_{ij} \text{ on M for } f^* g_{ij} = g_{ij} \circ f;$$
$$Q^i_{jk} \text{ on M for } Q^i_{jk} \circ f, \text{ etc.}$$

Therefore, $Dg_{ij} = 0$ becomes

$$(42) \qquad dg_{ij} = (g_{ik}\, \Gamma^k_{jl} + g_{jk}\, \Gamma^k_{il})\, f^l \qquad \text{on M},$$

and the definition of torsion is

$$(43) \qquad df^i + \Gamma^i_{jk}\, f^k \wedge f^j = \frac{1}{2} Q^i_{jk}\, f^j \wedge f^k \qquad \text{on M}.$$

Put

$$dg_{ij} = g_{ij/k}\, f^k, \qquad g_{ij/k} = g_{ji/k},$$
$$df^i = \frac{1}{2} f^i_{jk}\, f^j \wedge f^k, \qquad f^i_{jk} = -f^i_{kj},$$

and solve (42) and (43) to obtain

$$(44) \quad 2\,\Gamma^i_{jk} = g^{il}(g_{jl/k} + g_{kl/j} - g_{jk/l}) - g^{il}(Q_{ljk} + Q_{jkl} + Q_{kjl}) + g^{il}(f_{ljk} + f_{jkl} + f_{kjl}),$$

where

$$Q_{ljk} = g_{li}\, Q^i_{jk}, \quad f_{ljk} = g_{li}\, f^i_{jk}.$$

In a *Riemannian space*, $Q^i_{jk} = 0$, and
(a) if $f^i = dx^i$, then $df^i = 0$ and Γ^i_{jk} is the *Christoffel symbol*;
(b) if f is orthonormal, then $dg_{ij} = 0$ and Γ^i_{jk} is the *Ricci rotation coefficient*.

Another example of a manifold with a linear connection is a *Lie group* G, where *parallel transport* is *defined by left translations*. Let (f^i) be a frame in G′, then the connection form pulled back by f vanishes, $\Gamma = f^*\omega = 0$, therefore $R = 0$, whereas torsion may be obtained from the Maurer-Cartan equation (7) in Chapter IV:

$$Q^i_{jk} = -c^i_{jk} .$$

In a Riemannian space,

$$Dg_{ij} = 0 \Rightarrow \Omega_{ij} + \Omega_{ji} = 0 \Rightarrow R_{ijkl} + R_{jikl} = 0, \qquad (\alpha)$$
$$\Theta^i = 0 \Rightarrow \Omega^i_j \wedge \theta^j = 0 \Rightarrow R^i_{[jkl]} = 0, \qquad (\beta)$$

and for *any curvature tensor*,

$$R_{ijkl} + R_{ijlk} = 0. \qquad (\gamma)$$

Exercise. Form

$$R_{i[jlk]} + R_{j[kli]} + R_{k[lij]} + R_{l[kji]} \equiv 4R_{[ijlk]} = 0,$$

and use (α) and (γ) to prove

$$R_{ijkl} = R_{klij} . \qquad (\delta)$$

Example. Compute curvature of the Riemannian metric

(45) $$ds^2 = 2dudv - dx^2 - dy^2 - 2H(x, y, u)\, du^2.$$

First step: introduce an ‘orthonormal null’ field of frames:

$$f^1 = dx,\ f^2 = dy,\ f^3 = du,\ f^4 = dv - Hdu;$$

then

$$ds^2 = f^3 \otimes f^4 + f^4 \otimes f^3 - f^1 \otimes f^1 - f^2 \otimes f^2;$$

thus

$$(g_{ij}) = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) = (g^{ij}),$$

and

$$dg_{ij} = 0.$$

Second step: compute df^i and use (44) to evaluate the connection:

$$\Gamma_{ij} = \Gamma_{ijk}\, f^k \quad (= f^*\omega_{ij});$$
$$df^i = 0 \text{ for } i = 1, 2, 3; \quad df^4 = -H_x f^1 \wedge f^3 - H_y f^2 \wedge f^3;$$

therefore

$$f_{331} = -f_{313} = H_x,\ f_{332} = -f_{323} = H_y,\ \text{other f's} = 0$$

thus

$$(\Gamma_{ij}) = \left(\begin{array}{c|cc} 0 & H_x f^3 & 0 \\ & H_y f^3 & 0 \\ \hline \cdots & & \\ \cdots & 0 & \end{array}\right) \begin{array}{l} i=1 \\ i=2 \\ i=3 \\ i=4. \end{array}$$

Third step: compute the curvature two-form

$$\frac{1}{2} R_{ijkl} f^k \wedge f^l = d\Gamma_{ij} + \Gamma_{ik} \wedge \Gamma^k_j;$$

$$\frac{1}{2} R_{ijkl} f^k \wedge f^l = d\Gamma_{ij} = \left(\begin{array}{c|cc} 0 & (H_{xx} f^1 + H_{xy} f^2) \wedge f^3, & 0 \\ & (H_{xy} f^1 + H_{yy} f^2) \wedge f^3, & 0 \\ \hline \cdots & 0 & \end{array}\right).$$

Read off the essential components of R_{ijkl}:

$$R_{1313} = H_{xx},\ R_{1323} = H_{xy},\ R_{2323} = H_{yy}.$$

The Ricci tensor

$$R_{ij} = g^{kl} R_{iklj}$$

has only one non-vanishing component, $R_{33} = H_{xx} + H_{yy}$. Therefore: if $F(z,u)$ is an arbitrary function analytic in $z = x + iy$, then (45) with

$$H(x,y,u) = \mathrm{Re}F(x + iy, u)$$

satisfies the Einstein equations in empty space, $R_{ij} = 0$; this solution is non-flat iff $\partial^2 F/\partial z^2 \neq 0$. For example

$$F = (A(u) + iB(u))z^2$$

corresponds to a *plane gravitational wave* (these results are due to Ivor Robinson, 1956).

Gauge fields

We define a gauge configuration of type G to be a connection on a principal bundle P over space-time M; G is the structure group of the bundle. Thus, to define a gauge configuration on a given space-time, one should:

1. specify a group G; e.g.
   $G = U(1)$ corresponds to electromagnetism,
   $Sp(1) = SU(2)$ corresponds to a Yang-Mills field;
2. describe the G-bundle $P \to M$; in many cases, P is the trivial bundle, isomorphic to $M \times G$, but there are gauge configurations (at least among those considered by theoreticians) which require non-trivial bundles;
3. specify the field equation to be satisfied by the gauge field; these equations will restrict the class of all connections which can be introduced on P; a standard procedure is to derive the field equations from a variational principle;
4. the gauge field should be coupled to other particles; if they are described by wave functions, then one can use a generalized principle of minimal coupling. Any representation $\rho : G \to GL(V)$ defines 'particles of type ρ': their wave functions are maps $\varphi : P \to V$ such that $\varphi \circ \psi_a = \rho_{a^{-1}} \circ \varphi$, and $D\varphi = d\varphi + \rho'(\omega)\varphi$ (cf. equation (16)) is used to write an equation of motion for φ.

In fact, 3. and 4. cannot be considered separately, because the 'particles' play the role of sources for the gauge field.

In both Maxwell and Yang-Mills theories, the field equations are of the form

$$D^*R = -4\pi {}^*j, \tag{46}$$

where $R = f^*\Omega$ and $^*R$ is the dual of R with respect to the (Riemannian) metric g on M; $^*j$ is the dual of the G′-valued current 1-form j. In general, the current j is not conserved by itself; from (46) and (19) follows:

$$D^*j = 0,$$

and also

$$d\left({}^*j + \frac{1}{4\pi}[\Gamma, {}^*R]\right) = 0.$$

To summarize the correspondance between the *terminology* of fibre bundles and that of current physical literature, we give the following table, adapted, with modifications, from T. T. Wu and C. N. Yang (Physical Review D 12 (1975), 3845).

total space of the bundle	P	space of phase factors
base space	M	space-time
structure group	G	gauge group
local section of the principal bundle $M \supset U \xrightarrow[f]{} P$, $\pi \circ f = id_u$	f	local gauge
connection form on P	ω	gauge potential
curvature form on P	Ω	gauge field
pull-back of ω by f	$\Gamma = f^*\omega$	potentials in gauge f
pull-back of Ω by f	$R = f^*\Omega$	field strengths in gauge f
action of G in P, $a \in G$, $\psi_a : P \to P$, $\psi_a \circ \psi_b = \psi_{ba}$, etc.	ψ_a	gauge transformation of the first kind

V-valued function of type ρ, $\varphi : P \to V$, $\varphi \circ \psi_a = \rho_{a^{-1}} \circ \varphi$, $\rho : G \to GL(V)$	φ	wave function (of a particle of type ρ)
pull-back of φ by f	$\phi = \varphi \circ f$	wave function in gauge f
$S : U \to G, U \subset M$, defines a change of section from f to f': $f'(p) = \psi(f(p \cdot S(p)))$	S	gauge transformation of the second kind: $\Gamma' = S^{-1}\Gamma S + S^{-1}ds$ $\phi' = \rho_{S^{-1}} \circ \phi$
Bianchi identity	$D\Omega = 0$	Faraday part of the field equations.

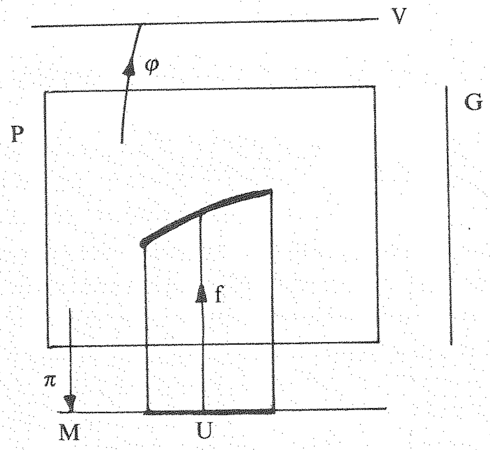


FIGURE 36

According to what has been said in Chapter V, a wave function of a particle interacting with a gauge field may be represented either as a map

$$\varphi : P \to V$$

satisfying

$$\varphi \circ \psi_a = \rho_{a^{-1}} \circ \varphi,$$

or as a section

$$\Phi : M \to E$$

of the bundle $\pi_E : E \to M$ associated with $\pi : P \to M$ by ρ, $E = P \times V/G$. (The notation here differs from that of Chapter V; present φ correspond to what was previously denoted by $\bar{s}$, etc.).

Gravitation may be – to some extent – considered as corresponding to a gauge field with the Lorentz group O(1,3) as the structure group. The bundle P is then the bundle of all orthonormal frames (tetrads, Vierbeins) on M. Since $P \subset FM$, this bundle has more structure than an abstract principal bundle: there is the canonical 1-form θ on P and, in addition to the curvature:

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j,$$

one has the torsion:

$$\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j$$

(which may be zero – but being zero is different from not existing at all: such is the status of torsion on the electromagnetic or Yang-Mills bundles). By analogy with other gauge theories, C. N. Yang and other authors have proposed to consider $*\Omega^i_j \wedge \Omega^j_i$ as the gravitational Lagrangian. Another possibility is to consider

$$*\Omega^i_j \wedge \Omega^j_i + \frac{1}{L^2} g_{ij} *\Theta^i \wedge \Theta^j.$$

Examples. In electromagnetism, all irreducible representations of U(1) are of the form

$$\rho_n : U(1) \to U(1), \quad \rho_n(u) = u^n, \quad n \in \mathbb{Z}.$$

A particle of type ρ_n is simply a particle of *electric charge* n,

$$D\varphi = d\varphi + n\omega\varphi \quad (\text{because } \rho'_n = n \cdot \mathrm{id}).$$

The Lie algebra of U(1) is $i\mathbb{R}$, therefore ω is pure imaginary.

In an SU(N) theory ($N \geq 2$), one often takes $\rho = \mathrm{Ad}$, and φ is then called a *Higgs field*. In this case, equations of motions are usually obtained from a principle of least action associated with the form on P:

$$h(*\Omega \wedge \Omega) + k(*D\varphi \wedge D\varphi) + U(k(\varphi, \varphi))\,\eta,$$

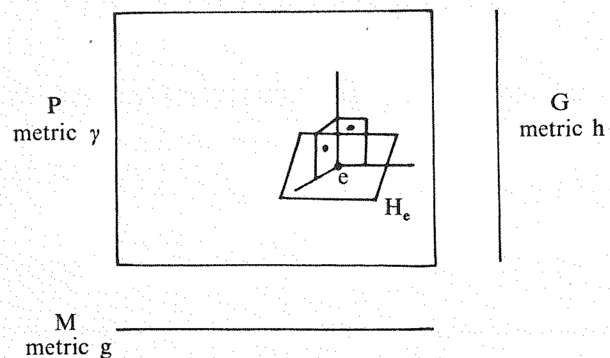
where h is a biinvariant metric on G (e.g. the Killing metric), k is an invariant metric on V (in fact, for $\rho = \mathrm{Ad}$, $V = G'$, and one takes $k = h$), and η is a volume element associated with a Riemannian metric on M, which is also used to define the duals.

Given a connection ω on P, a biinvariant metric h on G, and a Riemannian metric g on M, one defines a Riemannian metric γ on P as follows. Let $X \in TP$; then

$$\gamma(X, X) = g(T\pi(X),\ T\pi(X)) + h(\omega(X),\ \omega(X)).$$

The metric γ on P is invariant under the action of G and defines a *generalized Kaluza-Klein geometry* on P (the classical K – K geometry, considered in attempts to 'unify' gravitation with electromagnetism, corresponds to $G = U(1)$).

Conversely, a metric γ on P, invariant under the action of G, leads to a connection on $P: H_e \subset T_eP$ is defined as the vector space of all vectors at $e \in P$ orthogonal to $\mathrm{ver}_e P$.



Relative to γ, the horizontal substance $H_e \subset T_e P$ is orthogonal to the vertical substance $\mathrm{ver}_e P$.

FIGURE 37

Exercise. Consider the Ricci scalar $R^{AB}{}_{AB} = \mathfrak{R}$ corresponding to the Riemannian curvature tensor (R_{ABCD}) associated with the metric $\gamma = \gamma_{AB}\, dx^A dx^B$, where $A,B = 1,\ldots,\dim P$. Vary the action $(\int \mathfrak{R} \cdot$ volume element on P) with respect to g and ω to obtain the combined set of Einstein's equations (with a cosmological term) and the equation for the gauge field $D{}^*\Omega = 0$. Show that the cosmological term vanishes for an Abelian gauge group.

VII

EXAMPLES OF NON TRIVIAL BUNDLES OCCURRING IN PHYSICS

Several examples of non-trivial bundles are given in Chapter V; it is interesting that some of these bundles are relevant to physics. To see this, consider first the *theorem* due to C.N. Yang: If $P \to M$ is a trivial U(1)-bundle, Σ a closed 2-surface in M, and F any electromagnetic field on M corresponding to a connection ω on P, then the flux of the magnetic field through Σ vanishes,

$$\int_\Sigma F = 0.$$

Proof: since $P \to M$ is trivial, it has a section $f : M \to P$, and $iF = f^*\Omega = f^*d\omega = d(f^*\omega)$; therefore, by Stokes' theorem

$$\int_\Sigma iF = \int_\Sigma d(f^*\omega) = \int_{\partial\Sigma} f^*\omega = 0.$$

The last equality holds because Σ is closed, i.e. compact and has no boundary, $\partial\Sigma = \phi$.

A classical theorem in topology says that if M is contractible, then $P \to M$ is trivial. (A topological space M is *contractible* if the identity map id_M is homotopic to a constant map, i.e. if there exists $h : [0,1] \times M \to M$ which is continuous and such that $h(0,p) = p$ (any $p \in M$) and $h(1,p) = p_0$ for any p and a fixed point p_0; for instance $\mathbb{R}^n$ is contractible, $h(t,p) = (1-t)p$, $p_0 = 0$, but $\mathbb{S}_n$ is not).

Therefore, if a magnetic pole is found in nature, then at least one of the following is true:

(i) the interpretation of electromagnetism as a connection in a U(1)-bundle is incorrect;

(ii) space-time is not contractible; therefore, in particular, it is not homeomorphic to $\mathbb{R}^4$;

(iii) differentiable manifolds do not provide satisfactory models of space-time.

In the theoretical considerations of magnetic poles one assumes (ii) and constructs non-trivial bundles over the space-time:

$$\begin{aligned} M &= \mathbb{R}^4 - \{\text{world-line of the pole}\}, \\ &\cong \mathbb{R}^2 \times \mathbb{S}_2 \quad \text{which is not contractible.} \end{aligned}$$

The U(1)-bundles over $\mathbb{S}_2$ are known; they are labelled by integers; the simplest non-trivial among them is described in Chapter V:

$$U(2) \to U(2)/U(1) = \mathbb{S}_3 \to \mathbb{S}_2 .$$

The sphere $\mathbb{S}_3$ may be represented by pairs of *complex* numbers (z_0, z_1) subject to

$$|z_0|^2 + |z_1|^2 = 1, \tag{1}$$

and the map $U(2) \to \mathbb{S}_3$ sends the coset consisting of all elements of U(2) of the form:

$$a = \begin{pmatrix} z_0 & \cdot \\ z_1 & \cdot \end{pmatrix}$$

into $(z_0, z_1) \in \mathbb{S}_3$. Therefore, the canonical form on U(2)

$$\tilde{\omega} = a^{-1} da = a^{+} da = \begin{pmatrix} \bar{z}_0\, dz_0 + \bar{z}_1\, dz_1 & \cdot \\ & \end{pmatrix}$$

defines a 1-form on $\mathbb{S}_3$

$$\omega = \bar{z}_0\, dz_0 + \bar{z}_1\, dz_1 \tag{2}$$

which, expressed in terms of the Euler angles, becomes

$$\omega = \frac{i}{2}(d\chi + \cos\vartheta d\varphi) = i\alpha .$$

This is a connection form on the principal U(1)-bundle $\mathbb{S}_3 \to \mathbb{S}_2$ and the corresponding electromagnetic field

$$\Omega = d\omega = \frac{i}{2} \sin \vartheta d\varphi \wedge d\vartheta ,$$

or rather

$$F = \frac{1}{2} \sin \vartheta d\varphi \wedge d\vartheta,$$

describes a *magnetic pole* of strength $g = \frac{1}{2}$ (the units are such that the charge of the electron equals the fine structure constant).

The singularities ('strings') of the electromagnetic potentials corresponding to a magnetic pole are due to the non-trivial nature of the bundle $\mathbb{S}_3 \to \mathbb{S}_2$. If one removes from $\mathbb{S}_2$ the north pole $(\vartheta = 0)$, then

$$\mathbb{S}_2 \ni (\vartheta, \varphi) \underset{f}{\longmapsto} \left(z_0 = e^{i\varphi} \cos \frac{\vartheta}{2}, z_1 = \sin \frac{\vartheta}{2} \right) \in \mathbb{S}_3$$

is a local section, and

$$f^*\alpha = \frac{1}{2} (1 + \cos \vartheta)\, d\varphi = A$$

is the potential whose essential component, $A_\varphi = \dfrac{1 + \cos \vartheta}{2r \sin \vartheta}$, is singular at $\vartheta = 0$.

A completely analogous construction leads to the Belavin et al. *instanton* (pseudoparticle) solution of the Yang-Mills equations. Consider

$$Sp(2) \to Sp(2)/Sp(1) = \mathbb{S}_7 \to \mathbb{S}_4$$

and replace in both formulae (1) and (2) the complex numbers z_0, z_1 by *quaternions*. Equation (1) then defines $\mathbb{S}_7$. Instead of the Euler angles we introduce a unit quaternion $u \in Sp(1)$,

$\bar{u} = u^{-1}$, and one more quaternion $\zeta = z_1 z_0^{-1}$; then

$$z_0 = \rho u, \; z_1 = \rho \zeta u, \quad \text{where } \rho^2 = \frac{1}{1 + |\zeta|^2},$$

and the connection form (2) becomes

$$\omega = u^{-1}\, du + \frac{u^{-1}}{2} \rho^2 (\bar{\zeta}\, d\zeta - (d\bar{\zeta})\zeta)\, u.$$

The corresponding curvature form $\Omega = d\omega + \omega \wedge \omega$ is given by

$$u \Omega u^{-1} = \rho^4\, d\bar{\zeta} \wedge d\zeta,$$

and describes the BPST instanton on $\mathbb{S}_4$ with line-element given by

$$ds^2 = \rho^4\, d\bar{\zeta}\, d\zeta.$$

Since the Yang-Mills equations are conformally invariant for $\dim M = 4$, the solution on $\mathbb{S}_4$ may be transformed, by the stereographic projection, into a solution on $\mathbb{R}^4$.

Note an interesting analogy: Maxwell theory is to complex numbers what Yang-Mills theory is to quaternions.

Another example is provided by the Hopf fibration

$$U(3) \to U(3)/U(2) = \mathbb{S}_5 \to \mathbb{C}\,\mathbb{P}_2.$$

A construction similar to the one for the magnetic pole leads to

$$\omega = \bar{z}_0\, dz_0 + \bar{z}_1\, dz_1 + \bar{z}_2\, dz_2 = i\alpha,$$

where $z_0, z_1, z_2 \in \mathbb{C}$, and

$$|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$$

is the equation of $\mathbb{S}_5$. The parametrization

$$z_0 = e^{i\chi} \cos\vartheta, \; z_1 = e^{i(\chi+\mu)} \sin\vartheta \cos\varphi, \; z_2 = e^{i(\chi+\nu)} \sin\vartheta \sin\varphi$$

leads to the ' electromagnetic instanton ' solution

$$F = \sin 2\vartheta d\vartheta \wedge (\cos^2 \varphi d\mu + \sin^2 \varphi d\nu) - \sin^2 \vartheta \sin 2\varphi d\varphi \wedge (d\mu - d\upsilon)$$

defined over $\mathbb{C}\mathbb{P}_2$ with the metric given by

$$ds^2 = d\vartheta^2 + \sin^2 \vartheta [d\varphi^2 + \cos^2 \vartheta (\cos^2 \varphi d\mu + + \sin^2 \varphi d\upsilon)^2 + \sin^2 \varphi \cos^2 \varphi (d\mu - d\upsilon)^2].$$

The general case is described in my paper, "Solutions of the Maxwell and Yang-Mills Equations Associated with Hopf Fibrings", Intern. J. Theor. Phys, 16 (1977), 561-565.

VIII

TOPOLOGICAL INVARIANTS AND CHARACTERISTIC CLASSES

Basic reference: *Characteristic Classes* by J. W. Milnor and J. D. Stasheff, Princeton University Press, Princeton, N.J. 1974 (contains a comprehensive bibliography).

It is desirable to distinguish between conserved quantities, invariants, and topological invariants:

Example 1. Consider an isolated hypothetical family of electric and magnetic charges and the following statements about the system:

Statement (a) at any given instant of time, $t = \text{const}$, the electric charge inside

$$\Sigma \text{ equals } \frac{1}{4\pi} \oint_{\Sigma} \vec{E}\, d\vec{S} = \frac{1}{4\pi} \int \operatorname{div} \vec{E}\, dV$$

and is independent of Σ (provided all charges are enclosed).

This statement is equivalent to:

$$\operatorname{div} \vec{E} = 0$$

in empty space.

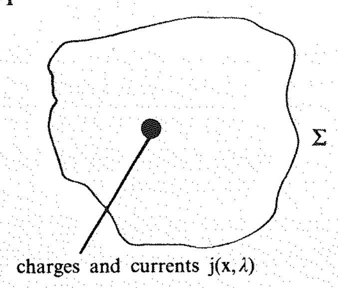


FIGURE 38

Statement (b) electric charge is conserved in time,

$$\frac{d}{dt} \oint_{\Sigma} \vec{E}\, d\vec{S} = 0 \Leftrightarrow \frac{\partial \vec{E}}{\partial t} = c \operatorname{curl} \vec{B}\,.$$

Assume now that the distribution of charges depends, in a *continuous* manner, on a parameter λ, labelling the system, then

$$\frac{1}{4\pi} \oint \vec{E}_\lambda\, d\vec{S} = q_\lambda$$

may depend arbitrarily on λ.

Statement (c)

$$\frac{1}{4\pi} \oint \vec{B}_\lambda\, d\vec{S} = \frac{1}{2} n_\lambda,\ n_\lambda = \text{integer};$$

but an integer depending continuously on λ must be constant, therefore

$$\oint \vec{B}_\lambda\, dS \text{ is a constant.}$$

Among these three statements, only the last refers to a 'topological' invariant!

Sometimes it is asserted that a 'topological invariant is obtained by integrating a total divergence'. This is not so; the electric charge is a counter example. Moreover, locally any function may be represented as a divergence, whereas globally, the integral of a true divergence over a closed manifold is zero.

Example 2. Consider a 2-sphere of radius r, then

$\oint dS = 4\pi r^2$ is an invariant (area of the sphere);

but $\oint k dS = 4\pi$ is a topological invariant (does not change under smooth deformations of the sphere).

Here k denotes the Gaussian curvature, and the invariance and value of the integral follows from the *Gauss-Bonnet theorem*.

CLOSED AND EXACT FORMS; DE RHAM COHOMOLOGY

Consider an n-dimensional differential manifold M and the vector spaces $C^k(M)$ of k-forms on M. *Closed* k-forms on M constitute a space

$$Z^k(M) = \{\alpha \in C^k(M) \mid d\alpha = 0\},$$

and the *exact* k-forms constitute a subspace of $Z^k(M)$,

$$B^k(M) = \{d\beta \mid \beta \in C^{k-1}(M)\}.$$

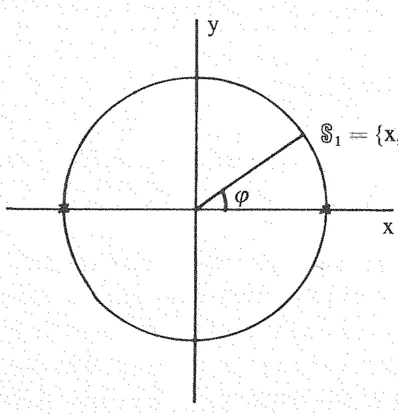
The quotient vector space

$$H^k(M) = Z^k(M)/B^k(M) \qquad (k = 0,1,\ldots,n)$$

is the k-th *cohomology* vector space of M.

Example. Consider $M = \mathbb{S}_1$ and the form $\alpha \in C^1(\mathbb{S}_1)$ defined by

$$\alpha = x dy - y dx \quad \text{on} \quad x^2 + y^2 = 1.$$



φ is a smooth function on $\mathbb{S}_1 - \{(1,0)\}$ and also on $\mathbb{S}_1 - \{(-1,0)\}$ but not on all $\mathbb{S}_1$; $\alpha = d\varphi/S_1 - \{(1,0)\}$ etc.

FIGURE 39

Clearly,

$$d\alpha = 0 \quad \text{on } \mathbb{S}_1,$$

but α is not exact because

$$\oint_{S_1} \alpha = \int_0^{2\pi} d\varphi = 2\pi,$$

whereas for any exact form $\alpha = d\beta$, the integral over a closed manifold is zero by Stokes' theorem. Note that

$\vec{B} = \text{curl}\,\vec{A}$ means $B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ is exact

$\text{div}\, B = 0$ means $B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ is closed.

Let $\omega \in Z^1(\mathbb{S}_1) = C^1(\mathbb{S}_1)$, then $\omega \in B^1(\mathbb{S}_1) \Leftrightarrow \oint \omega = 0$; in fact, if $\oint \omega = 0$, then $a(\varphi) = \int_0^\varphi \omega$ is well defined on $\mathbb{S}_1$, $a(2\pi) = a(0) = 0$, and $\omega = da$. Therefore, for any $\omega \in Z^1(\mathbb{S}_1)$, we have

$$\omega = \frac{\alpha}{2\pi} \oint \omega + da \quad \text{for some} \quad a \in C^0(\mathbb{S}_1),$$

and this shows

$$H^1(\mathbb{S}_1) = \mathbb{R}.$$

Since $\mathbb{S}_1$ is connected, constants are the only closed 0-forms, $B^0(\mathbb{S}_1) = \{0\}$, and

$$H^0(\mathbb{S}_1) = \mathbb{R}$$

The dimension $b_k(M)$ of $H^k(M)$ is called the *kth Betti number* of M.

Poincaré Lemma. If M is smooth and contractible, then $b_1 = b_2 = \ldots = b_n = 0$ (i.e. all closed forms on M are exact).

Sketch of proof: M is smooth contractible if there is a one-parameter group (φ_t) of transformations of M such that $\varphi_0 = id_M$ and $\lim \varphi_t(p) = p_0$ (any $p \in M$, fixed $p_0 \in M$). Let u be the vector field on M induced by (φ_t); then

$$L_u = d \circ i(u) + i(u) \circ d, \qquad d \circ \varphi_t^* = \varphi_t^* \circ d$$

$$\frac{d}{dt} \varphi_t^* \alpha = L_u \varphi_t^* \alpha = \varphi_t^* L_u \alpha \quad \text{for any } \alpha \in C(M).$$

Define

$$h_k : C^k(M) \to C^{k-1}(M)$$

by

$$h_k(\alpha) = -\int_0^\infty \varphi_t^* \, (i(u)\alpha) \, dt;$$

then

$$(h_{k+1} \circ d + d \circ h_k)(\alpha) = -\int_0^\infty \varphi_t^* \, (i(u) \, d\alpha +$$

$$+ \, di(u)\alpha) \, dt = -\int_0^\infty \frac{d}{dt} \, \varphi_t^* \alpha dt = \alpha.$$

Therefore

$$h_{k+1} \circ d + d \circ h_k = \mathrm{id},$$

and

$$\text{if } \alpha \in Z^k(M), \text{ then } \alpha = -d\int_0^\infty \varphi_t^* \, (i(u)\alpha) \, dt \in B^k(M).$$

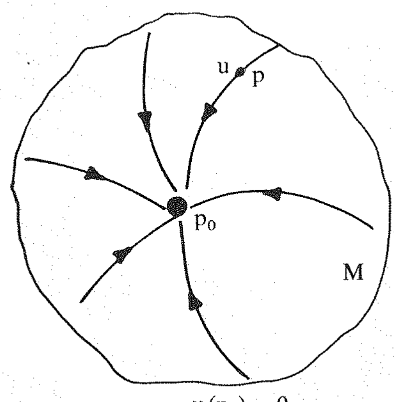


FIGURE 40

Exercise. Assuming $M = \mathbb{R}^n$ and $\alpha \in Z^k(\mathbb{R}^n)$, find β such that $\alpha = d\beta$. Hint: take $u = -x^i \dfrac{\partial}{\partial x^i}$, where (x^i) are Cartesian coordinates.

SINGULAR HOMOLOGY, CHAINS AND INTEGRATION ON MANIFOLDS

(No proofs are given here, very sketchy presentation).

The *standard* k-*simplex* in $\mathbb{R}^k$ is

$$\Delta^k = \{(q^1, \ldots, q^k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} q^i \leq 1; \quad 0 \leq q^i\}.$$

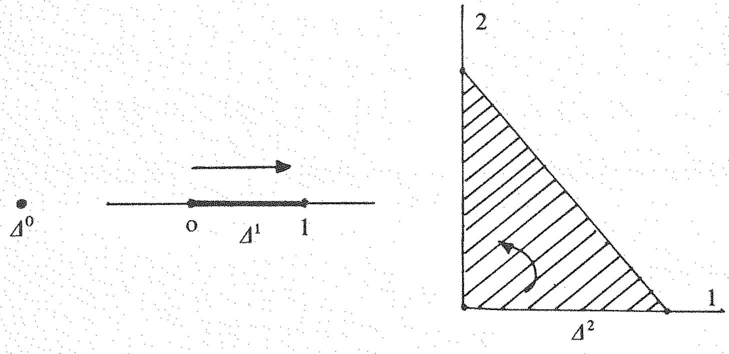


FIGURE 41

A differentiable (singular) k-*simplex* in a manifold M is a smooth map

$$\sigma : \Delta^k \to M.$$

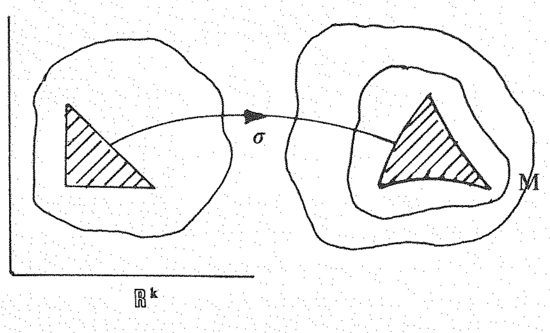


FIGURE 42

(extendable to a smooth map of a neighbourhood od Δ^k).

A k-chain in M is:

$$c = \sum_i a_i \sigma_i,$$

where $a_i \in \mathbb{R}$ and σ_i are k-simplices in M.

Face maps

$$F_i^k : \Delta^k \to \Delta^{k+1} \qquad i = 0, \ldots, k+1$$

are defined in an obvious manner (cf. figures):

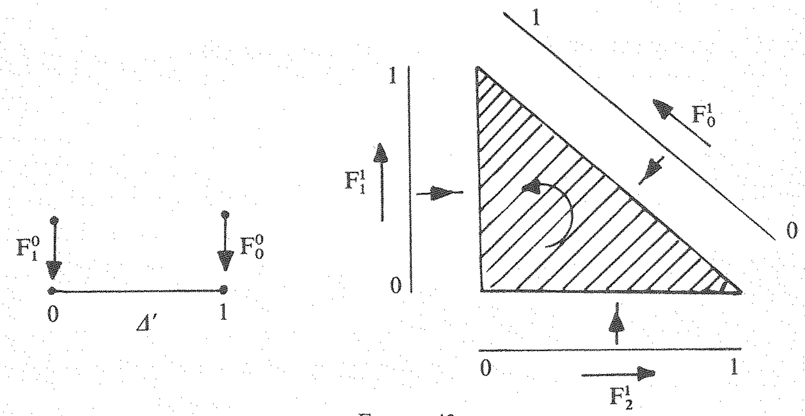


FIGURE 43

$$F_i^k (q^1, \ldots, q^k) = (q^1, \ldots, q^{i-1}, 0, q^i, \ldots, q^k);$$

for $i = 1, \ldots, k$, and with a suitable definition of F_0^k. Furthermore, the ith face of a simplex $\sigma : \Delta^k \to M$ is $\sigma^i = \sigma \circ F_i^k$.

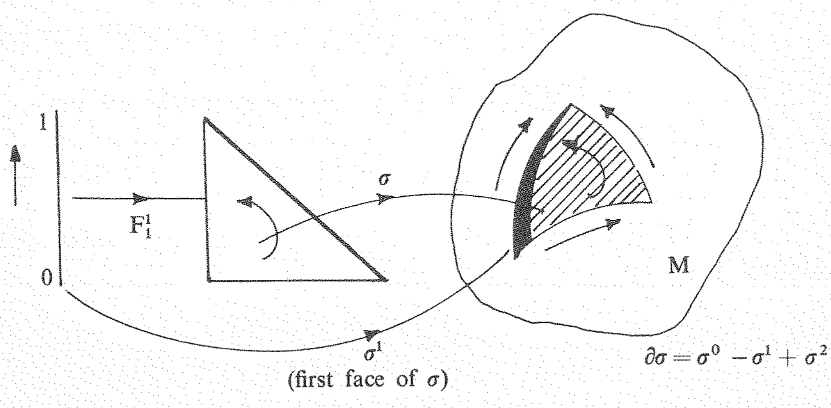


FIGURE 44

The *boundary* of σ is a (k − 1)-chain $\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma^i$ (look at the arrows to appreciate the signs).

If $c = \Sigma a_i \sigma_i$ then $\partial c = \Sigma a_i \partial\sigma_i$.

Theorem. $\partial \circ \partial = 0.$

Proof: compute or inspect the following figure.

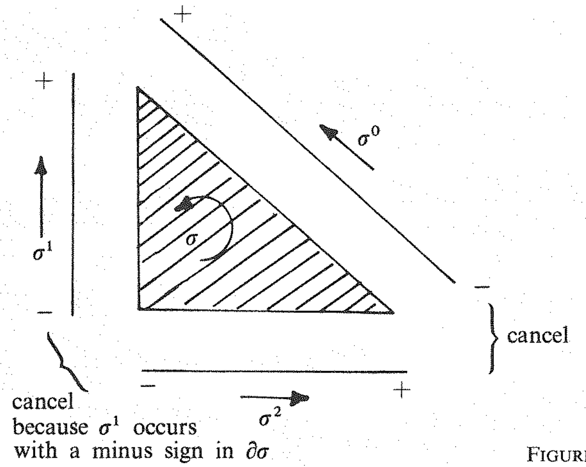


FIGURE 45

If $\omega \in C^k(M)$ and σ is a k-simplex on M, then:

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^* \omega.$$

If $c = \sum_i a_i \sigma_i$ is a k-chain on M, then

$$\int_c \omega = \sum_i a_i \int_{\sigma_i} \omega.$$

Stokes Theorem.

(1) $\int_{\partial c} \omega = \int_c d\omega$ for any $\omega \in C^k(M)$ and any k-chain c on M.

The set of all k-chains on M forms a vector space: the boundary operator singles out the subspace of k-*cycles*

$$Z_k(M) = \{c \text{ is a } k\text{-chain} \mid \partial c = 0\}$$

and k-*boundaries*

$$B_k(M) = \{\partial c \mid c \text{ is a } (k+1)-\text{chain}\} \subset Z_k(M)$$

by $\partial \circ \partial = 0$. The quotient

$$H_k(M) = Z_k(M)/B_k(M)$$

is the k-th (singular, real, differentiable) *homology group* of M (in fact, it is a real vector space).

If $\alpha \in Z^k(M)$, then we denote by $\bar{\alpha}$ its image in $H^k(M)$. In other words, $\bar{\alpha}$ is the equivalence class of all k-forms which differ from α by an exact form. Similarly, if $c \in Z_k(M)$ then $\bar{c}$ is its image in $H_k(M)$. There is a bilinear map

$$H^k(M) \times H_k(M) \to \mathbb{R}$$

given by

$$(\bar{\alpha}, \bar{c}) \mapsto \int_c \alpha = f_{\bar{\alpha}}(\bar{c})$$

(this is well defined by Stokes' theorem). $\int_c \alpha$ is called the *period* of $\alpha \in Z^k(M)$ over $c \in Z_k(M)$.

The de Rham theorem. The linear map.

$$H^k(M) \to H_k(M)^* \tag{2}$$

given by

$$\bar{\alpha} \mapsto f_{\bar{\alpha}}$$

is an isomorphism of vector spaces.

Corollary. If all periods of a closed form are zero, then the form is exact (injectiveness of (2)), moreover (2) is surjective:

given a set of periods for the generators of $Z_k(M)$, there is a closed form which assumes these periods as values.

Exercise. Show that if $\alpha \in Z^k(M)$ and $\beta \in Z^l(M)$, then $\alpha \wedge \beta \in Z^{k+l}(M)$.

If, moreover, either α or β is exact, then so is $\alpha \wedge \beta$. Therefore one can define a bilinear map

$$H^k(M) \times H^l(M) \to H^{k+l}(M)$$

by

$$(\bar{\alpha}, \bar{\beta}) \to \overline{\alpha \wedge \beta} \underset{\text{def}}{=} \bar{\alpha} \wedge \bar{\beta}.$$

Therefore, $H^*(M) = \bigoplus_k H^k(M)$ is an algebra (the *cohomology algebra* of M).

The Poincaré duality theorem. If M is compact, oriented, n-dimensional, then the bilinear map

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R} \tag{3}$$

given by

$$(\bar{\alpha}, \bar{\beta}) \mapsto \int_M \alpha \wedge \beta = p_{\bar{\alpha}}(\bar{\beta}) \qquad (\alpha \in Z^k, \beta \in Z^{n-k})$$

is non-singular, i.e.

$$H^k(M) \ni \bar{\alpha} \mapsto p_{\bar{\alpha}} \in H^{n-k}(M)^*$$

is an isomorphism.

By comparing (2) and (3) one obtains the isomorphism

$$H^k(M) \to H_{n-k}(M).$$

THE CHERN-WEIL CONSTRUCTION

Reference: S. S. Chern and J. Simons, Annals of Math. *99* (1974), 48.

Consider two smooth connections on $P \to M$ with structure group G:

$$\omega \quad \text{and} \quad \omega_1 = \omega + \alpha.$$

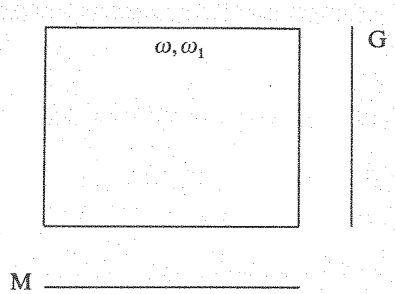


FIGURE 46

The 1-form α is G'-valued and horizontal. Construct a one-family of connections linking ω and $\omega_1 : \omega_t = \omega + t\alpha$ $(0 \le t \le 1)$. Using

$$(4) \qquad [[\alpha, \alpha], \alpha] = 0$$

which is a consequence of the Jacobi identity and holds for any G'-valued 1-form α, one computes

$$\Omega_t = d\omega_t + \frac{1}{2}[\omega_t, \omega_t] = \Omega + t\, D\alpha + \frac{1}{2} t^2 [\alpha, \alpha];$$

$$(5) \qquad D\Omega_t = t[\Omega, \alpha] + t^2 [D\alpha, \alpha] = t[\Omega_t, \alpha];$$

$$(6) \qquad \frac{d\Omega_t}{dt} = D\alpha + t[\alpha, \alpha].$$

Consider a k-linear symmetric map

$$w : G' \times G' \times \ldots \times G' \to \mathbb{R}$$

which is invariant under the adjoint action of G in G':

$$w(\mathrm{Ad}_a A_1, \ldots, \mathrm{Ad}_a A_k) = w(A_1, \ldots, A_k), \text{ any } a \in G,\ A_i \in G';$$

put $a = \exp t B$, then $\left.\dfrac{d}{dt}\right|_{t=0}$ gives

$$(7) \qquad w([A_1, B], A_2, \dots, A_k) + w(A_1, [A_2, B], \dots, A_k) + \\ + \dots + w(A_1, A_2, \dots, [A_k, B]) = 0.$$

If (α_i) are G′-valued forms on P, $\alpha_i = \alpha_i^{j_i} e_{j_i}$, where (e_j) is a frame in G′, then

$$w(\alpha_1, \alpha_2, \dots, \alpha_k) = \alpha_1^{i_1} \wedge \alpha_2^{i_2} \wedge \dots \wedge \alpha_k^{i_k} w(e_{i_1}, e_{i_2}, \dots, e_{i_k})$$

is a $\mathbb{R}$-valued form on P.

We shall consider forms such as

$$w(\alpha) = w(\alpha, \alpha, \alpha, \dots, \alpha);$$
$$w(\alpha, \beta) = w(\alpha, \beta, \beta, \dots, \beta);$$
$$w(\alpha, \beta, \gamma) = w(\alpha, \beta, \gamma, \dots, \gamma).$$

If α is a 1-form and β is a 2-form, then the invariance condition (7) applied to $w(\alpha, \beta)$ gives

$$(8) \qquad w([\alpha, \alpha], \beta) + (k-1)\, w(\alpha, [\beta, \alpha], \beta) = 0.$$

We can now prove

Theorem 1.

$$(9) \qquad \frac{d}{dt} w(\Omega_t) = k\, dw(\alpha, \Omega_t) \quad \text{on P.}$$

Proof:

The L.H.S. is

$$\frac{d}{dt} w(\Omega_t) = kw\left(\frac{d\Omega_t}{dt}, \Omega_t\right) = kw(D\alpha + t[\alpha, \alpha], \Omega_t);$$

the R.H.S. may be evaluated as follows (use $D = d$ for scalar-valued forms):

$$dw(\alpha, \Omega_t) = Dw(\alpha, \Omega_t) = w(D\alpha, \Omega_t) - (k-1)\, w(\alpha, D\Omega_t, \Omega_t) = \\ = w(D\alpha, \Omega_t) - (k-1)\, tw(\alpha, [\Omega_t, \alpha], \Omega_t) = \\ = w(D\alpha + t[\alpha, \alpha], \Omega_t).$$

This coincides with L.H.S.

Consider now local sections f, $f' = U \rightarrow P$ and $R_t = f^*\Omega_t$, $R'_t = f'^*\Omega_t$, $A = f^*\alpha$, $A' = f'^*\alpha$; then $R'_t = S^{-1}R_tS$, and $A' = S^{-1}AS$ where $S: U \rightarrow G$ is the 'gauge transformation' from f to f'.

The invariance of w gives $w(R_t) = w(R'_t)$, and

$$w(A, R_t) = w(A', R'_t).$$

Therefore (9) projects to M

$$\frac{d}{dt} w(R_t) = kdw(A, R_t) \quad \text{on M}.$$

If c is a closed 2k-chain in M, $\partial c = \emptyset$ then Stokes' theorem gives

$$\frac{d}{dt} \int_c w(R_t) = 0 \quad \text{implies}$$

(10) $$\int_c w(R_1) = \int_c w(R).$$

In words: a topological invariant is obtained by integrating over a cycle a closed form whose variation is exact.

Theorem 2.

The 2k-form $w(\Omega)$ *on* P is exact:

(11) $$w(\Omega) = kd \int_0^1 w(\omega, \Psi_t)\, dt,$$

where

$$\Psi_t = t\Omega + \frac{1}{2}(t^2 - t)[\omega, \omega].$$

Proof: Consider $w(\Psi_t)$ and compute

$$\frac{d}{dt} w(\Psi_t) = kw\left(\frac{d\Psi_t}{dt}, \Psi_t\right) = kw\left(\Omega + \left(t - \frac{1}{2}\right)[\omega, \omega], \Psi_t\right).$$

On the other hand (use $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, and $d\Psi_t = t[\Psi_t, \omega]$),

$$dw(\omega, \Psi_t) = w(d\omega, \Psi_t) - (k-1)\, w(\omega, d\Psi_t, \Psi_t) =$$

$$= w\left(\Omega - \frac{1}{2}[\omega, \omega], \Psi_t\right) - (k-1)\, w(\omega, t[\Psi_t, \omega], \Psi_t) =$$

$$= w\left(\Omega + \left(t - \frac{1}{2}\right)[\omega, \omega], \Psi_t\right).$$

Note: equation (11) does not project, in general, to M because ω and Ψ_t are not horizontal forms: their pull-backs to M by $f: U \to P$ have a 'bad' transformation law, e.g.

$$\Gamma' = S^{-1}\,\Gamma S + \underline{\underline{S^{-1} dS}}.$$

Corollary of Theorem 2: the form w(R) on M is *closed*.

Examples

1. In electromagnetism, all the k-fold exterior products

$$\Omega,\ \Omega \wedge \Omega, \ldots,\ \Omega \wedge \Omega \wedge \ldots \wedge \Omega \qquad \left(k \le \frac{1}{2} \dim M\right)$$

lead to topological invariants. If $\dim M = 4$, then only the first two are relevant

$$\int \Omega \sim \oint \vec{B}\ d\vec{S}, \quad \int \Omega \wedge \Omega \sim \int \vec{E} \cdot \vec{B}\, d^4x.$$

2. In an SU(N)-theory over a 4-dimensional space-time M, the only topological invariants formed from curvature are

$$w(\Omega) = h_{ij}\, \Omega^i \wedge \Omega^j,$$

where h_{ij} is an invariant metric on G′. By evaluating (11) one finds

$$h_{ij}\, \Omega^i \wedge \Omega^j = d\left(h_{ij}\, \omega^i \wedge \Omega^j + \frac{1}{3} c_{ijk}\, \omega^i \wedge \omega^j \wedge \omega^k\right),$$

where $h_{ij} = h(e_i, e_j)$, $\omega = \omega^i e_i$, $\Omega = \Omega^i e_i$, $[e_i, e_j] = c^k_{ij} e_k$ $c_{ijk} = h_{il} c^l_{jk}$. Note that invariance of h implies

$$h([e_i, e_j], e_k) + h(e_j, [e_i, e_k]) = 0 \Rightarrow c_{ijk} = c_{[ijk]}.$$

3. *Pontryagin classes.* Consider a *real*, n-dimensional vector bundle $E \to M$ associated with a principal bundle $P \to M$ with structure group $G = GL(n, \mathbb{R})$. The Lie algebra G' may be identified with $\mathfrak{L}(\mathbb{R}^n)$, the space of all $n \times n$ real matrices.

Consider the polynomial in $\lambda \in \mathbb{R}$,

$$\det\left(\lambda I_n - \frac{1}{2\pi} A\right) = \sum_{k=0}^{n} \lambda^{n-k} f_{k/2}(A),$$

where $A \in \mathfrak{L}(\mathbb{R}^n)$. From the polynomial $f_{k/2}(A)$ one can obtain by 'polarization' a k-linear symmetric map $f_{k/2}(A_1, A_2, \ldots, A_k)$. Since

$$\det\left(\lambda I_n - \frac{1}{2\pi} A\right) = \det a^{-1}\left(\lambda I_n - \frac{1}{2\pi} A\right) a =$$
$$= \det\left(\lambda I_n - \frac{1}{2\pi} a^{-1} A a\right)$$

for any $a \in GL(n, \mathbb{R})$, the k-linear map $f_{k/2}$ is invariant under the action of $GL(n, \mathbb{R})$ in $\mathfrak{L}(\mathbb{R}^n)$.

Let $[\gamma]$ denote the cohomology class of a form on M corresponding to a closed, invariant and horizontal form γ on P. If Ω is the curvature form of a connection on $P \to M$, then

$$p_{k/2}(E) = [f_{k/2}(\Omega)] \in H^{2k}(M, \mathbb{R})$$

is the (k/2)th Pontryagin class of the bundle E. The structure group G of $P \to M$ can be restricted to O(n) (by the introduction of a smooth metric in the fibres of $E \to M$). Therefore, any connection on $P \to M$ can be smoothly deformed into a Euclidean, i.e. O(n)'-valued connection. For such a connection, the curvature form Ω is also O(n)'-valued, i.e.

represented by 2-forms with values in the Lie algebra of $n \times n$ skew-symmetric matrices. If ${}^tA = -A$, then the equality

$$\det\left(\lambda I_n - \frac{1}{2\pi} A\right) = \det\left(\lambda I_n + \frac{1}{2\pi} A\right)$$

shows that $f_{k/2}(A)$ is an even function of A. Since for odd k it is also an odd function,

$$f_{k/2}(A) = 0 \quad \text{for} \quad k = 2s+1 \quad \text{and} \quad A \in O(n)'.$$

Therefore, all Pontryagin classes with odd k are zero, and one can put

$$k = 2s;\ s = 0, 1, \ldots, \left[\frac{1}{4} \dim M\right];$$

$$\text{(12)} \qquad p_s(E) = [f_s(\Omega)] \in H^{4s}(M, \mathbb{R}).$$

4. *Chern classes*. Similarly, if $E \to M$ is a *complex* n-dimensional vector bundle associated with a (faithful) representation of $G = GL(n, \mathbb{C})$ and a principal G-bundle $P \to M$, then one can define its Chern classes as follows. One first defines the invariant polynomials $g_k (k = 0, \ldots, n)$ by

$$\det\left(\lambda I_n - \frac{1}{2\pi i} A\right) = \sum_{k=0}^{n} \lambda^{n-k} g_k(A), \qquad A \in \mathfrak{L}(\mathbb{C}^n),$$

and puts

$$c_k(E) = [g_k(\Omega)] \in H^{2k}(M, \mathbb{C}).$$

The structure group G of the bundle may be restricted to U(n) by the introduction of a hermitean metric on the bundle $E \to M$. Therefore, it is sufficient to consider $U(n)'$-valued curvature forms Ω. Since the polynomials g_k restricted to $U(n)'$ are real, so are the Chern classes

$$\text{(13)} \quad c_k(E) = [g_k(\Omega)] \in H^{2k}(M, \mathbb{R}),\ k = 0, \ldots, \frac{1}{2} \dim M.$$

There is a simple relation between Pontryagin and Chern classes resulting from the possibility of complexifying any real vector bundle and embedding, in a trivial way, $GL(n, \mathbb{R})$ into $GL(n, \mathbb{C})$. It follows from the formula

$$f_s(A) = g_{2s}(iA) = i^{2s} g_{2s}(A) = (-1)^s g_{2s}(A)$$

that

$$p_s(E) = (-1)^s c_{2s}(\mathbb{C} \otimes E), \tag{14}$$

and

$$c_{2s+1}(\mathbb{C} \otimes E) = 0. \tag{15}$$

The last relation provides a simple obstruction to restricting $GL(n, \mathbb{C})$ to $GL(n, \mathbb{R})$: if one of the odd Chern classes of a complex vector bundle is non-zero, then the bundle cannot be made into a real bundle of the same dimension. (It can always be made, in a trivial manner, into a real bundle of double real dimension). For example, the first Chern class of the 1-dimensional complex vector bundle associated with the Hopf $U(1)$-bundle $\mathbb{S}_3 \to \mathbb{S}_2$ is different from zero. (In fact, its period corresponds to the Dirac monopole of lowest strength, cf. T. T. Wu and C. N. Yang, Physical Rev. D *14* (1976), 437).

Therefore this bundle cannot be restricted to a real line bundle. Magnetic poles, if found in nature, would justify the necessity of using complex wave functions to describe the quantum-mechanical behaviour of particles interacting with the poles.

5. *The Euler class.* Consider now the even-dimensional real vector space $\mathbb{R}^{2m}$ with the canonical frame $e^1, e^2, \ldots, e^{2m}$. Let α be a 2-form on $\mathbb{R}^{2m}$,

$$\alpha = \frac{1}{2} A_{ij}\, e^i \wedge e^j, \qquad A_{ij} + A_{ji} = 0.$$

One defines the Pfaffian of A by the formula

$$\underbrace{\alpha \wedge \alpha \wedge \ldots \wedge \alpha}_{\text{m factors}} = m!\ \mathrm{Pf}(A)e^1 \wedge e^2 \wedge \ldots \wedge e^{2m}.$$

If $B \in \mathfrak{L}(\mathbb{R}^{2m})$, then $\mathrm{Pf}({}^tBAB) = \mathrm{Pf}(A) \det B$, so that the Pfaffian is invariant under the action of $G = SO(2m)$. Moreover,

$$\mathrm{Pf}(A)^2 = \det A.$$

If $E \to M$ is a real, oriented, 2m-dimensional vector bundle and Ω a curvature form on its principal bundle, then

$$e(E) = \left[\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)\right] \in H^{2m}(M, \mathbb{R}) \tag{16}$$

is its Euler class. In particular, if $E = TM$ is the tangent bundle of an oriented 2m-dimensional manifold M, then the period of e(E) over M is equal to the Euler-Poincaré characteristic of M. For $m = 1$ this reduces to the classical *Gauss-Bonnet theorem*: if Ω_{12} is the component of the curvature 2-form of a closed oriented surface M, referred to a (local) field of orthonormal frames (e^1, e^2) on M, then

$$\frac{1}{2\pi}\int_M \Omega_{12} = b_0 - b_1 + b_2. \tag{17}$$

There is a simple relation between the top (= highest degree) Chern class and the Euler class of the real form of a complex vector bundle $E \to M$. Let E be m-dimensional and consider the embedding k of $GL(m, \mathbb{C})$ in $GL(2m, \mathbb{R})$ given by

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a and b are real $m \times m$ matrices. The map k is a homomorphism of groups, and

$$\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = |\det(a + ib)|^2 > 0.$$

This shows that the *real form* of a complex vector bundle E is orientable. Moreover, if the structure group of E is restricted to U(m), then its real form comes with a structure group restricted to SO(2m). If a + ib is in the Lie algebra of U(m),

$$(a + ib)^+ + a + ib = 0,$$

then the matrix k(a + ib) is in the Lie algebra of SO(2m),

$$(18) \qquad {}^t a + a = 0 \quad \text{and} \quad {}^t b = b.$$

Moreover, if the matrices a and b satisfy (18), then

$$(19) \qquad \det i(a + ib) = \mathrm{Pf} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

On the other hand, one has

$$g_m(a + ib) = \det \left(\frac{i}{2\pi} (a + ib) \right).$$

By comparing (13), (16) and taking into account (19), one obtains the formula

$$(20) \qquad c_m(E) = e(E_{real})$$

relating the mth Chern class of an m-dimensional complex vector bundle E to the Euler class of its real form E_{real}.

For example, the U(1)-bundle

$$\mathbb{S}_3/\mathbb{Z}_2 \to \mathbb{S}_2$$

has its first Chern class equal to the Euler class of the tangent bundle of $\mathbb{S}_2$, because

$$\mathbb{S}_3/\mathbb{Z}_2 = SO(3)$$

is the total space of the bundle of (oriented) orthonormal frames of $\mathbb{S}_2$.

LITERATURE

There is a wealth of literature on differential geometry and its applications to theoretical physics. The list given below is not exhaustive, but it contains information on the books and articles used by the author while preparing the lectures. It includes also references to publications which appeared after the lectures had been given: they extend and complement the material presented here.

A. BOOKS

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INDEX

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This little book is based on lectures given by the author, during the academic year 1976-77, at the Institute of Theoretical Physics, State University of New York at Stony Brook. The text closely follows the lecture notes and reflects their informal, sketchy character.

The book is oriented towards presenting differential geometry in a manner suitable for use in the study of classical gauge theory and general relativity. Emphasis is put on geometrical and intrinsic aspects of tensor algebra and the calculus on manifolds. Differential forms are much used and shown to be convenient to write the Maxwell and Yang-Mills equations as well as to find their solutions such as plane waves and the Liénard-Wiechert field. There is an elementary introduction to Lie groups, fibre bundles and connections followed by applications to physics. Many exercises and examples are presented, including a description of simple, topologically non-trivial gauge configurations (magnetic monopoles and the BPST instanton). The last chapter contains a very brief outline of some of the notions of algebraic and differential topology needed to define Chern, Pontryagin and Euler classes.

ANDRZEJ TRAUTMAN was born in Warsaw, Poland in 1933 and graduated from a high school in Paris (1949). He prepared a Ph. D. thesis under Leopold Infeld's supervision and obtained the degree in Warsaw in 1959. Since 1961 he has worked at Warsaw University where he is now Professor of physics and Director of the Institute of Theoretical Physics. Andrzej Trautman was Research Fellow and Visiting Professor at King's College (1958) and Imperial College (1959-60), London, at Syracuse University (1961 and 1967), Collège de France (1963 and 1981), University of Chicago (1971), SUNY at Stony Brook (1976-77) and Université de Montréal (1982). He is a member of the Polish and Czechoslovak Academies of Sciences. He wrote numerous articles on the theory of general relativity, applications of differential geometry to physics and on classical Yang-Mills fields.