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EINSTEIN AND THE GEOMETRIZATION OF PHYSICS

1. The idea of connecting the motion of matter to the geometric properties of space – such as curvature – may be traced back to Lobačevskiĭ [17], Riemann [20], Clifford [2] and Mach [18]. Einstein was the first to formulate a definite, precise, verifiable and verified theory which directly related the distribution and motion of matter to the geometry of spacetime. Moreover, his theory of general relativity reduces gravitation to geometry in the sense that all information about gravitational interactions is contained in the data on metrical relationships in spacetime. The simplicity and beauty of that theory encouraged mathematicians and physicists to look for its extension to electromagnetism, which, in the 1920s, was the only known, other than gravitation, domain of fundamental physical phenomena.

2. The search for a geometric and unified theory of physical field occupied a major place in the scientific activity of Albert Einstein. Shortly after the general theory of relativity had been formulated (1915), Hermann Weyl put forward a geometric model of gravitational and electromagnetic forces [26]. Einstein criticized [E1] the extension of Riemannian geometry proposed by Weyl, but he became fascinated with the idea of building a unified theory, providing a description of electromagnetic and gravitational fields in terms of differential geometry [E2].

3. Einstein's programme of constructing a unified theory has been often, and sometimes sharply, criticized by physicists. Einstein was accused of having ignored the growth of physics after 1935, when it became clear that there exist in nature fundamental interactions other than gravitational and electromagnetic forces. Moreover, all Einstein's attempts were in the framework of classical physics and he hoped to explain the discrete nature of matter within that scheme.

4. The original " Einstein Programme " of unifying physics can be summarized as follows:
– construct a theory of a classical geometric field without sources;

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– unify the basic variables of gravitation and electromagnetism into one geometric object;

– obtain unified field equations from a variational principle;

– describe charged particles as sourceless solutions and derive their motion from the field equations.

Einstein hoped that fundamental properties of elementary particles and their apparent quantum behaviour could also be somehow explained on the basis of a geometric, classical and unified theory.

5. The relativistic theory of gravitation is based on two fundamental geometric objects: a metric tensor g and a linear connection Γ. The metric is needed to measure distances, time intervals, relative velocities and angles. Based on the notion of parallel transport, due to Levi–Civita, the connection is a somewhat subtler concept needed to compare directions, forces and fields at points separated in space and time.

6. Essentially all attempts at unification rely on geometries based on g and Γ. In order to obtain new degrees of freedom needed to describe electromagnetism, one either relaxes the restrictions imposed on g and Γ in Riemannian geometry, or increases the number of dimensions of the underlying manifold, or both.

7. The first 'unified theory' was due to Weyl [26]: he observed that Einstein's theory of gravitation is based on a 'relativity of directions' and proposed to extend it to account for a 'relativity of magnitude' by allowing conformal transformations of the metric,

$$g'_{\mu\nu} = e^{f} g_{\mu\nu} \tag{1}$$

If

$$\nabla_\rho g_{\mu\nu} = A_\rho g_{\mu\nu}$$

then eq. (1) is compatible with the gauge transformation

$$A'_\mu = A_\mu + \partial f / \partial x^\mu$$

Einstein criticized Weyl's theory by pointing out that, according to (1), $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ is not well–defined and the rate of atomic clocks may depend on their history. Weyl's theory played a major role in the history of physics [30]: in 1929 Weyl interpreted gauge transformations as acting on wave-functions of charged particles, rather than on g. This idea led later to non-Abelian gauge theories [29] and to the interpretation of electromagnetic and Yang–Mills potentials as connections on principal fibre bundles (cf. § 12).

8. A class of unified theories, originated by Kaluza [8] and Klein [12], has been developed by Einstein and his coworkers [E3, E10, E26–29] and other authors [6, 15, 22]. These theories assume a 5–dimensional Riemannian

geometry of a special type. It has been recognized that the Riemannian metric on the Kaluza–Klein space is equivalent to the metric which may be introduced, in a natural manner, on the total space P of a U(1)–bundle over spacetime M, provided that M has a Riemannian metric and P carriers a U(1)–connection, i.e. an electromagnetic potential. This observation has been generalized to non–Abelian gauge configurations [1,6,9,13,24].

9. Eddington [4] proposed to consider Γ as a basic quantity and to derive from it both the metric tensor and the eletromagnetic field by splitting $R_{\mu\nu}$ into its symmetric and skew–symmetric parts. Einstein developed this idea by postulating that the Lagrange density should be proportional to $| \det R_{\mu\nu} |^{1/2}$. Unfortunately, this led to equations incompatible with experiments and Einstein abandoned this approach [E4–8]. Very recently, Kijowski and Tulczyjew were able to improve the Eddington–Einstein theory and to derive equations equivalent to the Einstein–Klein–Gordon and the Einstein–Maxwell systems without introducing the metric tensor as a fundamental variable [10,11].

10. The theories based on teleparallelism assume a vanishing curvature of the connection Γ, compatible with the metric tensor g. In other words, all geometry – and physics, as far as only gravitation and electromagnetism are concerned – is contained in g and the torsion tensor Q. These theories, developed by Einstein in collaboration with Mayer [E11–23, E25] also turned out to be non–viable, but they now play a role in the study of ' gauge theories of gravity ', developed by Hehl, Ne'eman, Von der Heyde and their coworkers [5].

11. In 1925 Einstein briefly considered [E9] a theory based on Γ and a non–symmetric density $\mathfrak{G}^{\mu\nu}$; he identified $\mathfrak{G}^{[\mu\nu]}$ with the elettromagnetic field. He returned to this idea in 1945 [E22–E42] and spent the last ten year of his life working on an ' asymmetric theory ' based on $g_{\mu\nu}$ and $\Gamma^{\rho}{}_{\mu\nu}$, both of which need not be symmetric in the pair (μ , ν). Schrödinger [21] and many other authors [7,23] contributed to this field of research, but no successful, unified theory emerged out of their work.

12. The development of physics in recent years throws a new light on the idea of geometrizing physics. It seems that all fundamental interactions exhibit similarities that are most easily perceived and described in the language of differential geometry. There is a prospect for achieving a unification of weak and electromagnetic interactions (the Weinberg–Salam theory). According to Chen Ning Yang [31],

> " Einstein's insistence on the importance of unification was a deep insight, which he courageously defended, against all spoken and unspoken criticism.... It turns out that the structure that Einstein was seeking was the gauge field ".

Indeed, it appears that the idea of a connection has a significance which goes well beyond the theory of gravitation. The potential of an electromagnetic field plays a role in comparing the phases of wave functions of charged particles at different points. This leads to the important idea a of a gauge field. Generalized by Yang and Mills to more complicated " phase factors " which can change by means of trasformations belonging to a non–Abelian group, gauge fields have become the most promising candidate to provide a geometric description of physics. From the point on view of mathematics they are connections on principal fibre bundles.

13. Gauge theories attract much attention because they are renormalizable. Moreover, the non–Abelian theories allow a spontaneous breaking of symmetries that leads to massive vector particles, needed to describe the short–range character of nuclear forces.

14. If the current views are confirmed, then the four fundamental interactions that (probably) underly all physical phenomena will allow a descriptions by means of gauge fields – or, equivalently, connections, – associated with Lie groups. In the case of gravitation, the Lorentz group plays the fundamental role, the Weinberg–Salam theory is based on $U(1) \times SU(2)$, whereas the current model of strong interactions (quantum chromodynamics) is based on $SU(3)$.

15. If the future development of physics confirms the hopes physicists now associate with gauge fields, then connections on principal bundles will provide the key to a geometrization of physics close in spirit, if not in detail, to Einstein's dream.

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Einstein's papers on unified field theories

This list is as complete as the author of the article has been able to make it. Many of Einstein's papers have been published in the Sitzungsberichte Preuss. Akademie der Wissenschaften (Berlin). The name of this journal is abbreviated here to SB. At the end of most listings there is a code letter relating the paper to one of five main tracks in the field of unified theories. The meaning of the code letters is as follows:

A = asymmetric theories, i.e. theories with an asymmetric metric tensor and a linear connection with torsion,

B = pure affine theories, initiated by Eddington and based on a symmetric linear connection,

K = theories of the Kaluza–Klein type,
T = theories with teleparallelism,
W = the Weyl theory.

References in the text to Einstein's papers listed below are given in the style [En].

1. *Eine naheliegende Ergänzung des Fundamentes der allgemein Relativitätstheorie.* SB (1921) 261–264 (W).
2. *Grundgedanken und Probleme der Relativitätstheorie,* in *Nobelstiftlesen, Les Prix Nobel en 1921–22.* Imprimerie Royale, Stockholm 1923 (general remarks on the programme of unification).
3. (with J. Grommer) *Beweis der Nichtexistenz eines überall regulären zentrisch symmetrischen Feldes nach der Feldtheorie von Th. Kaluza.* Jerusalem University, « Scripta », 1 (1923), No. 7 (K).
4. *Zur allgemeinen Relativitätsheorie.* SB, (1923), 32–38 (B).
5. *Bemerkung zu meiner Arbeit « Zur allgemeinen Relativitätstheorie ».* SB, (1923) 76–77 (B).
6. *Zur affinen Feldtheorie.* SB, (1923), 137–140 (B).
7. *The theory of the affine field.* « Nature », 112 (1923), 448–449 (B).
8. (*Anhang:*) *Eddingtons Theorie und Hamiltonsches Prinzip.* In the book by A. S. Eddington, *Relativitätstheorie in mathematischer Behandlung,* Springer, Berlin 1925 (B).
9. *Einheitliche Feldtheorie von Gravitation und Elektrizität.* SB, (1925) 414–419 (A).
10. *Zu Kaluzas Theorie des Zusammenhängs von Gravitation und Elektrozität.* SB, (1917) 23–25 and 26–30 (K).
11. *Riemann–Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelism.* SB, (1928) 217–221 (T).
12. *Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität.* SB, (1928) 224–227 (T).
13. *Über den gegenwartigen Stand der Feldtheorie.* « Festschrift Prof. Dr. A. Stodola zum 70. Geburstag », Füssli Verlag, Zürich 1929 (T).
14. *Zur einheitlichen Feldtheorie.* SB, (1929), 2–7 (7).
15. *The new field theory.* « Observatory », *52* (1929) 82–87 and 114–118 (T).
16. (with Th. de Donder) *Sur la théorie synthétique des champs.* « Revue gén. de l'électricité », 25 (1929) 35–39 (T).
17. *Einheitliche Feldtheorie und Hamiltonsches Prinzip.* SB, (1929) 156–159 (T).
18. *Théorie unitaire de champ physique.* « Ann. Inst. H. Poincaré », 1 (1930) 1–24 (T).
19. *Auf die Riemann–Metrik und den Fern–Parallelismus gegründete einheitliche Feldtheorie.* « Math. Ann. ». 102 (1930) 685–697 (T).
20. *Die Kompatibilität der Feldgleichungen in der einheitlichen Feldtheorie.* SB, «(1930) 18–23 (T).
21. (with W. Meyer) *Zwei strenge statische Lösungen der Feldgleichungen der einheitlichen Feldtheorie.* SB, (1930) 110–120 (T).
22. *Zur Theorie der Räume mit Riemann–Metrik und Fernparallelismus.* SB, (1930), 401–402 (T).
23. *Über den gegenwärtigen Stand der allgemeinen Relativitätstheorie.* Yale Univ. Library Gazette, 6 (1930) 3–6 (T).
24. *Gravitational and electrical fields,* « Science », 74 (1930) 438–439 (K).
25. (with W. Meyer) *Systematische Untersuchungen über kompatible Feldgleichungen, welche in einem Riemannschen Raume mit Fernparallelismus gesetzt werden können.* SB, (1931) 252–265 (T).
26. (with W. Meyer) *Einheitliche Theorie von Gravitation und Elektrozität.* SB, (1931) 541–557 and (1932) 130–137 (K).
27. *Der gegenwärtiger Stand der Relativitätstheorie.* « Die Quelle (Pedagogischer Führer) », 82 (1932) 440–442 (K).
28. (with P. G. Bergmann) *Generalisation of Kaluza's theory of electricity.* « Ann. of Math. », 39 (1938) 683–701 (K).

29. (with V. Bargmann and P. G. Bergmann) *On five-dimensional representation of gravitation and electricity*. In. *Th von Karman Anniversary Volume*, pp. 212–225. Caltech, Pasadena 1941 (K).
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31. *Bivector fields II*, « Ann. of Math. », (45 1944) 15–23.
32. *Generalisation of the relativistic theory of gravitation*. « Ann. of Math. », *46* (1945) 578–584 (A; g and Γ complex).
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