# The Einstein-Cartan Theory

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## Summary

(Full text of this lecture will be published elsewhere)

In 1923 Elie Cartan proposed to modify the Einstein theory of gravitation by allowing space-time to have torsion and relating it to the density of intrinsic angular momentum of a continuous medium. Cartan's idea did not attract any attention at that time. This may be due, on the one hand, to the fact that Cartan's paper had appeared before the spin of the electron was discovered, and, on the other, to Einstein's fascination with the problem of unifying gravity with electromagnetism.

The idea of connecting torsion to spin became alive again around 1960, mainly thanks to the work of D. W. Sciama and T. W. B. Kibble. There was considerable activity on this problem from 1965 to 1975.

There is no "logical" or experimental, compelling need to modify Einstein's theory, but one can advance good heuristic arguments in favour of the Cartan idea:

(i) The geometrical independence of the metric $g$ and linear connection $\Gamma$ leads to the idea of treating these quantities as independent variables in the sense of a principle of least action. If $g$ and $\Gamma$ are assumed to be compatible, then the freedom in the choice of $\Gamma$ reduces to that of the torsion tensor $Q$.

(ii) According to relativistic quantum theory, the Poincaré — or the inhomogeneous Lorentz group — is physically more significant than the Lorentz group itself. The Poincaré group has two fundamental invariants: mass and spin. The first among them is related to translations and to energy-momentum. In Einstein's theory, the density of energy-momentum is source of curvature whereas spin has no such direct dynamical significance. In a sense, the Einstein-Cartan theory restores — to some extent — the symmetry between mass and spin. It introduces also an unexpected "duality": via Noether's theorem, energy-momentum is generated by translations whereas Einstein's equation relates it to curvature, which is responsible for rotations of vectors undergoing parallel transport. Conversely, spin is generated by rotations, but torsion induces translations in the tangent spaces to a manifold ("Cartan displacement"). This duality can be traced to the fact that the Einstein-Cartan Lagrangian is linear in curvature, an assumption criticized by C. N. Yang. Recently, F. W. Hehl, Y. Ne'eman, N. Straumann and their coworkers have studied a theory of gravitation based on a Lagrangian quadratic in both curvature and torsion. It is clear however, that there are no compelling reasons to abandon the linear Lagrangian.

(iii) There is an interesting analogy between the description of magnetic moments in electrodynamics and spin in the theory of gravitation. In a phenomenological

description of electromagnetism, the external magnetic field produced by a ferromagnet may be obtained in at least three ways: by considering a surface current equivalent to the actual distribution of microscopic currents and magnetic moments, by replacing the latter by a volume distribution of "Ampère currents", or, finally, by introducing a smooth field of the magnetization vector. In the Einstein theory, there are analogues for the first two descriptions, whereas the Einstein-Cartan theory provides the third.

The Einstein-Cartan theory assumes, as a model of spacetime, a four-dimensional manifold with a linear connection $\Gamma$ compatible with a metric tensor $g$. The gravitational part of the Lagrangian, $\sqrt{-g}\,R$, is formed from the curvature tensor of $\Gamma$. This prescription leaves no room for new arbitrary constants. The left hand sides of the field equations are obtained by varying this Lagrangian with respect to $g$ and $Q$. Variation with respect to $g$ may be replaced by that relative to the field of frames (tetrads). The sources of the gravitational field are described by expressions resulting either from phenomenology or by varying an action integral obtained by applying the principle of minimal gravitational coupling to a special-relativistic Lagrangian. There are subtleties concerning the Maxwell and other gauge fields.

The Einstein-Cartan field equations are

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = \frac{8\pi G}{c^4}\,t_{\mu\nu} \tag{E}$$

$$Q^{\varrho}_{\mu\nu} - \delta^{\varrho}_{\mu}Q^{\sigma}_{\sigma\nu} - \delta^{\varrho}_{\nu}Q^{\sigma}_{\mu\sigma} = \frac{8\pi G}{c^3}\,s^{\varrho}_{\mu\nu} \tag{C}$$

The Cartan equation (C) is trivial in the sense that if the spin density vanishes, $s^{\varrho}_{\mu\nu} = 0$, then so does torsion, $Q^{\varrho}_{\mu\nu} = 0$. Quite independently of this, torsion is topologically trivial: any linear connection can be deformed into a connection without torsion.

The Bianchi identities for $R^{\mu}{}_{\nu\varrho\sigma}$ and $Q^{\varrho}_{\mu\nu}$ give two sets of constraints on the sources. One of them may be symbolically written as

$$\nabla t = R \cdot s + Q \cdot t \sim R \cdot Q$$

Without good reason, Cartan required $\nabla t = 0$ and was led to the algebraic constraint $R \cdot Q = 0$.

F. W. Hehl has shown that, by solving (C) for $Q$, one can reduce the system (E) — (C) to an equation with the Einstein tensor, built from $g$, on the left and an effective energy-momentum tensor,

$$T_{\text{eff}} = t + \operatorname{div} s + s^2 \tag{H}$$

on the right. The term quadratic in spin seems to be the only essential difference between the Einstein-Cartan and Einstein theories. Similar terms have also been obtained by B. M. Barker and R. F. O'Connell from Gupta's quantum theory of gravity.

On the basis of (H) one can argue that the Einstein-Cartan theory may be physically relevant only when the density of energy is of the same order of magnitude as the spin density squared. For matter consisting of particles of mass $m$ and spin $\hbar/2$, this will occur at densities of order $m^2c^4/G\hbar^2$. For nucleons, the density in question is $10^{54}$ g/cm$^3$, much less than the Planck density. Putting the same result in a slightly different way, one can say that a particle should have a radius of order

$$\left(\frac{Gm^2}{\hbar c}\right)^{1/3} \frac{\hbar}{mc}$$

for gravitational effects of spin to be comparable to those of mass.

The Einstein-Cartan theory is viable, but differs so slightly from Einstein's theory that it may take a very long time to confirm it or disprove.

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