Can poles change color?[a)]

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The definition of the total nonabelian charge ("color") in a classical Yang–Mills theory is shown to require a careful analysis of the boundary conditions at infinity imposed on the potentials and on gauge transformations. The color current of a nonabelian plane wave is found to be different from zero in the transverse gauge, though it vanishes in the null gauge. The color charge of a single pole, described by the Liénard–Wiechert potentials, is constant by virtue of the Yang–Mills equations. An approximate computation indicates that the total color charge of a system of particles may change in time, as a result of radiation. To make this result meaningful, it is necessary to find a method of fixing the allowed gauge transformations to those having a direction-independent limit at infinity.

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I. INTRODUCTION

In a classical gauge theory of the Yang–Mills type, sources of the field have a nonabelian charge density. More precisely, the sources are described by a 3-form j with values in the Lie algebra of the structure (gauge) group. If the group is nonabelian, the current j does not, by itself, satisfy a differential conservation law; it has to be supplemented by another current i constructed out of quantities referring only to the gauge configuration. The latter current may be interpreted as representing the density of the nonabelian charges residing in the gauge field itself. We use the name "color" for this nonabelian charge, but our considerations have little, if anything, to do with chromodynamics. Our nonabelian charges may equally well be associated with "flavors" and, in particular, the charges occurring in gauge theories of weak interactions.

One expects that, upon integration of $i + j$ over a space region Ω, it should be possible to find its total color content. By means of the Gauss law, the total color is represented as the flux, across the boundary of Ω, of the Lie-algebra-valued electric field. The problems considered in this paper are the following: What is the dependence of the total color on the choice of gauge? Can the total nonabelian charge of a system of particles change in time, as a result of radiation? Both these problems have analogs in Einstein's theory of gravitation. The question of color radiation is analogous to that of gravitational radiation. From this point of view, the Yang–Mills theory may be considered as a simplified model of general relativity.

The nature of the difficulties one encounters when trying to define total color, even in the static case, can be seen as follows. Consider a gauge potential A, function of the spherical coordinates r,θ,φ. For an isolated system, one expects that, at large distances,

$$A = O(r^{-1}) \tag{1.1}$$

and the corresponding electric field E is of order $O(r^{-2})$. If A is replaced by its gauge transform $A' = S^{-1}AS + S^{-1}dS$, then E changes into $E' = S^{-1}ES$. A gauge transformation function S satisfying

$$S(r,\theta,\varphi) = a(\theta,\varphi)[I + O(r^{-1})] \tag{1.2}$$

is compatible with (1.1) because $A' = a^{-1}Aa + a^{-1}da + O(r^{-2})$ and $a^{-1}da = O(r^{-1})$. This transformation, however, induces such a change in E,

$$E' = a(\theta,\varphi)^{-1}Ea(\theta,\varphi) + O(r^{-3}) \tag{1.3}$$

that the flux of E' bears no simple relation to that of E. In the theory of general relativity, a similar problem occurs, but is not as acute as in the case of the Yang–Mills theory.[1] In Einstein's theory, the coefficients Γ of a linear connection constitute an analog of A: they transform under changes of the local frames in a manner similar to A. There is also an important difference: the Yang–Mills equations contain second derivatives of A, whereas Γ appears in Einstein's equations differentiated only once. As a result, for a static configuration, Γ falls off faster, $\Gamma = O(r^{-2})$, and an arbitrary function a occurring in (1.2) is not allowed here.

Presumably, the arbitrary function a can be eliminated by a more detailed analysis of the gauge potentials. For example, if it can be shown that the $1/r$ part of a static potential is spherically symmetric, then one can restrict a by requiring that the spherical symmetry be explicit.

In this paper, we leave aside the question of how the direction-dependent gauge transformation can be eliminated and concentrate on the study of the dynamics of Yang–Mills fields in the wave zone. The purpose of the work is to determine the rate of change of a "retarded" total color charge. We use an asymptotic expansion method developed for the study of gravitational radiation by Bondi, *et al.*[2] and Sachs.[3] The method has also been used in the context of the Yang–Mills theory to prove peeling-off theorems for gauge fields.[4]

II. NOTATION[5]

All gauge configurations considered here are defined on the Minkowski space $\mathbb{R}^4$ with its standard metric $g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2$ and orientation given

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by the volume 4-form $dt\wedge dx\wedge dy\wedge dz$. A Lie group G is assumed to be the structure (gauge) group of the theory and $\mathfrak{g}$ denotes its Lie algebra. A gauge potential A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^4$,

$$A = A^i_{\ \mu} e_i \, dx^\mu, \tag{2.1}$$

where (e_i) is a linear basis in $\mathfrak{g}$. The field strengths are

$$F = dA + \tfrac{1}{2}[A,A] = \tfrac{1}{2} F^i_{\mu\nu} e_i \, dx^\mu \wedge dx^\nu. \tag{2.2}$$

The four-dimensional Hodge dual of F is denoted by $\check{F}$. If F is represented in terms of its electric and magnetic components,

$$F = dt \wedge (E_x\, dx + E_y\, dy + E_z\, dz) - B_x\, dy \wedge dz - B_y\, dz \wedge dx - B_z\, dx \wedge dy, \tag{2.3}$$

then

$$\check{F} = E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy + dt \wedge (B_x\, dx + B_y\, dy + B_z\, dz). \tag{2.4}$$

The Yang–Mills equations are

$$d\check{F} + [A,\check{F}] = 4\pi j, \tag{2.5}$$

where j is the $\mathfrak{g}$-valued 3-form describing the sources.

If $S:\mathbb{R}^4 \to G$ is a function corresponding to a gauge transformation, then

$$A' = S^{-1}AS + S^{-1}dS \quad \text{and} \quad F' = S^{-1}FS \tag{2.6}$$

are the transformed potential and field strengths, respectively.

If G is either abelian or semisimple and compact, then its Lie algebra $\mathfrak{g}$ is compact, i.e., it admits a scalar product, $\mathfrak{g}\times\mathfrak{g} \ni (X,Y) \mapsto (X\,|\,Y) \in \mathbb{R}$, which is invariant,

$$([Z,X]\,|\,Y) + (X\,|\,[Z,Y]) = 0 \quad \text{for any } X,Y,Z \in \mathfrak{g}, \tag{2.7}$$

and positive-definite. If G is semisimple and compact one can indeed take the (negative of the) Killing–Cartan form on $\mathfrak{g}$ as such a scalar product. By applying (2.7) it is straightforward to prove the following:

Lemma: If the elements X and Y of a compact Lie algebra are such that $[X,Y] = X$, then $X = 0$.

If (e_i) is a linear basis in $\mathfrak{g}$, then X may be written as $X^i e_i$ and

$$(X\,|\,Y) = h_{ij} X^i X^j, \tag{2.8}$$

where

$$h_{ij} = (e_i\,|\,e_j). \tag{2.9}$$

The structure constants of $\mathfrak{g}$ relative to (e_i) are defined by

$$[e_i, e_j] = c^k_{ij} e_k$$

and the condition of invariance (2.7) is equivalent to

$$h_{kl} c^l_{ij} + h_{jl} c^l_{ik} = 0. \tag{2.10}$$

The two-dimensional unit sphere S_2 has a metric $d\theta^2 + \sin^2\theta\, d\varphi^2$ and a surface element $d\theta \wedge \sin\theta\, d\varphi$. The two-dimensional Hodge dual on S_2 will be denoted by a star; thus

$$^\star 1 = d\theta \wedge \sin\theta\, d\varphi, \quad ^\star d\theta = \sin\theta\, d\varphi, \quad ^\star \sin\theta\, d\varphi = -d\theta. \tag{2.11}$$

There exist a few useful relations between four-dimensional and two-dimensional duals. Let $r = (x^2 + y^2 + z^2)^{1/2}$ and $u = t - r$ be the "retarded time." In coordinates (u,r,θ,φ) the Minkowski metric is $du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$ and $dt \wedge dx \wedge dy \wedge dz = du \wedge dr \wedge r d\theta \wedge r \sin\theta\, d\varphi$. One easily shows that, for any 1-form α on S_2, linear in $d\theta$ and $d\varphi$, the 4-dual of $du \wedge \alpha$ is equal to $du \wedge {}^\star\alpha$, whereas the 4-dual of $dr \wedge \alpha$ is $-(du + dr) \wedge {}^\star\alpha$. Similarly, the 4-dual of $du \wedge dr$ is ${}^\star r^2$.

III. THE CONSERVATION LAW

It is clear from the Yang–Mills field equation (2.5) that the current j of the sources is not conserved by itself: in general $dj \neq 0$. The "total current"

$$J = j - (1/4\pi)[A,\check{F}] \tag{3.1}$$

is conserved,

$$dJ = 0, \tag{3.2}$$

but contains a highly gauge-dependent field contribution,

$$i = -(1/4\pi)[A,\check{F}], \tag{3.3}$$

analogous to the pseudotensor of energy and momentum of the gravitational field in Einstein's theory. Let Σ_R be the surface (boundary) of a ball $\Omega_R \subset \mathbb{R}^3$ of radius R. The total nonabelian charge q contained in Ω_R at time t may be formally defined as

$$q(t,R) = \int_{\Omega_R} J \quad \text{at } t = \text{const.} \tag{3.4}$$

and, by virtue of Eq. (2.5), expressed as a surface integral,

$$q(t,R) = (1/4\pi) \int_{\Sigma_R} \check{F} \quad \text{at } t = \text{const.} \tag{3.5}$$

The Gauss law (3.5) is analogous to the expression of total energy and momentum of a gravitational configuration by means of a surface integral of the Von Freud superpotential. The rate of change of color, $\dot{q} = \partial q/\partial t$ is given by

$$\dot{q}(t,R) = \int_{\Sigma_R} \frac{\partial}{\partial t} \lrcorner\, i \quad \text{at } t = \text{const.} \tag{3.6}$$

provided that the sources are spatially bounded and R is sufficiently large so that $j = 0$ on Σ_R. If both A and F tend to 0 sufficiently fast as $R \to \infty$, then

$$q_\infty(t) = \lim_{R\to\infty} q(t,R) \tag{3.7}$$

is well-defined and conserved by (3.6), $\dot{q}_\infty(t) = 0$. It is known, however, that such a description is not adequate when radiation is present. In this case, one expects a suitably defined total charge to change in the course of time and both A and F to behave as $1/r$ at large r. Making use of the retarded time $u = t - r$, one can define

$$q_{\text{ret}}(u) = \lim_{R\to\infty} q(u + R, R). \tag{3.8}$$

The charge q_{ret}, which is defined on the "future null infinity" in the sense of Penrose,[6] may depend on u even though $\dot{q}_\infty(t) = 0$.

IV. TWO SIMPLE EXAMPLES

The current i, describing the color carried by a classical gluon wave, depends on the choice of gauge to such an extent that it may always be reduced to 0 at a spacetime point. Moreover, in special cases it may be zero throughout spacetime even though the corresponding solution of the Yang–Mills equations is believed to represent a truly colored wave. This difficulty will be illustrated on the following.

Example (i): Let $v = t - z$ and let H be a $\mathfrak{g}$-valued function of x, y, and v. The potential[7]

$$A = H\,dv \tag{4.1}$$

is a solution of the sourceless Yang–Mills equations if, and only if, H satisfies the Laplace equation in x and y,

$$\Delta H = 0. \tag{4.2}$$

In particular, the solution $H = a(v)x + b(v)y$ corresponds to Coleman's nonabelian plane waves.[8] It follows from (4.1) that the field and its dual are

$$F = \left(dx\,\frac{\partial H}{\partial x} + dy\,\frac{\partial H}{\partial y}\right) \wedge dv,$$
$$\check{F} = \left(dy\,\frac{\partial H}{\partial x} - dx\,\frac{\partial H}{\partial y}\right) \wedge dv \tag{4.3}$$

so that $[A,\check{F}] = 0$ and the current i vanishes. Consider now the same configuration in a different gauge, defined as follows. Let S be a G-valued function of x, y, and v, defined as a solution of the equation

$$S^{-1}\dot{S} + S^{-1}HS = 0, \quad \text{where } \dot{S} = \frac{\partial S}{\partial v}. \tag{4.4}$$

The potential A' obtained from the potential (4.1) by transforming it with S is

$$A' = M\,dx + N\,dy, \tag{4.5}$$

where

$$M = S^{-1}\frac{\partial S}{\partial x} \quad \text{and} \quad N = S^{-1}\frac{\partial S}{\partial y}. \tag{4.6}$$

The transformed field strengths are

$$F' = dv \wedge (\dot{M}\,dx + \dot{N}\,dy),$$
$$\check{F}' = dv \wedge (\dot{M}\,dy - \dot{N}\,dx) \tag{4.7}$$

and

$$4\pi i' = -[A',\check{F}'] = ([M,\dot{M}] + [N,\dot{N}])dv \wedge dx \wedge dy. \tag{4.8}$$

In the transverse gauge (4.5) the current i' has a structure similar to that of the stress–energy vector-valued 3-form t_μ, given by

$$8\pi t_\mu = -((\dot{M}|\dot{M}) + (\dot{N}|\dot{N}))\frac{\partial v}{\partial x^\mu}\,dv \wedge dx \wedge dy. \tag{4.9}$$

The solution given by Eqs. (4.2)–(4.6) may be thus interpreted as representing a wave, moving with the velocity of light, endowed with energy and color densities proportional to $(\dot{M}|\dot{M}) + (\dot{N}|\dot{N})$ and $[M,\dot{M}] + [N,\dot{N}]$, respectively. From the appearance of the commutators in (4.8) one infers that radiation of color—if it exists—is a nonabelian phenomenon, requiring time-dependent and noncommuting sources.

A single pole particle of color q might radiate its charge away if $[q,\dot{q}]$ could be different from 0, but this is not the case, as follows from:

Example (ii)[9]: Let $z^\mu(s)$ be the coordinates of a timelike word line z, parametrized by its proper time s. One defines two functions u and r on the Minkowski space $\mathbb{R}^4$ as follows. For any $\xi \in \mathbb{R}^4$, let $u(\xi)$ be the value of s corresponding to the intersection of the word line z with the past-oriented light cone of vertex at ξ and let

$$r(\xi) = g_{\mu\nu}(\xi^\mu - z^\mu(u))\dot{z}^\nu(u). \tag{4.10}$$

Consider a pole particle of an *a priori* time-dependent color $q(u) \in \mathfrak{g}$ moving along z. The Liénard–Wiechert potential,

$$A = q(u)r^{-1}\dot{z}_\mu(u)\,d\xi^\mu, \tag{4.11}$$

is well-defined outside z, i.e., for $r \neq 0$, and the Yang–Mills equation in that region implies

$$\dot{q} + [q,\dot{q}] = 0. \tag{4.12}$$

Assuming that G compact, one obtains from the lemma

$$\dot{q} = 0. \tag{4.13}$$

This implies $i = 0$ so that the wave corresponding to (4.11) is colorless, although it transports energy.

V. BOUNDARY CONDITIONS AND GAUGE TRANSFORMATIONS

The formal definitions of q, Eqs. (3.4) and (3.5), have little meaning because of their unwieldy gauge dependence.[1,10] The total nonabelian charge should be an element of the Lie algebra $\mathfrak{g}$, defined up to "global gauge transformations," i.e., up to replacements of $q \in \mathfrak{g}$ by $a^{-1}qa$, where $a \in G$. If q is so defined, then one can construct out of it invariants, such as $(q|q)$, which provide gauge-independent, global characteristics of the system. A possible way of obtaining such a definition is suggested by the theory of general relativity where one considers total energy contained in "all of space" and expresses it by a surface integral over "a sphere at infinity." This is equivalent, in our case, to taking limits of q as $R \to \infty$, such as those given by Eqs. (3.7) and (3.8). If it can be shown that gauge transformation functions S may be meaningfully restricted to those having for $R \to \infty$ a limit independent of the direction along which one goes to infinity, then the limit of the surface integral (3.5) provides a total charge transforming, under changes of gauge, in the desired manner. The class of allowed gauge transformations depends on gauge configurations under study. In particular, in the case we consider, the asymptotic behavior ($R \to \infty$) of the gauge transformation functions should be adapted to that of the potentials.[11,12]

We shall make a specific assumption about the behavior of the potentials at large distances from the sources. The assumption may be justified by reference to what is known in linear field theories and by the successes of a similar hypothesis in the theory of gravitation.

From now on we shall use exclusively coordinates $u = t - r$, r, θ, and φ, with respect to which the Minkowski metric is $du^2 + 2du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$. We consider gauge configurations which may be described by potentials of the form[13]:

$$A = \sum_{k=1}^{\infty} r^{-k} A_k, \tag{5.1}$$

where each A_k is a g-valued form linear in du, dr, $r\,d\theta$, and $r \sin\theta\, d\varphi$, with coefficients depending only on u, θ, and φ. In particular,

$$A_1 = K\,du + L\,dr + r\omega, \tag{5.2}$$

where

$$\omega = M\,d\theta + N \sin\theta\, d\varphi \tag{5.3}$$

and K, L, M, and N are g-valued functions of u, θ, and φ. The form (5.1) is related to, but not equivalent with, the property of the solution to represent outgoing waves and to satisfy the Sommerfeld radiation condition.[3,14] The field strengths corresponding to (5.1) admit a similar expansion,

$$F = \sum_{k=1}^{\infty} r^{-k} F_k, \tag{5.4}$$

where

$$F_1 = du \wedge (\dot{L}\,dr + r\dot{\omega}) \tag{5.5}$$

and the dot denotes, from now on, a derivative with respect to u. We might have included in (5.1) a term of the form $H(u,\theta,\varphi)\,du$, which would have contributed to F_1. Such a term can, however, be gauge transformed, and absorbed in ω, in a manner similar to what has been done in Example (*i*) of Section IV.

A gauge transformation induced by a function of the form

$$S = a(\theta,\varphi)[I + r^{-1}\alpha(u,\theta,\varphi) + \cdots] \tag{5.6}$$

preserves (5.1). If gauge-transformed quantities are distinguished by primes, then

$$K' = a^{-1}Ka + \dot{\alpha}, \quad L' = a^{-1}La, \tag{5.7}$$

$$\omega' = a^{-1}\omega a + a^{-1}\,da. \tag{5.8}$$

It is important to note that, *a priori*, a may be an arbitrary smooth function on S_2. The occurrence of such a *function* makes it difficult to define the total nonabelian charge: at large distances, the field strengths F transform according to $F' = a^{-1}Fa$. In particular, the radial component θ of the $1/r^2$ part of the electric field transforms in this way. Therefore, its surface integral may be changed, essentially at will, by choosing an appropriate function a.[10,12]

In addition to the "generic" transformation functions (5.6) there may be some special ones, also preserving (5.1). For example:

(i) If c is a central element of g, then $S = \exp(c \log r)$ induces the change $A \mapsto A' = A + cr^{-1}\,dr$. Transformations of this form are used to eliminate the $r^{-1}L\,dr$ term from the electromagnetic potential.

(ii) If G contains SL(2,$\mathbb{R}$) as a subgroup, then its Lie algebra admits two nonzero elements X and Y such that $[X,Y] = X$. The potential $A = r^{-1}(Xu + Y)\,dt$ is a solution of the Yang–Mills equations of the form (5.1).[9] The function $S = \exp(tX)$ transforms it, however, to the Coulomb form $A' = r^{-1}Y\,dt$.

VI. THE ASYMPTOTIC EXPANSION

We now consider in more detail the asymptotic behavior of a gauge configuration produced by localized, time-dependent sources.[15] We assume that the current j falls off, at large distances, as r^{-4} or faster. In analogy with (5.2) we write

$$A_2 = P\,du + Q\,dr + r\pi, \tag{6.1}$$

where π is a g-valued 1-form linear in $d\theta$ and $d\varphi$; P, Q, and the coefficients of π depend on u, θ, and φ.

Under the generic gauge transformations (5.6), the form ω behaves like a gauge potential on S_2. One can associate with it a field strength,

$$d'\omega + \tfrac{1}{2}[\omega,\omega] = -B_r d\theta \wedge \sin\theta\, d\varphi, \tag{6.2}$$

where d' denotes the restriction of the exterior derivative to S_2,

$$-B_r = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} N \sin\theta - \frac{1}{\sin\theta}\frac{\partial M}{\partial\varphi} + [M,N]. \tag{6.3}$$

Moreover, for any u-dependent, g-valued differential form $\Phi = \Phi^i e_i$ on S_2, transforming as $\Phi \mapsto a^{-1}\Phi a$ under (5.6), one defines its gauge derivative with respect to ω as

$$D\Phi^i = d'\Phi^i + c^i_{jk}\omega^j \wedge \Phi^k. \tag{6.4}$$

Let Ψ denote the left-hand side of the Yang–Mills equations (2.5). The g-valued 3-form Ψ may be written as

$$\Psi = r^2(R\,du + U\,dr) \wedge d\theta \wedge \sin\theta\, d\varphi + du \wedge dr \wedge r\Xi, \tag{6.5}$$

where R and U are g-valued functions and Ξ is a g-valued form, linear in $d\theta$ and $d\varphi$. If the expansions (5.1) and (5.4) are introduced, then

$$R = \sum_{k=1}^{\infty} r^{-k} R_k,$$

$$U = \sum_{k=1}^{\infty} r^{-k} U_k, \quad \text{and} \quad \Xi = \sum_{k=1}^{\infty} r^{-k}\Xi_k, \tag{6.6}$$

where the quantities with subscripts are constant along the null rays $u,\theta,\varphi =$ const. The Yang–Mills equations without sources are now equivalent to the infinite system $R_k = U_k = \Xi_k = 0$, $k = 1,2,\cdots$. Among these equations we consider all those which involve only A_1 and A_2.

In the lowest order ($k = 1$), U_1 and Ξ_1 are identically zero and

$$R_1 = \ddot{L}. \tag{6.7}$$

For $k = 2$ one obtains the following:

$$U_2 = \dot{L} + [L,\dot{L}]. \tag{6.8}$$

Let us assume from now on that g admits a positive-definite, invariant scalar product. Remembering the lemma, one obtains from $U_2 = 0$

$$\dot{L} = 0. \tag{6.9}$$

Equation (6.9) will be assumed to hold from now on. The equation $R_2 = 0$ is equivalent to

$$^{\star}\dot{E}_r = D\,^{\star}\dot{\omega}, \tag{6.10}$$

where

$$E_r = \dot{Q} + K + [K,L]. \tag{6.11}$$

The function α occurring in the gauge transformation (5.7) may be used to reduce K' to zero so that $E'_r = \dot{Q}'$. Such a choice of gauge may be convenient in computations, but will not be adopted here because it requires time-dependent potentials for a static, Coulomb-like configuration. The equation $\Xi_2 = 0$ is equivalent to

$$[L,\dot{\omega}] = 0 \qquad (6.12)$$

and $U_3 = 0$ gives

$$D\,{}^\star DL = [L,{}^\star E_r]. \qquad (6.13)$$

Since the invariant scalar product of L with [L, anything] vanishes, Eq. (6.13) implies

$$h_{ij}L^iD\,{}^\star DL^j = 0 \qquad (6.14)$$

or, equivalently,

$$d(h_{ij}L^i\,{}^\star DL^j) - h_{ij}DL^i \wedge {}^\star DL^j = 0. \qquad (6.15)$$

The invariance condition (2.10) has been used, in conjunction with Eq. (6.4), to go over from (6.14) to (6.15). By integrating both sides of Eq. (6.15) over S_2 and taking into account that h is positive definite, one obtains

$$DL = 0 \qquad (6.16)$$

so that Eq. (6.13) reduces to

$$[L,E_r] = 0. \qquad (6.17)$$

The last equation one has to consider is $\Xi_3 = 0$, or

$$\dot{\pi} + [\dot{\pi},L] = \tfrac{1}{2}(DK + [DK,L] + {}^\star DB_r - \frac{\partial}{\partial u}DQ - [\dot{\omega},Q]). \qquad (6.18)$$

It follows from (6.9) and (6.16) that $\|L\|^2 = (L|L)$ is constant; if it is nonzero then one can define the projection q_L of color in the direction of L by

$$4\pi q_L = \int_{S_2} ({}^\star E_r|L)/\|L\|. \qquad (6.19)$$

Since L transforms according to (5.7), the integral (6.19) is well-defined (gauge-independent). Moreover, from Eqs. (6.10) and (6.16) is follows that q_L does not depend on u. The same is true of the "magnetic" (dual) charge m_L,

$$4\pi m_L = \int_{S_2} ({}^\star B_r|L)/\|L\|. \qquad (6.20)$$

For completeness, we give the explicit form of the $1/r$ and $1/r^2$ terms in the field strengths after the field equations (6.9) and (6.16) have been taken into account:

$$F = du \wedge \dot{\omega} + r^{-2}E_r\,du \wedge dr + r^{-1}\,du \wedge (\dot{\pi} - DK) - {}^\star B_r + O(r^{-3}). \qquad (6.21)$$

It is clear from (6.21) that $r^{-2}E_r$ and $r^{-2}B_r$ are the radial components of, respectively, the electric and magnetic $1/r^2$ parts of the field strengths.

All solutions to our equations can be divided into two classes depending on whether $L \neq 0$ or $L = 0$.

(i) If $L \neq 0$ then one can choose (6.9), (6.10), and (6.16)–(6.18), with E_r defined by (6.11), as independent equations. This is a rather strong system of equations: e.g., for $G = \mathrm{SU}(2)$ it implies that E_r, B_r and $\dot{\omega}$ are all parallel to L.

E. T. Newman raised the following problem: Are there solutions of the Yang–Mills equations, of the form considered in this paper, for which L cannot be reduced to zero by a gauge transformation? We have no complete answer to this question, but wish to make the following comments.

(a) The Lorentz condition $d\check{A} = 0$ implies the following restriction

$$E_r = L + [K,L] + {}^\star d'{}^\star\omega \qquad (6.22)$$

which is incompatible with $L = 0$ if $q \neq 0$.

(b) If the gauge potential is of the form

$$A = r^{-1}L\,dr + \omega + \overline{A}, \qquad (6.23)$$

Eq. (6.16) holds, and if

$$[L,\overline{A}] = 0, \qquad (6.24)$$

then the gauge transformation induced by

$$S = \exp(-L\log r) \qquad (6.25)$$

eliminates the L term without affecting ω and $\overline{A}$, i.e.,

$$S^{-1}AS + S^{-1}dS = \omega + \overline{A}. \qquad (6.26)$$

One cannot however, expect (6.24) to hold in general and $S^{-1}\overline{A}S$ may contain $\log r$ terms prohibited by the assumption inherent in Eq. (5.1).

(c) The analog of L vanishes for the Robinson–Trautman solutions[16] of Einstein's field equations.

(ii) If $L = 0$ then the field equations reduce to only two,

$$\dot{Q} + \dot{K} = {}^\star D^\star\dot{\omega}, \qquad (6.27)$$

$$\dot{\pi} = \frac{1}{2}(DK + {}^\star DB_r - \frac{\partial}{\partial u}DQ - [\dot{\omega},Q]). \qquad (6.28)$$

Given arbitrary K and ω one can integrate (6.27) and (6.28) to find Q and π, respectively. Formally, the total (retarded) color charge and its rate of change may be computed from

$$4\pi q_{\text{ret}}(u) = \int_{S_2} {}^\star E_r = \int_{S_2} (\dot{Q} + K)d\theta \wedge \sin\theta\,d\varphi, \qquad (6.29)$$

$$4\pi\dot{q}_{\text{ret}}(u) = \int_{S_2} D^\star\dot{\omega} = \int_{S_2} [\omega,{}^\star\dot{\omega}]. \qquad (6.30)$$

The significance of these formulas is limited by the occurrence of the arbitrary function a: $S_2 \to G$ in the gauge transformation (5.6). The function a can be restricted to be a constant if (i) the solution is spherically symmetric, or (ii) $A = O(r^{-2})$, i.e., $A_1 = 0$. In the case of the Liénard–Wiechert solution one also does not encounter a difficulty because of the spherical symmetry of the Coulomb, r^{-2} part of the field in each of the instantaneous rest systems of the particle. These simple examples suggest that it may be possible to eliminate the direction-dependent function a by reference to some properties of the gauge configurations, e.g., those at past or future infinity.

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