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GEOMETRICAL ASPECTS OF GAUGE CONFIGURATIONS+

by

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SUMMARY

These notes contain an informal description of the geometrical foundations of gauge theories. The theory of gravitation is compared to theories of the Yang-Mills type. Space-time symmetries of gauge configurations are defined in terms of automorphisms of principal bundles. Symmetry breaking is related to restricting the structure group of the bundle. The Liénard-Wiechert solution of the Yang-Mills equations is discussed in some detail. An approximate solution of the Yang-Mills equations is shown to allow for the phenomenon of radiation of the colour charge by a classical gluon field.

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INTRODUCTION

The validity and usefulness of a geometrical view of gauge configurations have been accepted by the physics community [1] and a fair number of surveys written on the description of Yang-Mills theories in terms of infinitesima connections on principal bundles over spacetime [2-10].

The present notes are based on the series of four lectures given by the author in February, 1981, in Schladming. They contain also a few changes and additions made after a similar series of lectures was delivered in March at the Collège de France. A part of the introductory material is omitted from these notes as it is included in a joint paper by M.E. Mayer and A. Trautman, appearing in this volume.

In my lectures I have tried to emphasize those aspect of the geometry of gauge configurations which I consider to be fundamental and important, but which have not received, so far, as much attention as they deserve. In particular, a considerable amount of time is devoted here to subtletie of symmetry breaking and of spacetime symmetries of gauge configurations. In the theory of gravitation, I like to stress the role of soldering and the rather different natur of translations. In the last lecture, I give a rather detailed analysis of the Liénard-Wiechert solution for the non-Abelian case, intended to show that geometrical methods provide convenient tools for the solution of concrete problems. In the last part of the notes, I discuss also the analogy between gravity and theories of the Yang-Mills type regarding the possibility of secular changes of their sourc At the classical level, one can consider the (hypothetical) phenomenon of radiation of the non-Abelian charge, such as colour, as an analogue to the phenomenon of change of the total energy and momentum due to gravitational waves.

I feel that the most important outcome of the bundle-theoretic approach to gauge fields may be a unification of fundamental interactions and an explanation of their hierarchy and symmetry breaking. This idea, which may be traced back to Weyl, Einstein, Kaluza, Klein, Pauli, Bargmann, Bergmann, and many other authors, was clearly formulated by B.S. DeWitt [11]. The Kaluza-Klein construction for a Yang-Mills field was later considered by several authors [6,12-15]. It was briefly described in my lectures, but is not included in the written text.

The notes should be read in conjunction with the "Brief introduction to the geometry of gauge fields" and other reviews listed in the Bibliography.

PURE GAUGE CONFIGURATIONS

A pure gauge configuration is given by a connection form ω on a principal bundle $\pi : P \to M$ with structure group G. The base space (spacetime) M is usually assumed to be oriented and endowed with a Riemannian (pseudo-Riemannian) metric ds^2. If M is 2n-dimensional, then the Hodge dual $*\alpha$ of an n-form α on M is invariant under the conformal transformations of the metric $ds^2 \to \sigma\, ds^2$. The curvature

$$\Omega = d\omega + \frac{1}{2}\,[\omega,\omega]$$

is a form defined on P, but since it is horizontal, its dual $*\Omega$ may be computed by reference to the metric and orientation on M. Since Ω is a 2-form, its dual is conformally invariant iff M is 4-dimensional and, in this case, the sourceless Yang-Mills equations

$$D * \Omega = 0$$

are also invariant under conformal changes of ds^2.

In physics, one works most often with local section of π. If U is an open subset of M and

$$s : U \to P , \quad \pi \circ s = id_U$$

is such a local section, then

$$A = s^*\omega \quad \text{and} \quad F = s^*\Omega$$

are the potential and the field strength of the gauge configuration given by ω, relative to s. If s' is another section over the same domain as s, then there exists a map

$$g : U \to G$$

such that

$$s'(x) = s(x)g(x) \quad \text{for any} \quad x \in U .$$

Putting

$$A' = s'^*\omega \quad \text{and} \quad F' = s'^*\Omega$$

one obtains the classical transformation formulae

$$A' = g^{-1}Ag + g^{-1}dg \quad \text{and} \quad F' = g^{-1}Fg$$

where dg is understood as follows: embed G in a group of matrices and compute the derivatives of g entry by entry, i.e. if $g = (g^i_j)$ then $dg = (dg^i_j)$.

INTERACTIONS AND ADDITIONAL STRUCTURE

Clearly, pure gauge fields do not suffice to describe all physics. Moreover, even in such a "pure" case as gravity in empty spacetime, one needs the metric tensor in addition to the linear connection which - in this case - may be identified with the gauge degrees of freedom.

In general, one considers a representation ρ of G in a (finite-dimensional, real or complex) vector space V, i.e. a homomorphism of Lie groups,

$$\rho : G \to GL(V) \ .$$

This defines a vector bundle $\rho(P) \to M$, associated with P by ρ. One can also form the tensor-fibre product

$$\rho^k(P) = \rho(P) \otimes_M \Lambda^k T^* M, \ k = 1,2,\ldots,\dim M.$$

It is known (cf., for example [16], p.76) that sections of $\rho^k(P) \to M$ are in a bijective and natural correspondence with horizontal k-forms of type ρ on P. For any such k-form ϕ one defines its covariant exterior derivative by

$$D\phi = \text{hor } d\phi = d\phi + \rho'(\omega) \wedge \phi \tag{1}$$

where

$$\rho' : G \to L(V)$$

is the homomorphism of the Lie algebra G of G into $L(V)$, derived from ρ,

$$\rho'(A)v = \frac{d}{dt} \rho(\exp tA)v\Big|_{t=0}, \ v \in V, \ A \in G \ .$$

Clearly, $D\phi$ corresponds to a section of $\rho^{k+1}(P) \to M$ and

$$D^2\phi = \rho'(\Omega) \wedge \phi \ . \tag{2}$$

If $s : U \to P$ is a local section, then $s^*\phi$ is a local section of $V \otimes \Lambda^k T^*M \to M$, i.e., a V-valued k-form on U.

Quantities of this type occur in both differential geometry and the description of gauge fields interacting with other kinds of matter.

Example 1. Let $G = U(1)$ so that $P \to M$ is an electromagnetic bundle. If $\rho_n : U(1) \to U(1) \subset GL(1,C)$ is the representation given by

$$\rho_n(z) = z^n , \quad z \in U(1) ,$$

then sections of $\rho_n(P)$ may be identified with wavefunctions of scalar particles of charge equal to n times the elementary charge. If $\phi : P \to C$ is the equivariant map corresponding to such a section,

$$\phi(pz) = z^{-n} \phi(p),$$

then

$$D\phi = d\phi + n\,\omega\,\phi$$

corresponds to the "minimal coupling prescription".

Example 2. If $G = SU(N)$ and $\rho = \mathbf{Ad}$ is the adjoint representation of G in its Lie algebra $\mathcal{G}$, then $\phi : P \to \mathcal{G}$ is a standard Higgs field.

Example 3. Let LM be the frame bundle of an n-dimensional manifold. Its structure group is $GL(n,R)$. If ρ is a tensor representation of $GL(n,R)$ in a vector space V, then a zero-form of type ρ, $\phi : P \to V$, corresponds to a tensor field of type ρ on M. In particular, if $V = R^n$ and $\rho = id$ then ϕ corresponds to a vector field. If $V = L(R^n)$ and $\rho = Ad$, then ϕ corresponds to a tensor field of valence (1,1). A metric tensor corresponds to the natural representation of $GL(n,R)$ in $L^2_s(R^n,R)$, i.e. in the vector space of symmetric $n \times n$ matrices, etc..

Example 4. Let $P \to M$ be a $GL(n,R)$-bundle over an n-dimensional manifold M. The bundle P is isomorphic (in the category of principal fibre bundles over M) to LM iff it admits a soldering form, i.e. a horizontal 1-form $\theta : TP \to R^n$ of type id [17].

Example 5. If M is n-dimensional, oriented and has a metric tensor, then the Hodge dual $*$ can be defined on horizontal forms on LM. In particular, to any integer k, $0 \leq k \leq n$, there corresponds the horizontal $\Lambda^k R^n$-valued (n-k)-form with components

$$\eta_{\mu_1 \mu_2 \ldots \mu_k} = * (\theta_{\mu_1} \wedge \theta_{\mu_2} \wedge \ldots \wedge \theta_{\mu_k})$$

where $\theta = (g^{\mu\nu}\theta_\nu)$ is the soldering form. In particular, $\eta = * 1$ is the volume n-form.

<u>Example 6.</u> The torsion form

$$\Theta = D\theta = d\theta + \omega \wedge \theta$$

corresponding to a linear connection ω on LM is a horizontal 2-form of type id. From (2) there follows the Bianchi identity for torsion

$$D\Theta = \Omega \wedge \theta \ .$$

<u>Example 7.</u> The curvature Ω of a connection ω on any bundle P is a horizontal 2-form of type Ad.

A scalar product h on V is invariant under the action of G on V defined by ρ if, for any $a \in G$ and $u,v \in V$,

$$h(\rho(a)u, \rho(a)v) = h(u,v) \ .$$

By differentiation, this implies

$$h(\rho'(A)u,v) + h(u,\rho'(A)v) = 0 \ , \ A \in G \ .$$

In particular, the Killing-Cartan form on G,

$$k(A,B) = \mathrm{Tr}\, \mathrm{Ad}'(A) \circ \mathrm{Ad}'(B) \ , \quad A,B \in G \ , \tag{3}$$

is invariant under the adjoint action of G in G, and, since

$$Ad'(A)B = [A,B] \ , \tag{4}$$

the property of invariance implies

$$k([A,B],C) + k(B,[A,C]) = 0 \ , \tag{5}$$

for any $A,B,C \in G$.

Let (e_i) and (e_a) be linear frames (bases) in G and V, respectively. The components of k and h in these frames are, respectively,

$$k_{ij} = k(e_i,e_j) \quad \text{and} \quad h_{ab} = h(e_a,e_b).$$

If

$$[e_i,e_j] = c^k_{ij} \ e_k$$

then, from (3) and (4) there follows the formula

$$k_{ij} = c^k_{i\ell} \ c^\ell_{jk} \ .$$

It is known that (k_{ij}) is non-singular iff G is semi-simple and, if, moreover, G is compact, then the quadratic form $k(A,A)$ is negative-definite. The connection and curvature forms may be represented by their components relative to (e_i),

$$\omega = \omega^i \ e_i, \ \Omega = \Omega^i \ e_i, \ \Omega^i = d\omega^i + \frac{1}{2} \ c^i_{jk}\omega^j \wedge \omega^k.$$

Similarly, $\phi : P \to V$ is written as

$$\phi = \phi^a \ e_a$$

and

$$D\phi = D\phi^a \ e_a$$

so that eq.(1) becomes

(4) $$D\phi^a = d\phi^a + \rho^a_{bi}\, \omega^i \wedge \phi^b$$

where

(5) $$\rho'(e_i)\, e_b = \rho^a_{bi}\, e_a \, .$$

If U is any function of the invariant

$$\phi^2 = h_{ab}\, \phi^a \phi^b$$

and η is a volume element on M, then the form on P given by

$$k_{ij} * \Omega^i \wedge \Omega^j + h_{ab} * D\phi^a \wedge D\phi^b + U(\phi^2)\pi^* \eta \qquad (6)$$

is horizontal and invariant under the action of G. Therefore, it defines a form on M, denoted by L, and used to formulate the principle of least action

$$\delta \int_M L = 0 \, .$$

This is the starting point of classical gauge theories of the Yang-Mills type. The field ϕ is referred to as a (generalized) Higgs field and the property of (6) which consists in the appearance of the derivatives of ϕ only through the form $D\phi$ is a reflection of the principle of minimal coupling between the gauge configuration and the matter fields interacting with it. Incidentally, the scalar product k occurring in (6) need not be given by the Killing-Cartan form (3): it may be any scalar product on G invariant under the adjoint action of G.

BREAKING OF SYMMETRY

The mechanism of spontaneous symmetry breaking has a simple interpretation in terms of restrictions of principal bundles [9,18]. It is discussed in considerable detail in the lectures by Meinhard E. Mayer appearing in this volume. In this short section I wish only to stress the analogy between the role of the metric tensor in general relativity theory and the breaking of symmetry by a normalized Higgs field. As an illustration, the restriction of the SO(3)-bundle over S_2, induced by the 't Hooft-Polyakov solution, is discussed in considerable detail.

Consider a G-bundle $\pi : P \to M$ and a representation ρ of G in a vector space V. Let

$$\phi : P \to W \quad V$$

be a field of type ρ with values in an orbit W of G. In other words, G acts transitively on the manifold W of values of ϕ. Therefore, given a fixed point $w_o \in W$, in each fibre $\pi^{-1}(x)$ of P there is at least one point q such that $\phi(q) = w_o$. The set of all such points,

$$Q = \{q \in P : \phi(q) = w_o\} ,$$

is a submanifold of P, and the restriction of π to Q defines a fibre bundle $Q \to M$. It is a principal bundle: its structure group H is the isotropy (stability) or little group of w_o,

$$H = \{a \in G : \rho(a) w_o = w_o\} .$$

The embedding $Q \to P$, $H \to G$ defines a restriction of P to H in the sense described in the "Brief Introduction".

Conversely, if such a restriction is given, one can define $\phi : P \to G/H$ by putting $\phi(qa) = a^{-1}H$ for $q \in Q$ and $a \in G$. Under rather general assumptions the "non-linear realization" of G in $W = G/H$ can be extended to a linear representation of G in a vector space V containing W.[+]

[+] I am indebted to Profs.L.Michel and A.Sparzani for having drawn my attention to the last problem and explained its subtleties.

Example 8. Let H be a closed subgroup of $GL(n,R)$. A restriction of the bundle LM of linear frames of an n-dimensional manifold M defines an H-structure on M. In particular, a Riemannian geometry on M is an $O(n)$-structure. In other words, the introduction of a metric tensor of Lorentz signature on spacetime is equivalent to breaking the symmetry from $GL(4,R)$ down to the Lorentz group.

Example 9. Consider the static, spherically symmetric solution of the Yang-Mills equations with $G = SO(3)$ and a source corresponding to a standard Higgs field. The 't Hooft-Polyakov Ansatz assumes a regular potential A; therefore, the corresponding bundle is trivial. Removing the time axis from R^4, one can represent

$$M = \{(t,x,y,z) \in R^4 : x^2 + y^2 + z^2 > 0\}$$

as the product $R^2 \times S_2$. Because of spherical symmetry it suffices to consider the trivial bundle

$$P = S_2 \times SO(3) \to S_2 \quad . \tag{7}$$

The (normalized) Higgs field

$$\phi : S_2 \times SO(3) \to S_2 \quad R^3 = \text{Lie algebra of } SO(3)$$

is given by

$$\phi(r,a) = a^{-1}\, r \tag{8}$$

where

$$r = (x,y,z) \in S_2 \text{ and } a \in SO(3).$$

Let

$$r_0 = (0,0,1) \; ;$$

then

$$H = \{a \in SO(3) : ar_o = r_o\} = SO(2)$$

and

$$Q = \{(r,a) \in P : a^{-1} r = r_o\} \approx SO(3).$$

Therefore ϕ reduces the trivial SO(3)-bundle (7) to a non-trivial SO(2)-bundle

$$\pi : SO(3) \to S_2 , \qquad \pi(a) = ar_o . \tag{9}$$

The latter bundle is isomorphic to the bundle of oriented dyads of S_2 and admits a canonical SO(2)-connection corresponding to a magnetic pole with a charge equal to twice the lowest (Dirac) value [19]. This connection is obtained by projecting the one on P onto the direction of r_o, considered as an element of the Lie algebra of SO(3).

The bundle (7) admits an obvious section

$$s : S_2 \to P , \qquad s(r) = (r, I)$$

where I is the unit element of SO(3). The pull-back $s^*\phi$,

$$s^*\phi(r) = r ,$$

corresponds to the "hedgehog" representation. On the other hand, if s' is any (local) section of (9), then

$$s'^*\phi(r) = r_o$$

corresponds to the description of the same Higgs field in a gauge exhibiting an alignment of the field along the third axis.

The field ϕ given by (8) is spherically-symmetric. To make this statement meaningful it is necessary to lift the action of SO(3) from the base, S_2, to the bundle,

$P = S_2 \times SO(3)$. This can be done in many ways. For example, if $a,b \in SO(3)$ and $r \in S_2$, then the formula

$$\gamma_a^o(r,b) = (ar,b)$$

defines a lift, but

$$\phi \circ \gamma_a^o \neq \phi \text{ , unless } a = I.$$

However, the action given by

$$\gamma_a^1(r,b) = (ar,ab)$$

leaves ϕ invariant,

$$\phi \circ \gamma_a^1 = \phi \quad .$$

SYMMETRIES OF GAUGE CONFIGURATIONS

The last example leads to the following general question: how to define space time symmetries of gauge configurations, and, in particular, of infinitesimal connections and of Higgs fields?

To appreciate the subtleties of the problem, consider first a simple, local situation. Let A be the G-valued 1-form of potential defined on M and $f : M \to M$ a transformation (diffeomorphism). One says that A is invariant under f if there exists (a gauge transformation) $g : M \to G$ such that

$$f^*A = g^{-1} Ag + g^{-1} dg \quad . \tag{10}$$

Clearly, eq.(10) implies

$$f^*F = g^{-1}\ Fg\ , \text{ where } F = dA + \frac{1}{2}\ [A,A]\ . \tag{11}$$

The converse is true if $H^2(M,R) = O$ and G is Abelian. Indeed, if G is Abelian, then (11) reads

$$f^*F = F \qquad \text{and} \qquad F = dA\ ,$$

therefore

$$O = f^*dA - dA = d(f^*A - A)$$

and $f^*A - A$ is exact by virtue of the topological assumption. However, for a non-Abelian G, the implication (11) → (10) is false even in the case $M = R^n$. For example, let G = SO(3) and consider a "plane wave" in Minkowski space [20,21],

$$A(t,x,y,z) = (a(u)x + b(u)y)du\ ,$$

where

$$u = t - z\ , \quad a,b : R \to G \approx R^3 \quad \text{and} \quad [a,b] \neq O\ .$$

The field

$$F = (adx + bdy) \wedge du$$

is invariant under the translation $(t,x,y,z) \to (t,x+\lambda,y,z)$, but the potential is not.

The topological condition

$$H^2(M,R) = O$$

is essential for the implication (11) → (10) to be true in the Abelian case. For example, the field strength of a magnetic pole, $\vec{B} = q\vec{r}/r^3$, exhibits spherical symmetry for any q, but the corresponding potential is spherically symmetric - in the sense of eq.(10) - iff the magnetic charge q satisfies the Dirac quantization condition.

A section $s : M \to P$ defines an isomorphism of bundles $i : M \times G \to P$ given by

$$i(x,a) = s(x)\,a \quad \text{where} \quad x \in M \quad \text{and} \quad a \in G.$$

A connection form ω on P, pulled-back by this isomorphism, assumes the form

$$\omega_s = i^*\omega = a^{-1}(da + A(x)a)$$

where

$$A = s^*\omega \quad .$$

An automorphism $h : P \to P$ (cf. the Brief Introduction) covering $f : M \to M$, pulled-back by i to $M \times G$, becomes $h_s = i^{-1} \circ h \circ i$ where

$$h_s(x,a) = (f(x), g(x)^{-1}a) \text{ and } h(s(x))g(x) = s(f(x)).$$

A simple computation gives

$$h_s^* \, \omega_s = a^{-1} \, (da + A'a)$$

where

$$A' = g f^* A g^{-1} - (dg) g^{-1} \quad .$$

Therefore, condition (10) is equivalent to

$$h_s^* \, \omega_s = \omega_s$$

and this, in turn, is equivalent to

$$h^*\omega = \omega \quad . \qquad (12)$$

To summarize, a gauge configuration described by a connection form ω on a principal bundle $P \to M$ is defined to be invariant under a diffeomorphism $f : M \to M$ if there is a lift h of f to Aut P such that (12) holds. The analogous condition of invariance for a Higgs field $\phi : P \to V$ is

$$\phi \circ h = \phi . \tag{13}$$

It is instructive to list typical situations when diffeomorphisms can be lifted and to give examples showing that this cannot always be done.

Example 10. If $P = M \times G$ then $f : M \to M$ lifts to $h: M\times G \to M\times G$ given by $h(x,a) = (f(x),a)$. Moreover, the group Aut P is a semi-direct product of G^M by Diff M [17].

Example 11. If P is a natural bundle - a bundle L^rM of frames of M of differential order r - then f can be lifted to P by the very definition of a natural bundle.

Example 12. If both M and G are compact, then any one-parameter group (f_t) of diffeomorphisms of M can be lifted to a one-parameter group (h_t) of automorphisms of $\pi : P \to M$ in such a way that $\pi \circ h_t = f_t \circ \pi$. It suffices to take a connection on P and the flow generated by the horizontal lift of the vector field induced on M by (f_t).

However, a diffeomorphism which is not homotopic to the identity need not lift:

Example 13. Complex conjugation on U(1), $f(z) = \bar{z}$, does not lift to the exponential bundle $\pi : R \to U(1)$, $\pi(t) = \exp(2\pi\, it)$, considered as a principal Z-bundle. (It does, however, lift to an automorphism of the bundle structure.)

Example 14. Similarly, the space inversion $f : S_2 \to S_2$, $f(r) = -r$, does not lift to the principal Hopf bundle $S_3 \to S_2$.

From the point of view of physical applications, we are most often interested in lifting a Lie group of transformations of M to a Lie group of automorphisms of $\pi: P \to M$, covering the given action in M. There are subtleties, as shown by the following

Example 15. Any rotation of S_2 lifts to the Hopf bundle $S_3 \to S_2$ (this follows from example 12), but the natural action of SO(3) on S_2 does not lift. However, the action of SU(2) does lift, as is obvious from the identification $S_3 \approx SU(2)$.

Example 16. There is a simple construction of all G-bundles $\pi : P \to M$ admitting a Lie group K of automorphisms transitive on the fibres of π (cf., e.g. [16] p. 105 and [22]). Let J be the subgroup of K, leaving invariant a point $x_o \in M$. If $p_o \in P$ is such that $\pi(p_o) = x_o$, then for any $a \in J$, the point ap_o is in the same fibre as p_o. There is thus an element $\lambda(a)$ of G such that

$$a\ p_o = p_o\ \lambda(a)$$

and

$$\lambda : J \to G \tag{14}$$

is a homomorphism. Moreover, K × G acts transitively on P by

$$(a,b)\ p = apb^{-1}$$

and

$$(a,b)\ p_o = p_o \leftrightarrow a \in J \text{ and } b = \lambda(a)\ .$$

Therefore, P is diffeomorphic to the quotient (K × G)/J, where J is assumed to act on K × G by

$$(a,b)c = (ac,\ b\lambda(c))\ , \quad c \in J\ .$$

Conversely, given a group K acting transitively on M, and a homomorphism (14) of the stability group J of a point $x_o \in M$, one constructs the bundle by taking

$$P_\lambda = (K \times G)/J$$

and defining the action of K × G on P_λ in the standard manner.

For example, if $K = SU(2)$, $G = U(1)$ and $M = S_2$, then $J = SO(2) = U(1)$ and all homomorphisms $\lambda : J \to G$ are of the form

$$\lambda_n(z) = z^n \quad \text{for some} \quad n \in Z .$$

It is seen by inspection that P_{λ_n} is isomorphic to the lens space

$$SU(2)/Z_n \quad , \quad n \in Z . \tag{15}$$

If one starts,however, with $K = SO(3)$, then one gets in this manner only the even lens spaces,

$$SO(3)/Z_n \approx SU(2)/Z_{2n} . \tag{16}$$

In terms of magnetic monopoles, the bundles (15) and (16) correspond to the Dirac and Schwinger quantization conditions, respectively.

GRAVITATION+

The similarities and differences between gravitation and theories of the Yang-Mills type have been discussed by many authors (cf., e.g.,[9], [23] and the references given there). In this brief section, I intend only to summarize my views and to comment on the "Abelian nature of gravitational waves".

1. Gravitation is a theory based on fibre bundles wich are

+This section has been influenced, in part, by a discussion with R.P.Wallner. I am indebted to Peter Aichelburg and Roman Sexl of the University of Vienna for hospitality and stimulating conversations. I wish also to acknowledge discussions on this matter held at various occassions with Jürgen Ehlers, Friedrich Hehl, J.Nitsch and M. Schweizer.

"concrete" and, as such, have a richer structure (cf. Example 4) than "abstract" bundles underlying gauge theories of the Yang-Mills type.

2. The soldering form on LM leads to the notion of torsion which has no analogue in electromagnetism or the Yang-Mills theory.

3. The metric tensor is somewhat analogous to a Higgs field: it breaks down the full linear symmetry to the Lorentz group (cf. Example 8). There is a complete analogy between the condition of compatibility between the metric tensor ($g_{\mu\nu}$) and a linear connection, expressed by

$$Dg_{\mu\nu} = 0 \tag{17}$$

and a similar equation

$$D\phi^a = 0 \tag{18}$$

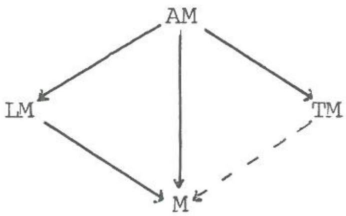
assumed to be satisfied by the ground state of a gauge configuration.

4. There has been a lot of discussion on the choice of the structure group G for a theory of gravitation. Essentially, there are two (minor) problems to consider:

(i) whether one wishes to incorporate translations - or even take them as the starting point;

(ii) whether the metric is to be introduced ab initio, as part of the definition of the bundle, or rather as an additional structure, restricting the linear or the affine group to its Lorentz or Poincaré subgroup, respectively.

Concerning the first problem, it is convenient to summarize the situation by reference to the following diagram of bundles over an n-dimensional manifold:



(19)

Here TM is the tangent bundle and

$$AM = LM \times_M TM$$

is the affine bundle. Its structure group is the general affine group,

$$GA(n,R) = GL(n,R) \times R^n$$

which acts on AM as follows: if $e : R^n \to T_xM$ is a linear frame at $x \in M$, considered as an isomorphism of vector spaces, $u \in T_xM$, $a \in GL(n,R)$ and $b \in R^n$, then

$$(e,u)(a,b) = (ea,\ u + e(b)) \ .$$

All <u>solid</u> arrows in (19) denote projections of <u>principal</u> bundles with appropriate groups. The tangent bundle, however, does not admit a natural structure of principal bundle. It is easy to see that the introduction of an R^n-action on TM, to make out of it a principal bundle, is equivalent to giving a global section of $LM \to M$, i.e. a teleparallelism structure on M [24]. A connection on $TM \to M$, compatible with such a structure, has torsion but no curvature.

There is no essential difference between considering connections on LM and AM. According to a classical theorem (cf.[16], p. 127), any affine connection is defined by a linear connection and a tensor field of type Ad.

5. An essential difference between theories of the Yang-Mills type and gravitation is in the choice of the Lagrangian leading to the field equations. In the Yang-Mills case, there is only one kind of duality that can be used to construct the left-hand side of the field equations: the Hodge dual operating on Ω (or F) considered as a G-valued 2-form. In the gravitational case, there is the Levi-Civita $\Lambda^2 R^n$-valued (n-2)-form $\eta_{\mu\nu}$ (cf. Example 5) which leads to the Einstein(-Cartan) Lagrangian

$$\frac{1}{2}\,\eta_{\mu\nu} \wedge \Omega^{\mu\nu} \quad .$$

6. The vanishing of torsion, assumed in Einstein's theory, considerably reduces the "degrees of freedom" inherent in a metric connection. In particular, plane gravitational waves are, in a well-defined sense, Abelian. To see this, consider the metric

$$ds^2 = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^4 + e^4 \otimes e^3 \qquad (20)$$

where

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = du, \quad e^4 = dv + H\,du \qquad (21)$$

and H is a function of the coordinates u = t-z, x and y. The connection 1-form Γ, referred to the coframe (e^μ),

$$\Gamma_{\mu\nu} = (a_{\mu\nu}(u)\,x + b_{\mu\nu}(u)\,y)\,du \qquad (22)$$

is compatible with (20) iff

$$a_{\mu\nu} + a_{\nu\mu} = 0 \qquad \text{and} \qquad b_{\mu\nu} + b_{\nu\mu} = 0\,, \qquad (23)$$

i.e. iff the polarization "vectors" $a = (a_{\mu\nu})$ and $b = (b_{\mu\nu})$ have values in the Lie algebra SO(1,3) of the Lorentz group. The vanishing of torsion,

$$de^\mu + \Gamma^\mu_{\ \nu} \wedge e^\nu = 0 \quad , \qquad (24)$$

restricts the values of a,b to the commutative Lie subalgebra n of $SO(1,3)$, isomorphic to R^2 [25]+. Therefore

$$[a,b] = 0$$

and the gravitational wave, described completely by eqs. (20) - (24), is really plane, in contradistinction to a plane-fronted Yang-Mills wave with $[a,b] \neq 0$.

GROUPS OF GAUGE TRANSFORMATIONS [21]

Gauge transformations may be defined as automorphisms of a principal bundle preserving the absolute elements of a gauge theory. Putting it in a slightly different way, a gauge theory is based on a category C, which is a subcategory of the category of principal G-bundles over M. Gauge transformations are simply isomorphisms in C.

For any principal bundle $P \to M$, there is the exact sequence

$$I \to \mathrm{Aut}_0 P \to \mathrm{Aut}\, P \to \mathrm{Diff}\, M \qquad (25)$$

where $\mathrm{Aut}_0 P$ is the group of vertical (based) automorphisms of P. If $\mathcal{G}$ is the subgroup of Aut P preserving the absolute elements of $P \to M$ and

$$\mathcal{G}_0 = \mathcal{G} \cap \mathrm{Aut}_0 P ,$$

then one can form the exact sequence

$$I \to \mathcal{G}_0 \to \mathcal{G} \to \mathcal{G}/\mathcal{G}_0 \to I .$$

+Moshe Flato pointed out that n corresponds to the nilpotent part of the Iwasawa decomposition of the Lorentz group. I gratefully acknowledge the hospitality extended to me by M. Flato during my visit to Dijon.

The elements of $\mathcal{G}_o$ are pure gauge transformations, whereas the elements of $\mathcal{G}$ can be referred to as gauge transformations. In the case of a Yang-Mills theory over S_4, the group $\mathcal{G}/\mathcal{G}_o$ coincides with O(5) whereas $\mathcal{G}_o$ is "large". In Einstein's theory, $\mathcal{G}_o = \{id\}$, but $\mathcal{G}$ = Diff M is "large". The sequence (25) splits if (i) $P \to M$ is trivial [17] or (ii) P is natural.[+] I do not know whether it splits in any case not covered by (i) or (ii).

TIME-DEPENDENT GAUGE CONFIGURATIONS

There are not many time-dependent exact solutions of the Yang-Mills equations. Coleman's plane-fronted waves have been already briefly discussed here. Waves with spherically-symmetric wave fronts have wire singularities [25,26]. In this section, I analyze in some detail the Liénard-Wiechert solution, adapted by Arodź [27] to the Yang-Mills case. A method used by Roskies [28] to study the asymptotic properties of Yang-Mills configurations leads to a simple estimate of the rate of radiation of the colour charge carried by a classical gluon field. There is an analogy between the energy-momentum vector in Einstein's theory and the colour charge in chromodynamics. Physically, the analogy is related to a presumed similarity in the self-interaction of gravitons and gluons. To appreciate the analogy formally, it is convenient to write both the Einstein and Yang-Mills equations in terms of differential forms,

$$dU - 4\pi i = 4\pi j$$

where j is a vector-valued 3-form describing the sources. In the Yang-Mills case, U is the Hodge dual, $*F$, of the field strength, i.e. a G-valued 2-form. The 3-form

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[+]I am indebted to Ivan Kolàr for having pointed out this to me.

$$i = -\frac{1}{4\pi}[A,*F]$$

is the gluon contribution to the conserved total current $i + j$.

Similarly, in the case of gravitation, $U = (U^{\mu})$ is the 2-form of Freud's "superpotential" [29]

$$4U^{\mu} = *(dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}) \wedge \omega_{\nu\rho} \quad ,$$

where $\omega^{\nu}_{\rho} = \Gamma^{\nu}_{\rho\sigma}\, dx^{\sigma}$ are the connection 1-forms. The currents $i = (i_{\mu})$ and $j = (j_{\mu})$ correspond, respectively, to the energy-momentum densities of the gravitational field and of its source,

$$i_{\mu} = *dx^{\nu}\, t_{\mu\nu} \quad , \quad j_{\mu} = *dx^{\nu}\, T_{\mu\nu} \quad ,$$

where $(t_{\mu\nu})$ is the Einstein "pseudotensor" and $(T_{\mu\nu})$ is the stress tensor of the source [10].

In both cases the field contribution i to the total current is highly gauge-dependent: no physical meaning can be attached to the notion of a local distribution of either gravitational energy or colour charge of the gluon field. If the fields satisfy suitable boundary conditions at large distances, say F or $\Gamma = O(r^{-2})$, one can compute the total charge q (energy-momentum or colour) from the Gauss law

$$4\pi q = 4\pi \int_{B_R} (i + j) = \oint_{S_R} U \quad (R \to \infty) \quad , \tag{26}$$

where S_R is the surface (boundary) of a ball B_R of radius R. The surface integral converges for $R \to \infty$ even if F or $\Gamma = O(r^{-1})$, provided that the "electric" component of the $1/r$ part of the field is tangent to S_R. Such is the case of pure outgoing waves. The boundary conditions are presumed to fix the gauge at large distances so that q is well-defined, up to the transformation $q \to g^{-1} qg$, $g \in G$ (up to a Lorentz transformation in the case of gravity). It should be noted, however, that imposing suitable boundary conditions is a subtle matter.

To construct the Liênard-Wiechert solution of the Yang-Mills equations with group G, consider a point particle of colour charge q, whose history is represented by a time-like world-line z in Minkowski space. A priori, the charge may depend on time. It is, therefore, a (dimensionless) function $q : R \to G$.

Let $z^{\mu}(s)$ be the Cartesian coordinates of z. The world-line is parametrized by its proper time, $g_{\mu\nu} \dot{z}^{\mu} \dot{z}^{\nu} = 1$, $\dot{z}^{\mu} = dz^{\mu}/ds$, $\dot{z}^{0} > 0$. One associates with z a system of comoving spherical coordinates (t,r,θ,ϕ) by writing [30]

$$x^{\mu} = z^{\mu}(u) + r\ell^{\mu}(\theta,\phi)/p(u,\theta,\phi) ,$$

where $u = t-r$ is a retarded time,

$$r = g_{\mu\nu}(x^{\mu} - z^{\mu}(u))\dot{z}^{\nu}(u) \geq 0$$

is a radial distance measured in the rest frame of an observer moving along z, and $\ell = (\ell^{\mu})$ is the null vector field,

$$\ell = (1, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) .$$

These definitions imply

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = (1-rp^{-1}\dot{p})\, du^2 + 2\, du\, dr - p^{-2}\, r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where a dot denotes differentiation with respect to u. The function $p = g_{\mu\nu}\, \dot{z}^{\mu}\, \ell^{\nu}$ has a simple physical interpretation: if ω is the frequency of a beam of light moving in the (θ,ϕ)-

direction, as seen from rest in the coordinate system (x^μ), then $\omega p(u,\theta,\phi)$ is the frequency of the same beam measured at $z^\mu(u)$ by the observer moving along z. The world-line z is straight, $\ddot{z}^\mu = 0$, if and only if $\dot{p} = 0$.

The Liénard-Wiechert potential may be written as

$$A = q(u)\ r^{-1}\ \dot{z}_\mu(u)\ dx^\mu \quad . \tag{27}$$

This form of the potential fixes the gauge almost completely; the only remaining freedom is of constant gauge transformations, $q \to g^{-1}\ qg$, $g \in G$. Such global gauge transformations are not enough to align a time-dependent colour along a fixed direction.

The field strengths corresponding to (27),

$$F = qr^{-2}\ du \wedge dr + r^{-1}\ du \wedge \{\dot{q}dr + grd(p^{-1}\dot{p})\} \ , \tag{28}$$

satisfy the Yang-Mills equation $D * F = 0$ $(r \neq 0)$ if and only if

$$\dot{q} + [q,\dot{q}] = 0 \tag{29}$$

and

$$\frac{\partial}{\partial u}\ (p^{-1}\ \dot{q}) = 0 \quad . \tag{30}$$

If G is either (i) Abelian, or (ii) compact and semi-simple, then eq.(29) implies $\dot{q} = 0$. It is worth noting that, in the important case (ii), strict conservation of colour follows from the Yang-Mills equation alone. Moreover, the field (28) has the same structure as in electrodynamics; it contains a Coulomb-like r^{-1} term and a radiative r^{-2} term, linear in $(\ddot{z}^\mu)$. Clearly, the latter term gives rise to outgoing radiation of energy. The expressions for the Poynting vector and the total intensity may be obtained from the corresponding formulae derived in electromagnetism by replacing the square of the electric charge by $- \mathrm{Tr}\, q^2$. Moreover, it is easy to see that, if $\dot{q} = 0$, then the colour current j corresponding to (27) is a distribution with support on the world-line z.

If $\dot{q} \neq 0$, then eq. (30) leads to

$$\dot{p} = 0 \qquad \text{and} \qquad \ddot{q} = 0 .$$

The particle is thus unaccelerated and its colour changes linearly with time,

$$q(t) = at + b , \tag{31}$$

where, by virtue of eq. (29),

$$[a,b] = a \neq 0 .$$

The x-coordinates may now be adjusted so that p = 1 and the solution assumes a manifestly spherically-symmetric form,

$$A = r^{-1} (au + b)\, dt , \tag{32}$$

$$F = r^{-2} (at + b)\, dt \wedge dr . \tag{33}$$

In this case, colour appears to change, but there is no transfer of energy by the gluon field. Incidentally, the field strengths (33) may also be derived from the potential $r^{-1} (at + b)\, dt$ which is not gauge equivalent to (32).

Gauss's law applied to the field strengths (33) gives the time-dependent charge (31). A closer analysis shows, however, that the gauge configuration described by (32) and (33) is, in fact, time-independent. This is easily seen in the simple, but typical, case when G = SL(2,R) and

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $S = I + au$, then $S^{-1} = I - au$ and

$$A' = S^{-1} AS + S^{-1} dS = r^{-1} b\, dt + a\, du ,$$

$$F' = S^{-1} FS = r^{-2} (ar + b)\, dt \wedge dr \quad . \tag{34}$$

The gauge-transformed field (34) is explicitly time-independent, but contains an unexpected r^{-1} term which makes it impossible to apply Gauss's law.

Returning to the analogy between the Yang-Mills and Einstein theories, one should bear in mind that a single, spherically symmetric body cannot radiate gravitational waves, but a system of bodies, moving under their mutual attraction, is believed to lose energy due to gravitational radiation. One is thus led to consider the asymptotic behaviour of a bounded, time-dependent source of a Yang-Mills field [28]. An approximate computation, described below, shows that the total colour charge of such a source may indeed change as a result radiation of colour, in analogy to the gravitational case.

Consider a classical Yang-Mills field in Minkowski space. Introduce a system of spherical coordinates (r,θ,ϕ) and put $u = t - r$. Assume that there is a gauge such that

$$j = O(r^{-3}) \text{ and } A = A_o + O(r^{-3}), \text{ for large } r,$$

where

$$r^2 A_o = (Kr+P)du + (Lr+Q)dr + (Mr+R)r d\theta + (Nr+S)r \sin\theta\, d\phi, \tag{35}$$

and the Lie algebra-valued functions $K, L, \ldots, S$ depend on u,θ, and φ only. It follows from these assumptions that the field strength is

$$F = F_o + O\,(r^{-3})$$

where

$$F_o = dA_o + \frac{1}{2}\,[A_o, A_o] = O(r^{-1})\ .$$

The Yang-Mills equation

$$d * F + [A, * F] = 4\pi j$$

is seen to be equivalent, to order r^{-3}, to

$$d * F_o + [A_o, * F_o] = O(r^{-3})\,.$$

The last equation reduces to the system

$$\ddot{L} = 0 \quad , \quad \dot{L} + [L,\dot{L}] = 0 \quad , \tag{36}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial L}{\partial \theta} - [L,M]\right) = [L,\dot{M}] \quad , \tag{37}$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin\theta}\frac{\partial L}{\partial \phi} - [L,N]\right) = [L,\dot{N}]\ , \tag{38}$$

$$\frac{\partial}{\partial u}\,(K + [K,L] + \dot{Q}) + [K,\dot{L}] = [M,\dot{M}] + [N,\dot{N}] + $$

$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial u}\left(\frac{\partial}{\partial \theta} M \sin\theta + \frac{\partial N}{\partial \phi}\right) \quad , \tag{39}$$

where dots denote again derivatives with respect to u. Any solution to this system gives an approximate solution (35) of the Yang-Mills equation. Asymptotically, the solution is an outgoing (retarded) wave. Eqs. (36) can be solved, $L = au + b$, where a and b depend on θ and φ, and

$$[a,b] = a\ . \tag{40}$$

For G semi-simple and compact, the only solution to (40) is $a = 0$. Therefore $\dot{L} = 0$, and if $L = 0$ is assumed, then the system of equations reduces to (39) [28]. The field strength is now

$$F = du \wedge (\dot{M} d\theta + \dot{N} \sin\theta\, d\phi) + r^{-2}(K + \dot{Q})\, du \wedge dr + \dots \quad (41)$$

where dots stand for terms which do not contribute to the surface integral (26). The first term in (41) is of order $1/r$ and represents the radiative, purely transverse part of the field. Total colour charge can be now evaluated by computing (26) for $u = \text{const.}$ and $r = R \to \infty$. Using $* (du \wedge dr) = r^2 (d\theta \wedge \sin\theta\, d\phi)$ and equ. (39) one obtains

$$4\pi q = \oint (K + \dot{Q})\, d\theta \wedge \sin\theta\, d\phi \ ,$$

$$4\pi \dot{q} = \oint ([M,\dot{M}] + [N,\dot{N}])\, d\theta \wedge \sin\theta\, d\phi \ , \quad (42)$$

where the integrals are taken over a unit sphere. It is clear from (42) that radiation of colour is a truly non-linear and non-Abelian phenomenon requiring at least two particles with non-commuting charges. Radiated energy is computed from the Yang-Mills Poynting vector,

$$4\pi \dot{E} = \oint \mathrm{Tr}(\dot{M}^2 + \dot{N}^2)\, d\theta \wedge \sin\theta\, d\phi \ . \quad (43)$$

Here E is the total energy of the system and Tr denotes the scalar product in the Lie algebra of G, defined by its (negative-definite) Killing form k.

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