We assume that the left hand side of the gravitational equations can be obtained by varying the integral

$$J=\int_{\Omega} K\sqrt{-g}\,d_4x.$$

K is an invariant built from $g_{\mu\nu}$ and their derivatives. The gravitational field equations are

(17) $$K_{\alpha\beta} \underset{\text{def.}}{=} \frac{\delta J}{\delta g^{\alpha\beta}} = \varrho u_\alpha u_\beta + T_{\alpha\beta}.$$

For $K=\varkappa R$, we have $K_{\alpha\beta}=\varkappa(R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R)$, and (17) becomes Einstein's equation. In a manner similar to that used in section 3, we get $K^{\alpha\beta}{}_{;\beta}=0$. Equation (17) can be integrated only when

(18) $$(\varrho u^\alpha u^\beta + T^{\alpha\beta})_{;\beta}=0.$$

Assuming $L^A=j^A$, we obtain by virtue of (14), and $u^\alpha{}_{;\beta}u^\beta=Du^\alpha/ds$:

(19) $$\varrho\frac{Du_\alpha}{ds}+u_\alpha(\varrho u^\beta)_{;\beta}=j^A\psi_{A;\alpha}+F_{A\alpha}{}^{B\beta}(j^A\psi_B)_{;\beta}$$

Transvecting (19) with u^α, we get the law of "conservation of mass":

(20) $$(\varrho u^\beta)_{;\beta}=u^\alpha(j^A\psi_{A;\alpha}+F_{A\alpha}{}^{B\beta}(j^A\psi_B)_{;\beta}).$$

From (19) and (20) we have the *equations of motion*:

(21) $$\varrho\frac{Du^\alpha}{ds}=(g^{\alpha\nu}-u^\alpha u^\nu)\left((j^A\psi_B)_{;\beta}F_{A\nu}{}^{B\beta}+j^A\psi_{A;\nu}\right).$$

The equations of motion appear as necessary *integrability conditions* for the gravitational field equations.

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REFERENCES

[1] L. Infeld, *Equations of motion for linear field theories*, Bull. Acad. Polon. Sci. Cl. III (1955), 213.
[2] E. Noether, Nachr. Ges. Göttingen (1918), 235.
[3] E. L. Hill, Rev. Mod. Phys. **23** (1951), 253.
[4] J. A. Schouten, *Ricci Calculus*, 2-nd edition, Berlin 1954.
[5] F. J. Belinfante, Physica **7** (1940). 449.
[6] P. G. Bergmann, Phys. Rev. **75** (1949), 680.
[7] D. Hilbert, Math. Ann. **92** (1924), 1.
[8] J. S. de Wet, Proc. Cambr. Phil. Soc. **43** (1947), 511.
[9] L. Infeld, Acta Phys. Polon. **13** (1954), 187.

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THEORETICAL PHYSICS

Killing's Equations and Conservation Theorems

by

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Presented by L. INFELD on August 14, 1956

1. The principle of general covariance leads, for a given field theory, to four differential identities [1]. When the field equations (I. 4)* are fulfilled, these identities yield four conservation laws

(1) $$T^\beta_{\alpha;\beta}=0.$$

It is known, however, that a Lorentz-covariant theory in a flat space-time possesses 10 conservation laws [2], [3]. A theory which is general-covariant is also Lorentz-covariant, but no angular momentum tensor exists in a Riemannian space-time (apart from special cases).

Thus we see that the properties of the space-time continuum are essential for deriving the conservation theorems. The purpose of this note is to reveal some of the relations between covariance, space-time structure and conservation laws. The notation of [1] will be used in this paper.

2. We shall deal with field theories, the equations of which can be obtained from (I. 1, 2), where

(2) $$L=L(g_{\mu\nu},\psi_A,\psi_{A,\nu}).$$

In general, the behaviour of the Lagrangian density L can be investigated under certain groups of transformations of the

(a) independent variables $x^\nu\to x^{\nu\prime}$;
(b) metric tensor $g_{\alpha\beta}\to{}'g_{\alpha\beta}$;
(c) field variables $\psi_A\to{}'\psi_A$ ("gauge transformations").

We have dealt with the transformations (a) in [1]. This group will be investigated from another point of view in sections 3 and 5. The gauge transformations and related properties of the field equations are examined in [2] and [3].

*) The formulae from [1] we shall refer to as I.

Let us fix our attention on the group (b). The simplest transformations of the metric are the *conformal transformations*:

(3) $g_{\alpha\beta} \to {}'g_{\alpha\beta} = \sigma g_{\alpha\beta}$, where $\sigma(x^\nu_{\ast})$ is a scalar.

Taking the infinitesimal conformal transformation

$$\delta g^{\alpha\beta} = {}'g^{\alpha\beta} - g^{\alpha\beta} = g^{\alpha\beta}\delta\eta,$$

we obtain for the variation of $\mathfrak{L}$ (cf. (I. 5) and (2)):

$$\delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial g^{\alpha\beta}} g^{\alpha\beta}\delta\eta = \frac{1}{2}\sqrt{-g}\,T_{\alpha\beta}g^{\alpha\beta}\delta\eta = \frac{1}{2}\mathfrak{T}\delta\eta.$$

Thus, *a necessary and sufficient condition for the field equations derived from a Lagrangian density* (2) *to be conform-invariant is*

(4) $$T^\alpha_\alpha = T = 0.$$

Equation (4) holds for the electromagnetic field [4] which is intimately connected with the vanishing of the rest-mass of photons.

3. Equation (1) does not allow a direct physical interpretation (in a curved space-time), because of the term $\Gamma^\nu_{\alpha\beta}\mathfrak{T}^\beta_\nu$ in $\mathfrak{T}^{\,\beta}_{\alpha;\beta}$.

For physical inerpretation, conservation laws in the form of an *ordinary* divergence are of interest. Our purpose is to give a general method for constructing such laws.

Let us rewrite equation (I. 11) profiting from (2):

(5) $$\delta^* I = \int_\Omega \left(\frac{\partial\mathfrak{L}}{\partial g^{\alpha\beta}}\delta^* g^{\alpha\beta} + \frac{\partial\mathfrak{L}}{\partial\psi_A}\delta^*\psi_A + \frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\delta^*\psi_{A,\nu} + (\mathfrak{L}\delta\zeta^\nu)_{,\nu}\right) d_4x = 0.$$

Let us now assume that in the Riemannian space-time V_4 there exists a vector field $\delta\zeta^\nu$ fulfilling the *Killing equations*:

(6) $$\delta^* g^{\alpha\beta} \equiv \delta\zeta^{\alpha;\beta} + \delta\zeta^{\beta;\alpha} = 0.$$

Each solution of (6) generates a one-parameter group of transformations leaving the metric $g^{\alpha\beta}$ *invariant in form*. These transformations are called *motions* [5] and are a generalisation of Lorentz transformations in a pseudo-Euclidean space. The number of independent solutions of (6) in a V_4 is $\leqslant 10$. Let us by $\underset{i}{\delta}\zeta^\nu$ $(i = 1, 2, \ldots, k \leqslant 10)$ denote these solutions, then, writing $\underset{i}{\delta^*}\varphi$ for variations corresponding to $\underset{i}{\delta}\zeta^\nu$, we obtain from (5):

$$\underset{i}{\delta^*} I = \int_\Omega \left(\frac{\partial\mathfrak{L}}{\partial\psi_A}\underset{i}{\delta^*}\psi_A + \frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\underset{i}{\delta^*}\psi_{A,\nu} + (\mathfrak{L}\underset{i}{\delta}\zeta^\nu)_{,\nu}\right) d_4x = 0.$$

If the field equations (I. 4) are satisfied, we get

(7) $$\underset{i}{\delta^*} I = \int_\Omega \left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\underset{i}{\delta^*}\psi_A + \mathfrak{L}\underset{i}{\delta}\zeta^\nu\right)_{,\nu} d_4x = 0.$$

Owing to the arbitrariness of Ω, (7) yields

(8) $$\left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\underset{i}{\delta^*}\psi_A + \mathfrak{L}\underset{i}{\delta}\zeta^\nu\right)_{;\nu} = 0, \quad i = 1, \ldots, k.$$

Equation (7) when applied to a region bounded by two space-like hypersurfaces, gives:

(9) $$\int_\infty \left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\underset{i}{\delta^*}\psi_A + \mathfrak{L}\underset{i}{\delta}\zeta^\nu\right) dS_\nu = \underset{i}{\mathrm{const}}, \quad i = 1, \ldots, k$$

(we have assumed the vanishing of the field in space infinity). The assumption that $\mathfrak{L}$ does not depend on $g_{\alpha\beta,\nu}$ played no role in this section.

4. Examples. a) When the space-time is *flat* (pseudo-Euclidean, R_4), there exist 10 independent solutions of (6). In a Galilean (pseudo-Cartesian) co-ordinate systems, we can take as solutions of (6):

(10) $$\delta\zeta^\nu = \delta a^\nu, \qquad \delta\zeta^\nu = x^\mu\delta\omega^\nu{}_\mu.$$

δa^ν and $\delta\omega^{\mu\nu}$ are constant, arbitrary vector and bivector respectively. Writing (8) explicitly for $\delta\zeta^\nu$ given by (10), we get 4 laws of conservation of energy and momentum and 6 laws of conservation of angular momentum [3]:

(11) $$t^\nu_{\mu,\nu} = 0, \quad \text{where} \quad t^\nu_\mu = -\delta^\nu_\mu L + \frac{\partial L}{\partial\psi_{A,\nu}}\psi_{A,\mu},$$

(12) $$J^\nu_{\alpha\beta,\nu} = 0, \quad \text{where} \quad J^\nu_{\alpha\beta} = x_{[\alpha}t^\nu_{\beta]} + \frac{\partial L}{\partial\psi_{A,\nu}}\psi_B F_{A[\alpha}{}^B{}_{\beta]}.$$

b) We call a space-time *stationary*, when there exists a co-ordinate system in which $g_{\alpha\beta,0} = 0$. The vector field $\delta\zeta^\nu$ with components $\delta\zeta^0 = 1$, $\delta\zeta^k = 0$ fulfills (6), and we obtain from (8) a law of "conservation of energy":

$$\left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\psi_{A,0}\right)_{,\nu} = \mathfrak{L}_{,0}.$$

c) In a V_4 of *constant curvature* equations, (6) have 10 independent solutions and (8) gives 10 conservation laws.

5. Co-ordinate transformations generated by solutions of

(13) $$\delta^* g^{\alpha\beta} \equiv \delta\zeta^{\alpha;\beta} + \delta\zeta^{\beta;\alpha} = 2g^{\alpha\beta}\delta\chi, \quad (\delta\chi = \text{scalar})$$

are called *conformal co-ordinate transformations* (not to be confused with conformal transformations of metric, section 2). They constitute a generalisation of motions ($\delta\chi = 0$). For a solution $\underset{i}{\delta}\zeta^\nu$ of (13), equation (5) becomes:

$$\underset{i}{\delta^*} I = \int_\Omega \left(\mathfrak{T}\underset{i}{\delta}\chi + \frac{\partial\mathfrak{L}}{\partial\psi_A}\underset{i}{\delta^*}\psi_A + \frac{\partial\mathfrak{L}}{\partial\psi_{A,\nu}}\underset{i}{\delta^*}\psi_{A,\nu} + (\mathfrak{L}\underset{i}{\delta}\zeta^\nu)_{,\nu}\right) d_4x = 0.$$

When $L^A = 0$ we get:

$$\delta^* I = \int_\Omega \left(\underset{i}{\mathfrak{T}} \delta\chi + \left(\frac{\partial \mathfrak{L}}{\partial \psi_{A,\nu}} \underset{i}{\delta^*} \psi_A + \mathfrak{L} \underset{i}{\delta} \zeta^\nu \right)_{,\nu} \right) d_4 x = 0$$

or:

$$(14) \qquad \left(\frac{\partial \mathfrak{L}}{\partial \psi_{A,\nu}} \underset{i}{\delta^*} \psi_A + \mathfrak{L} \underset{i}{\delta} \zeta^\nu \right)_{,\nu} = - \underset{i}{\mathfrak{T}} \delta\chi .$$

For $T = 0$ (conform-invariant field theory), (14) becomes identical in form with (8).

As an example, we can take a flat space-time and the Maxwell field [4]. Equations (13) in R_4 have 15 independent solutions: 10 motions (10) and 5 infinitesimal conformal transformations which are not motions, e. g.:

$$\delta\zeta^\nu = x^\nu \delta a , \qquad \delta\zeta^\nu = 2x^\nu x^\mu \delta a_\mu - x_\mu x^\mu \delta a^\nu \qquad (\delta a, \delta a^\nu = \mathrm{const}).$$

There is no difficulty in writing corresponding conservation laws.

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REFERENCES

[1] A. Trautman, *On the conservation theorems and equations of motion in co-variant field theories*, in this issue p. 677.

[2] W. Pauli, Rev. Mod. Phys. **13** (1941), 203.

[3] E. M. Corson, *Introduction to tensors, spinors and relativistic wave-equations*, London, 1954.

[4] E. Bessel-Hagen, Math. Ann. **84** (1921), 258.

[5] L. P. Eisenhart, *Riemannian geometry*, Princeton 1949.

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THEORETISCHE PHYSIK

Über die Bewegungsgleichungen des Pol-Dipol-Teilchens

von

K. NAGY

Vorgelegt von L. INFELD am 8. September 1956

Mit den Bewegungsgleichungen des Pol-Dipol-Teilchens haben sich mehrere Forscher von verschiedenen Gesichtspunkten aus beschäftigt [1]—[7]. Unser Ziel ist die einfache Aufstellung der Bewegungsgleichungen des Pol-Dipol-Teilchens auf Grund der Infeldschen Methode. die für Pol-Teilchen ausgearbeitet wurde [8].

Wir betrachten das Teilchen als ein Probeteilchen. Den Spin des Teilchens beschreiben wir durch einen antisymmetrischen Tensor $S_{\alpha 4}$.

$$(1) \qquad S_{23} = \sigma_x ; \qquad S_{31} = \sigma_y ; \qquad S_{12} = \sigma_z .$$

$\sigma(\sigma_x, \sigma_y, \sigma_z)$ bedeuten hier den Spinvektor. Wir nehmen an, dass die $S_{\alpha 4}$ in dem Koordinatensystem in dem das Teilchen ruht gleich Null sind Diese Bedingung kann man in der folgenden kovarianten Form anschrieben:

$$(2) \qquad S_{\alpha\beta} u_\beta = 0 ,$$

wo u_β die Vierergeschwindigkeit des Teilchens bedeutet.

Die Bewegungsgleichungen kann man aus den Feldgleichungen nur in nichtlinearen Feldtheorien herleiten. Nach dem Gedankengang von L. Infeld nehmen wir zu den Gleichungen des linearen Feldes (z. B. des elektro-magnetischen-, oder Mesonfeldes) noch die Gleichungen des Gravitationsfeldes hinzu, und bekommen so ein nichtlineares Gleichungsystem, aus dem man die Bewegungsgleichungen herleiten kann. Gehen wir endlich zu der speziellen Relativitätstheorie durch den Grenzübergang $g_{\alpha\beta} \to \sigma_{\alpha\beta}$, $x \to 0$ (x bedeutet hier die Gravitationskonstante) über, so bekommen wir die Bewegungsgleichungen in den linearen Feldtheorien:

$$(3) \qquad \frac{dt}{ds} \int T_{\alpha\beta,\beta}\, dv = 0 .$$

Der Energie-Impuls-Tensor $T_{\alpha\beta}$ ist

$$(4) \qquad T_{\alpha\beta} = t_{\alpha\beta} + E_{\alpha\beta} ,$$