COMMENTS ON THE PAPER BY ELIE CARTAN: SUR UNE GENERALISATION DE LA NOTION DE COURBURE DE RIEMANN ET LES ESPACES A TORSION

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This paper is a real gem, written in s style characteristic of Elie Cartan: it contains important new ideas, but no precise definitions, theorems or equations.

The author generalizes the notion of parallel transport of vectors, introduced by Levi Civita. This generalization is motivated by physical considerations: Cartan refers to his earlier paper on the stress-energy tnesor in Einstein's theory and to the work of the brothers E. and F. Cosserat on continuous media with an intrinsic angular momentum.

The geometry considered by Cartan is that of a three-dimensional manifold with a metric tensor g and a linear connection ω which is Euclidean - or metric - i.e. compatible with g. The condition of compatibility may be written as

$$Dg_{\mu\nu} = 0 \quad , \tag{1}$$

where D is the exterior covariant derivative, $Dg_{\mu\nu} = dg_{\mu\nu} - \omega^{\rho}{}_{\mu}g_{\rho\nu} - \omega^{\rho}{}_{\nu}g_{\rho\mu}$, and $\mu,\nu,\rho = 1,2,3$.

Cartan emphasizes that eq.(1) does not completely define the connection form $\omega^{\mu}{}_{\nu}$, namely

$$\omega^{\mu}{}_{\nu} = \gamma^{\mu}{}_{\nu} + \kappa^{\mu}{}_{\nu} \quad , \tag{2}$$

where γ is the Levi Civita connection and the tensor-valued form $\kappa_{\mu\nu} = g_{\mu\rho}\kappa^{\rho}{}_{\nu}$ is skew in the pair (μ,ν) , but otherwise arbitrary. He interprets formula (2) by saying that infinitesimal parallel transport defined by ω consists of a translation, i.e. parallel transport defined by the Levi Civita connection, and of a rotation given by κ .

To understand the rest of the paper, it is convenient to introduce, following Cartan, a moving frame, i.e. a field (θ^μ) of three linearly independent 1-forms (triad). In terms of these basis forms, the curvature and torsion 2-forms may be written as

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} \theta^\rho \wedge \theta^\sigma \quad ,$$

$$\Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = \frac{1}{2} Q^\mu{}_{\rho\sigma} \theta^\rho \wedge \theta^\sigma \quad .$$

Cartan considers next a field of triads, defined by parallel transport along a closed curve (loop), and a radius vector along the loop. A vector field (u^μ) is parallel if it is covariantly constant,

$$Du^\mu = 0 \quad . \tag{3}$$

The integrability condition of (3) is

$$\Omega^\mu{}_\nu u^\nu = 0 \quad .$$

In a curved space, in general there are no parallel vector fields, but eq.(3) can always be integrated along a curve. When this is done along a small loop, the vector u changes, approximately, by

$$\Omega^\mu{}_\nu u^\nu$$

times the surface element spanned by the loop. Similarly, a radius (position) vector field (r^μ) satisfies

$$\nabla_\mu r^\nu = \delta^\nu_\mu \quad \text{or} \quad Dr^\mu = \theta^\mu \quad , \tag{4}$$

and the integrability condition

$$\Omega^\mu{}_\nu r^\nu - \Theta^\mu = 0$$

determines the change in r when the position vector field is built along a loop by integration of eq.(4). Cartan points out that the position vector is not only rotated - as is a vector undergoing parallel transport - but suffers also a shift or translation proportional to Θ^μ times the surface element spanned by the loop.

Introducing the completely skew tensor $(\eta_{\mu\nu\rho})$, $\eta_{123} = \sqrt{\det(g_{\mu\nu})}$, one can represent the density of rotation by the vector (-valued 2-form)

$$t_\mu = \frac{1}{2} \eta_{\mu\nu\rho} \wedge \Omega^{\nu\rho} \tag{5}$$

and the density of translation by the "couple"

$$s_{\mu\nu} = -\eta_{\mu\nu\rho} \wedge \theta^\rho . \tag{6}$$

The Bianchi identities

$$D\Omega^\mu{}_\nu = 0 \qquad \text{and} \qquad D\theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu$$

together with eq.(1), imply the conservation laws italicized by Cartan in the sixth paragraph of the paper

$$Dt_\mu = 0 \quad , \tag{7}$$

$$Ds_{\mu\nu} = \theta_\nu \wedge t_\mu - \theta_\mu \wedge t_\nu \quad . \tag{8}$$

These equations are interpreted as conditions of equilibrium of a continuous medium under the action of elastic forces with a non-vanishing (if $s_{\mu\nu} \neq 0$) density of moments, i.e. under the action of <u>torsion</u>-inducing forces (seventh paragraph).

Vanishing of torsion is equivalent to $s_{\mu\nu} = 0$ and $\kappa^\mu{}_\nu = 0$, and, by virtue of eq.(6), implies the symmetry of the stress tensor $t^{\mu\nu}$, defined by

$$t^\mu = \frac{1}{2} t^{\mu\nu} \eta_{\nu\rho\sigma} \wedge \theta^\rho \wedge \theta^\sigma \quad .$$

Incidentally, all of these considerations generalize to higher-dimensional spaces, and in particular, to the four-dimensional spacetime. In the latter case, $\eta_{\mu\nu\rho}$ is replaced by $\eta_{\mu\nu\rho\sigma}\theta^\sigma$, where $(\eta_{\mu\nu\rho\sigma})$ is the completely skew tensor in four dimensions. An essential change occurs in eq.(7), which is replaced by

$$Dt_\mu = \frac{1}{2} \eta_{\mu\nu\rho\sigma} \theta^\sigma \wedge \Omega^{\nu\rho} \quad . \tag{9}$$

In a subsequent paper [1], Cartan develops a theory of space, time and gravitation based on the geometry of a four-dimensional space with a metric compatible with its linear connection, which need not be symmetric. In that theory, $t_{\mu\nu}$ is interpreted as the stress-energy tensor. Led by analogies with special and general relativity, Cartan imposes on it a conservation law of the form (7); together with the identity (9) this results in a highly restrictive algebraic constraint. Presumably, the constraint discouraged Cartan, who later never returned to his theory of 1923. According to our present views, conservation laws in any geometric theory of gravitation result from Bianchi identities. Therefore, eq.(9) should be accepted as a differential (local) conservation law. It has been shown [2] that from an isometry in spacetime, and eqs.(8) and (9), there follows a global conservation law, similar to the law known in the Riemannian case.

Let us now return to Cartan's short note in the Comptes Rendus. In its tenth paragraph Cartan defines straight (autoparallel) lines and points out that, in general, they are different from Riemannian geodesics (shortest curves). These two sets of lines coincide if and only if the torsion tensor $Q_{\mu\nu\rho} = g_{\mu\sigma} Q^{\sigma}{}_{\nu\rho}$ is completely skew. Such is the case of R^3 with a linear connection defined by helicoidal displacements. If $\theta^1 = dx$, $\theta^2 = dy$ and $\theta^3 = dz$, then the coefficients of the connection are $\omega_{\mu\nu} = \alpha\eta_{\mu\nu\rho}\theta^{\rho}$, where α is the pitch of the helicoidal motion. Both curvature and torsion are constant. This is a 'parity-violating' space: the torsion tensor defines a preferred orientation.

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References

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2. A. Trautman, The Einstein-Cartan theory of gravitation, in "Ondes et Radiations Gravitationnelles", Colloque Intern. du CNRS, No.220, CNRS, Paris (1973).