47.

GENERALITIES ON GEOMETRIC THEORIES OF GRAVITATION

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All classical, local theories of spacetime and gravitation are based on a rather small number of assumptions about the geometry, the form of the field equations and the nature of the sources. The basic assumptions may be formulated in such a way as to allow an easy comparison between the theories. To achieve this, it is convenient to distinguish the 'kinematic' part of the assumptions, referring to the type of geometry, from the 'dynamic' part, which consists in specifying the form of the field equations.

The kinematics of essentially all theories is based on a four-dimensional differentiable manifold M as the model of spacetime. The manifold is endowed with at least two geometric structures: a connection and a metric structure. The connection is necessary to compare - and, in particular, to differentiate - objects such as vectors and tensors, needed to describe momenta, forces, field strengths, etc. In most cases a linear connection is used, but it is possible to develop all or parts of physics on the basis of other connections (affine, conformal). For example, to compare directions and to define straight (autoparallel) lines it suffices to consider a projective connection, defined as the equivalence class of linear connections whose coefficients $\Gamma^{\mu}_{\nu\rho}$ are related by

$$\bar{\Gamma}^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} + \delta^{\mu}_{\nu} \lambda_{\rho} \quad .$$

A metric structure is needed to measure distances, time intervals, angles and relative velocities. A theory is relativistic if its metric structure is given by a metric tensor g of signature (+++-). A somewhat weaker metric structure is called conformal geometry: it is given by an equivalence class of metric tensors, two tensors g and $\bar{g}$ being considered as equivalent if and only if they differ by a point-dependent factor,

$$\bar{g} = e^{\sigma} g \ .$$

Conformal geometry is enough to write 'gauge equations' such as source-free Maxwell and Yang-Mills equations.

In most theories, the metric structure and the connection are assumed to be compatible. For example, a conformal geometry is compatible with a projective connection if the property of being a null direction is preserved under parallel transport. In a relativistic theory, the metric tensor is compatible with a linear connection iff the latter is metric, $\nabla_{\mu} g_{\nu\rho} = 0$.

From the kinematic point of view, the majority of viable theories falls into one of the following classes:
(i) Newtonian theories based on a linear connection Γ and a Galilean metric structure, given by a symmetric tensor $(h^{\mu\nu})$ of signature (+++0). A suitably normalized zero eigenform (τ_{μ}) of h is the 1-form of absolute simultaneity. In the standard theory, (τ_{μ}) is the gradient of absolute time.
(ii) Bimetric theories have, in addition to the metric tensor g , another symmetric tensor $(h_{\mu\nu})$ as a basic variable. Linear connection(s) are built from g (and/or h) by the Christoffel formula or its modifications. There are two main subclasses:

1. Linearized theories of gravity interpret h as the gravitational potential meaningful only up to transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_{\mu} a_{\nu} + \nabla_{\nu} a_{\mu} \quad ,$$

where (a_{μ}) is arbitrary and ∇ is defined by the Levi Civita connection of the metric g ;

2. Bimetric theories in the strict sense had been proposed by N. Rosen and recently considered also by A.A. Logunov. In these theories, matter variables couple to h only, but the Lagrangian of gravitation is allowed to depend on both g and h .

(iii) Riemann-Cartan theories assume a compatible pair (Γ, g) as determining the underlying geometry. The most important among them are:
1. Einstein's theory of 1915, based on Riemannian geometry, and
2. the Einstein-Cartan theory, which is a slight modification and generalization of Einstein's theory, obtained by allowing a non-zero torsion $Q^{\mu}{}_{\nu\rho}$.

Other possibilities are:

3. the Nordström theory, based on conformally flat Riemannian geometry;
4. theories with distant parallelism (teleparallelism); they are dual to Riemannian theories in the sense that they assume vanishing curvature, but $Q^{\mu}{}_{\nu\rho} \neq 0$.
They are also called tetrad theories (C. Møller) as they admit a family of preferred fields of tetrads, defined up to constant Lorentz rotations.

All theories listed under (ii) and (iii) are relativistic: the tangent spaces to the spacetime manifold in any of these theories have a geometry equivalent to that of Minkowski space.

The principle of equivalence played a heuristic role in arriving at Einstein's relativistic theory of gravitation. One is tempted to formulate it today in the following, rather sharp, way: in the vacuum, the geometry of spacetime defines in lowest differential order only one linear connection. This principle, if accepted, may be used to rule out many of the bimetric theories listed under (ii.2).

To develop a definite theory of gravitation it is necessary to

A. Specify its kinematics, i.e. the type of geometry;
B. Write the field equations of gravitation and the equations of motion of other types of matter;
C. Give a physical interpretation to the quantities occurring in, or derivable from, the geometrical model;
D. Study the consequences of the kinematic and dynamic aspects of the theory.

These general guidelines require comments which can be here only very brief. The field equations of gravitation usually follow the pattern of the Poisson equation

$$\Delta \varphi = 4\pi\rho$$

of the Newtonian theory: there is a "left-hand side" constructed from the potentials and a "right-hand side" describing the sources. In most cases, the left-hand side is obtained from a variational principle whereas the form of the sources follows from either
(a) a Lagrangian L depending on both gravitational and matter variables, or
(b) phenomenological considerations.
In the first case, the interaction of matter with gravitation is achieved by imposing a principle of minimal coupling which is a prescription how to go over from the special-relativistic Lagrangian to its general-relativistic counterpart without introducing explicitly the curvature tensor. The second approach is less satisfactory from the point of view of foundations, but more

suitable to astrophysical applications. In either case, it is necessary to specify which geometric elements of the theory have a dynamical significance, i.e. are determined from the field equations and which are 'absolute', independent of the particular physical situation. For example, the metric tensor is absolute in special relativity and dynamical in the general theory. Torsion is absolute - and zero - in Einstein's theory, but acquires a dynamical significance in generalized theories such as the Einstein-Cartan theory. The group of symmetries of a theory preserves the absolute elements.

It is important to remember that the physical interpretation of the mathematical notions occurring in a physical theory must be compatible with the equations of the theory. For example, it follows from Einstein's equations that the worldlines of 'dust particles' are geodesics; this determines the physical interpretation of the linear connection. When one goes over to a theory with torsion, it is not possible to "generalize" this result by postulating that test particles move along the autoparallels. It turns out that in the Einstein-Cartan theory spinless particles still move along the geodesics of the Riemannian connection. To measure torsion, one has to consider particles with spin.

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