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ON GROUPS OF GAUGE TRANSFORMATIONS

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0. Summary

Groups of gauge transformations (gauge groups) are defined in the framework of principal bundles. The gauge group of a trivial bundle is exhibited and the gauge aspect of gravitation is compared to that of Yang-Mills theories.

1. Introduction

Recent developments in theoretical physics indicate a wide-ranging importance of gauge fields. There are reasons to believe that all fundamental forces are mediated by particles which are quanta of appropriate gauge fields. In the approximation of classical physics, gauge configurations are described best by connections on principal bundles over spacetime. The mathematical framework of fibre bundles provides precise definitions of the notions used in classical gauge theories. Among them are the notions of gauge transformations. In the older literature, a distinction was made between transformations of the first and second kind [1], whereas in recent works one refers to global and local gauge transformations [2]. There is also considerable interest in gauge transformations in the theory of gravitation [3] and its 'supersymmetric' modification [4].

Extending an earlier note [5], this paper contains the definitions and elementary properties of gauge groups. The theory of gravitation is contrasted to a Yang-Mills theory over Minkowski spacetime. The paper follows the standard notation and terminology used in differential geometry and applications of fibre bundle theory to physics (see, for example, [6] and the references given therein). All manifolds and maps are assumed to be of class C^∞. A principal fibre bundle includes in its definition a projection π of the total space of the bundle P on the base M and an action of a Lie group G on P to the right. The action is free and transitive on the fibres of π. If $\delta: P\times G \to P$ is the map defining the action, then one writes $\delta(p,a) = \delta_a(p)$ or pa, for simplicity. A connection is given by a one-form ω on P, with values in the Lie algebra of G. A (local) section s of π, $s: M \to P$, $\pi \circ s = \mathrm{id}_M$, corresponds to the physicists' idea of choosing a gauge. Let $\rho: G\times N \to N$ be a map defining a (left) action of G in a manifold N. A (generalized) Higgs field of type ρ is a map $\varphi: P \to N$, equivariant under the action of G, i.e. such that $\varphi\circ\delta_a = \rho_{a^{-1}}\circ\varphi$, where $\rho_a(n) = \rho(a,n)$, $a \in G$ and $n \in N$. The pullback $A = s^*\omega$ is the gauge

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potential in that gauge.

2. Automorphisms of principal bundles

A diffeomorphism $u:P \to P$ is an automorphism of the bundle $\pi:P \to M$ if there is a diffeomorphism $v:M \to M$ such that $\pi \circ u = v \circ \pi$ and $u(pa) = u(p)a$ for any $p \in P$ and $a \in G$. The set AutP of all automorphisms of P is a group under composition of maps. The diffeomorphism v is uniquely determined by the automorphism u and there is a homomorphism of groups $j:\mathrm{AutP} \to \mathrm{DiffM}$ given by $j(u) = v$. An automorphism u is called vertical (or based) if $j(u) = id_M$; the set Aut_0P of all vertical automorphisms is a normal subgroup of AutP and the sequence

$$1 \to \mathrm{Aut}_0P \to \mathrm{AutP} \xrightarrow{j} \mathrm{DiffM}$$

is exact. There are natural bijections among the following three sets

(i) Aut_0P;

(ii) the set of all maps $U:P \to G$ such that $U(pa) = a^{-1}U(p)a$ for any $p \in P$ and $a \in G$;

(iii) the set of all sections of the bundle $E \to M$, associated to $\pi:P \to M$ by the adjoint action of G on itself [7].

The correspondence between (i) and (ii) is given by

$$u(p) = pU(p).$$

Let $k:P\times G \to E$ be the canonical map, $k(p,a) = k(pb,b^{-1}ab)$, where $a,b \in G$. If U is as in (ii), then $k(p,U(p))$ depends only on $\pi(p)$ and defines a section $\tilde{u}$ of $E \to M$.

Example 1.

If c is a central element of G, then the constant map $U:P \to G$, $U(p) = c$, defines a vertical automorphism.

Example 2.

Let M be n-dimensional, $x \in M$ and let T_xM be the tangent space to M at x. A (linear) frame at x is a (linear) isomorphism $e:R^n \to T_xM$. The set LM of all such frames at all points of M gives rise, in a natural manner, to the principal bundle of frames; $\pi(e) = x$. The action of its group, $GL(n,R)$, is by composition of linear maps, $ea = e\circ a$. The bundle E associated to LM by ad consists of all linear automorphisms of the tangent spaces to M; a section of $E \to M$ is a field of invertible tensors of mixed valence on M. Any vertical automorphism of LM is given by a tensor field of this kind.

Example 3.

Let $T_xv:T_xM \to T_{v(x)}M$ denote the tangent map to $v:M \to M$ at $x \in M$. For any $v \in \mathrm{DiffM}$, one defines its lift $Lv:LM \to LM$ by

$$Lv(e) = T_xv\circ e, \quad \text{where } x = \pi(e). \qquad (1)$$

Clearly, Lv is an automorphism of $LM \to M$ and $jL = id_{\mathrm{DiffM}}$. For any $u \in \mathrm{AutLM}$, the composition $u\circ(Lj(u^{-1}))$ is a vertical automorphism.

For any manifold M and (Lie) group G, one defines the group G^M of all maps from M to G; the composition in G^M is induced pointwise from G. There is a natural homomorphism τ of DiffM into the automorphism group of G^M given by

$$\tau_v(w) = w\circ v^{-1}, \quad \text{where } v \in \text{DiffM}, \; w:M \to G.$$

Proposition 1.

The group of all automorphisms of the trivial bundle $pr_1:M\times G \to M$ is isomorphic to the semi-direct product of DiffM and G^M relative to τ.

Indeed, any automorphism $u:M\times G \to M\times G$ may be represented by the pair (v,w), where $v = j(u) \in \text{DiffM}$ and $w:M \to G$ is such that

$$u(x,a) = (v(x),w(v(x))a) \quad \text{for any } x \in M \text{ and } a \in G.$$

Moreover, if the automorphism u' is represented in this way by (v',w'), then

$$u\circ u' \text{ is represented by } (v\circ v', w\circ\tau_v(w')).$$

3. Gauge groups and symmetries

In any physical theory, besides dynamical variables which are subject to equations of motion, there occur absolute elements, such as external forces or the metric tensor in special relativity. In a gauge theory, the absolute elements are often given by geometric objects, defined on the bundle $\pi:P \to M$, in addition to the connection and the Higgs field which play a dynamical role. It is reasonable to define the gauge group of such a theory as the subgroup G of AutP, consisting of all automorphisms of π which preserve the absolute elements. The elements of G are called gauge transformations. A pure gauge transformation is a vertical element of G. The pure gauge group

$$G_0 = G \cap \text{Aut}_0P$$

is a normal subgroup of G and there is the exact sequence

$$1 \to G_0 \to G \to G/G_0 \to 1. \tag{2}$$

Gauge transformations act on sections and connections: if s is a section of $\pi:P \to M$ and $u \in G$, then $s' = u\circ s\circ v^{-1}$ is another section. Similarly, the pullback $\omega' = u^*\omega$ of a connection form is another connection form and there is the equality of potentials

$$s^*\omega' = v^*s'^*\omega.$$

This can be interpreted as follows: the form ω describes the same geometry and physics as ω' does, only 'translated' by the diffeomorphism v. In other words, any invariant constructed from ω' and the absolute elements at $x \in M$ is equal to the corresponding invariant constructed from ω at $v(x)$.

The gauge group of a Yang-Mills theory over Minkowski space is easily obtained on the basis of Proposition 1: the group G_0 is isomorphic to G^M, whereas G is isomorphic to the semi-direct product of the inhomogeneous Lorentz group and G^M relative to τ.

The following example shows that the gauge exact sequence need not split for a non-trivial bundle.

Example 4.

Consider the Z-bundle $\pi:R \to U(1)$, $\pi(t) = \exp 2\pi it$, and assume its total space R to have the standard metric and orientation. If these two elements are considered as absolute, then G reduces to R, the group of translations, and the sequence (2) becomes

$$1 \to Z \to R \to U(1) \to 1.$$

By definition, a diffeomorphism $v:M \to M$ is a symmetry of a gauge configuration given by ω on P if there is a gauge transformation $u:P \to P$ which covers v, i.e. $j(u) = v$, and

$$u^*\omega = \omega,$$

Similarly, a Higgs field of type ρ given by the map $\varphi:P \to N$ admits v as a symmetry if it is invariant under $u \in G$,

$$u^*\varphi = \varphi,$$

and $j(u) = v$. If N is an orbit of G, then φ restricts the bundle $\pi:P \to M$ to the little group of $\varphi_0 \in N$,

$$H = \{a \in G: \rho(a,\varphi_0) = \varphi_0\}.$$

The total space Q of the restricted bundle is

$$Q = \{p \in P: \varphi(p) = \varphi_0\}$$

and it is straightforward to prove

Proposition 2.

A Higgs field with values in an orbit of G is invariant under $u \in \mathrm{Aut}P$ if and only if $u \in \mathrm{Aut}Q$.

4. Gravitation

The 'kinematic' aspect of gravitation is described by a connection ω on the bundle LM of linear frames of an $n(=4)$-dimensional manifold and by a metric which may be considered as a generalized Higgs field $g:LM \to N \subset L^2_s(R^n,R)$, where N is an orbit of $GL(n,R)$ in the space of symmetric, n n matrices. According to the theorem on inertia of quadratic forms on R^n there is a one-to-one correspondance between the set of all such orbits and the collection of all possible signatures of these forms. The 'dynamics' consists of differential equations for ω and g.

An important aspect of gravitation is the 'concrete' nature of LM: its elements are linear frames on M whereas not much can be said about the elements of an 'abstract' bundle P. The bundle $\pi:LM \to M$ is 'richer' than an abstract bundle. Its additional structure is completely described by the canonical one-form $\theta:TLM \to R^n$ defined by

$$\theta_e = e^{-1} \circ T_e \pi \tag{3}$$

where θ_e is the restriction of θ to $T_e LM$ and $e \in LM$ is interpreted as an isomorphism from R^n to $T_{\pi(e)}M$. Clearly, $\theta_{ea} \circ T_e \delta_a = a^{-1} \circ \theta_e$, thus proving

$$\delta_a^* \theta = a^{-1} \circ \theta \tag{4}$$

and, for any $u \in TLM$,

$$\theta(u) = 0 \Leftrightarrow T\pi(u) = 0. \tag{5}$$

Proposition 3.

A principal bundle $\pi : P \to M$, with an n-dimensional base and structure group $GL(n,R)$ is isomorphic to the bundle of linear frames $LM \to M$ if and only if there is a map $\theta : TP \to R^n$, linear on the fibres of $TP \to P$, and satisfying (4) and (5) for any $a \in GL(n,R)$ and $u \in TP$.

Indeed, if there is such a θ on P, then the (based) isomorphism $h : P \to LM$ is determined as follows. Condition (5) means that, for any $p \in P$, the linear map $T_p\pi : T_pP \to T_{\pi(p)}M$ factors through $\theta_p : T_pP \to R^n$, i.e. there is a linear map

$$h(p) : R^n \to T_{\pi(p)}M$$

such that $h(p) \circ \theta_p = T_p\pi$. The map $h(p)$ is uniquely defined; moreover, it is an isomorphism and, therefore, an element of LM lying over $\pi(p)$. The equivariance of h follows from (4).

The canonical ('soldering') form θ plays the role of an absolute element in the theory of gravitation. The following two propositions are useful in determining the gauge groups in gravity:

Proposition 4.

If $u \in Aut_0 LM$ and $u^*\theta = \theta$, then $u = id$. This follows directly from the definition (3) of θ: $u^*\theta = \theta$ is equivalent to $\theta_{u(e)} \circ T_e u = \theta_e$, any e. Using (3) and $\pi \circ u = \pi$, one obtains $u(e) = e$.

Proposition 5.

If $u : LM \to LM$ is a diffeomorphism such that $\pi \circ u = v \circ \pi$ for some diffeomorphism $v : M \to M$, then the following conditions are equivalent:

(i) $u = Lv$,

(ii) $u^*\theta = \theta$.

Indeed, it follows from the definition of θ and (1) that

$$(u^*\theta)_e = u(e)^{-1} \circ Lv(e) \circ \theta_e \quad \text{for any } e \in LM.$$

(i) $\Rightarrow$ (ii) is now obvious and (ii) $\Rightarrow$ (i) follows from the surjectivity of θ_e.

Let Ω and Θ be, respectively, the curvature and torsion two-forms of a linear connection ω. Denoting by Ω' and Θ' the forms corresponding to $\omega' = u^*\omega$, $u \in AutLM$, one obtains

Proposition 6.

For any ~~u ∈~~ AutLM,

$$\Omega' = u^*\Omega.$$

If, moreover, $u = Lv$, then

$$\theta' = u^*\theta. \qquad (6)$$

It is important to realize that (6) does not, in general, hold unless u is the lift of a diffeomorphism; one can 'generate torsion' by applying a suitable vertical automorphism to a symmetric connection. These remarks are intended to justify our definition of the group $\mathcal{G}$ of gauge transformations in theories of gravity based on LM:

$$\mathcal{G} = \{u \in \text{AutLM}: u^*\theta = \theta\}.$$

By Proposition 5 this group is isomorphic to DiffM and by Proposition 4 the group $\mathcal{G}_0$ of pure gauge transformations reduces to {id}. This should be contrasted with the case of a Yang-Mills theory over Minkowski space, for which $\mathcal{G}_0 = G^M$ is 'large' and $\mathcal{G}/\mathcal{G}_0$ is finite-dimensional ('small').

Incidentally, the lift $L:\text{DiffM} \to \text{AutLM}$ defines a splitting of the sequence

$$1 \to \text{Aut}_0\text{LM} \to \text{AutLM} \to \text{DiffM} \to 1$$

and the representation $u \to (v, u\circ Lv^{-1})$, $v = j(u)$, yields an isomorphism of AutLM on the semi-direct product of DiffM and Aut_0LM, relative to the homomorphism $\sigma:\text{DiffM} \to \text{Aut}(\text{Aut}_0\text{LM})$, where $\sigma_v(w) = Lv\circ w\circ Lv^{-1}$ for $w \in \text{Aut}_0\text{LM}$.

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