

LETTER TO THE EDITOR

A class of null solutions to Yang-Mills equations

Andrzej Trautman

Institute of Theoretical Physics, Warsaw University, Hoza 69, 00-681 Warsaw, Poland

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Abstract. A class of null solutions of Maxwell's equations is generalised to Yang-Mills theory. The plane-fronted, non-Abelian waves, recently found by Coleman, are among the null solutions.

Consider a plane electromagnetic wave, propagating in the z direction in Minkowski space. Its electric and magnetic fields are

$$\mathbf{E} = a\mathbf{e}_x + b\mathbf{e}_y, \quad \mathbf{B} = a\mathbf{e}_y - b\mathbf{e}_x, \quad (1)$$

where a and b are arbitrary functions of $u = t - z$, $\mathbf{e}_x = \text{grad } x$ and $\mathbf{e}_y = \text{grad } y$. This field may be derived from a potential (A_μ) , $\mu = 1, 2, 3, 4$, which will be represented by the 1-form $A = A_\mu dx^\mu$. A possible choice is

$$A = (ax + by + c) du \quad (2)$$

where c is another function of u . Indeed, the electromagnetic field is given by the 2-form

$$F = (a dx + b dy) \wedge du. \quad (3)$$

Clearly, the function c occurring in (2) may be eliminated by a gauge transformation without affecting either a or b .

The potential (2) is suitable for a generalisation to non-Abelian gauge fields, including gravitation, and to a class of null 'spherical' waves.

Consider the line element

$$2 du(dv + H du) - P^2(dx^2 + dy^2) \quad (4)$$

on a four-dimensional manifold M , referred to coordinates x, y, u and v . It includes Minkowski space in various ways, i.e. in different coordinate systems. In particular, if

$$P = 1 \quad H = 0 \quad u = t - z \quad v = \frac{1}{2}(t + z) \quad (5a)$$

then the line element reduces to

$$dt^2 - dx^2 - dy^2 - dz^2. \quad (5b)$$

Similarly, if

$$P = v[1 + \frac{1}{4}(x^2 + y^2)]^{-1} \quad H = \frac{1}{2} \quad u = t - r \quad v = r \quad (6a)$$

$$x + iy = 2e^{i\varphi} \cot \frac{1}{2}\vartheta$$

then the line element is

$$dr^2 - d\vartheta^2 - r^2(d\varphi^2 + \sin^2 \vartheta d\varphi^2). \quad (6b)$$

The line element (4), considered by Ivor Robinson as early as 1956, also includes the Schwarzschild solution ($H = \frac{1}{2} - m/r$, other identifications as in (6a)), plane gravitational waves, and many other solutions of Einstein's equations (Robinson and Trautman 1962).

Let A be the potential of a gauge configuration in a theory of the Yang-Mills type, based on a Lie group G . The potential is a 1-form on M , with values in \mathfrak{g} , the Lie algebra of G . The gauge field corresponding to A is

$$F = dA + \frac{1}{2}[A, A]$$

where the bracket denotes both the Lie algebra product and the exterior (wedge) product of forms. Introducing the Hodge (dual) operator $*$ one can write source-free Yang-Mills equations as

$$D * F = \underset{\text{def}}{d * F} + [A, * F] = 0. \quad (7)$$

Assuming the metric on M to be of the form (4) and orientation to be given by the coframe θ ,

$$\theta^1 = P dx \quad \theta^2 = P dy \quad \theta^3 = du \quad \theta^4 = dv + H du, \quad (8)$$

the duals of 2-forms may be evaluated from

$$*(du \wedge dx) = du \wedge dy \quad *(du \wedge dy) = -du \wedge dx \quad \text{etc.}$$

Let the potential be

$$A = \alpha(x, y, u) du$$

where α is a \mathfrak{g} -valued function of the variables indicated. The bracket $[A, A]$ vanishes because $du \wedge du = 0$. Therefore

$$F = (\alpha_x dx + \alpha_y dy) \wedge du$$

and

$$*F = (\alpha_x dy - \alpha_y dx) \wedge du$$

where subscripts x and y denote partial derivatives with respect to the variables indicated.

Since $[A, *F] = 0$, the Yang-Mills equation (7) reduces to

$$\alpha_{xx} + \alpha_{yy} = 0$$

so that the general solution is $\alpha = \text{Re } w(x + iy, u)$, where w is an arbitrary \mathfrak{g} -valued function of u , analytic in $x + iy$.

In the coordinate system (5), the solution

$$\alpha = a(u)x + b(u)y + c(u)$$

is a 'non-Abelian plane wave' (Coleman 1977). But is it really plane, i.e. invariant under translations in the x and y directions? By looking at the field, given by formula (3), one is tempted to give an affirmative answer to this question. It should be remembered, however, that in a non-Abelian gauge theory, F does not provide a complete description of the gauge configuration.

A transformation (diffeomorphism) f of M is a symmetry of a gauge configuration if the transformed (pulled-back) potential f^*A differs from A only by a change of gauge,

i.e. if there is a G -valued function g on M such that

$$f^*A = g^{-1}Ag + g^{-1}dg. \quad (9)$$

An infinitesimal version of the condition of invariance,

$$\mathcal{L}_\xi A = D\eta, \quad (10)$$

is obtained from (9) by considering the one-parameter groups $(f_t), (g_t), t \in \mathbf{R}$, where (f_t) is generated by a vector field ξ on M and $g_t = \exp t\eta$ with $\eta: M \rightarrow \mathfrak{g}$ (Bergmann and Flaherty 1978, Trautman 1979). Differentiation of both sides of equation (10) leads to

$$\mathcal{L}_\xi F = [F, \eta] \quad (11)$$

where, as in (10), \mathcal{L}_ξ is the Lie derivative with respect to ξ .

If ξ is a generator of a translation in the (x, y) plane, say $\xi = \partial/\partial x$, then equation (11) may be satisfied for a field of the form (3), but equation (10), in general, cannot. Indeed, let $G = \text{SO}(3)$, so that \mathfrak{g} is isomorphic to \mathbf{R}^3 and the Lie bracket corresponds to the vector product. Equation (11) implies

$$[a, \eta] = 0 = [b, \eta] \quad (12)$$

and equation (9) reduces to

$$a du = d\eta + [c, \eta] du. \quad (13)$$

If $[a, b] \neq 0$, then conditions (12) imply $\eta = 0$, in contradiction to (13). Moreover, the function c cannot be changed by a gauge transformation without affecting a and b . Therefore, in an $\text{SO}(3)$ theory, two potentials of the form (2), with different functions c , and the same, but non-commuting a and b , are gauge-inequivalent.

If the values of a, b and c belong to an Abelian Lie subalgebra \mathfrak{h} of \mathfrak{g} , then c can be reduced to 0 without changing either a or b ; in this case translations in the (x, y) plane are symmetries.

Plane-fronted gravitational waves may be described by the line element (4) with $P = 1$ and a linear connection of the form

$$\omega = \alpha du \quad (14)$$

when referred to the frame (8); α is now a $\mathfrak{gl}(4, \mathbf{R})$ -valued function of x, y and u , $\alpha = (\alpha^\mu{}_\nu)$. The connection is metric if and only if

$$\alpha_{\mu\nu} + \alpha_{\nu\mu} = 0.$$

It is moreover torsionless if the only non-vanishing elements of the matrix $(\alpha_{\mu\nu})$ are

$$\alpha_{31} = -\alpha_{13} = H_x \quad \alpha_{32} = -\alpha_{23} = H_y.$$

In other words, the conditions of symmetry and metric compatibility, imposed on the connection 1-form (14), imply that the values of α belong to the *Abelian*, two-dimensional Lie algebra \mathfrak{h} of the group of null Lorentz transformations which leave invariant the null vector $\partial/\partial v$ (see, for example, Ehlers *et al* 1966).

The only essential components of the curvature two-form

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$$

are

$$\Omega_{31} = (H_{xx} dx + H_{xy} dy) \wedge du$$

$$\Omega_{32} = (H_{xy} dx + H_{yy} dy) \wedge du.$$

The Einstein equation for empty space

$$\theta_{[\mu} \wedge \Omega_{\nu\rho]} = 0$$

reduces to

$$H_{xx} + H_{yy} = 0,$$

whereas the Kilmister–Yang equation (Kilmister 1962, Yang 1974)

$$D * \Omega_{\mu\nu} = 0$$

is equivalent to the weaker condition

$$H_{xx} + H_{yy} = \text{arbitrary function of } u.$$

Both of these field equations admit plane waves as solutions. A pure gravitational plane-wave solution of Einstein's equations is given by (Bonnor 1956, Peres 1959)

$$H = \text{Re} \{w(u)(x + iy)^2\}.$$

The corresponding metric is plane because it admits a group of isometries transitive on the null hyperplanes $u = \text{constant}$ (Bondi *et al* 1959, Ehlers and Kundt 1962).

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