Thus the solution of the quantum field problem with exponential interaction in R^3 provides a solution for the classical gas problem with pair interactions. Consequently, the prediction of quantum field theory can be, in principle, tested experimentally.

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THE GEOMETRY OF GAUGE FIELDS*) **)

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Principal fibre bundles with connections provide geometrical models of gauge theories. Bundles allow for a global formulation of gauge theories: the potentials used in physics are pull-backs, by means of local sections, of the connection form defined on the total space P of the bundle. Given a representation P of the structure (gauge) group G in a vector space V, one defines a (generalized) Higgs field α as a map from P to V, equivariant under the action of G in P. If the image of α is an orbit $W \subset V$ of G, then α breaks (spontaneously) the symmetry: the isotropy (little) group of $w_0 \in W$ is the "unbroken" group H. The principal bundle P is then reduced to a subbundle Q with structure group H. Gravitation corresponds to a linear connection, i.e. to a connection on the bundle of frames. This bundle has more structure than an abstract principal bundle: it is soldered to the base. Soldering results in the occurrence of torsion. The metric tensor is a Higgs field breaking the symmetry from $GL(4, \mathbf{R})$ to the Lorentz group.

INTRODUCTION

It has been known for some time that principal bundles with connections provide adequate geometrical models for classical gauge theories such as electromagnetism and the Yang-Mills theory [1–4]. Gravitation also fits in this scheme, but the corresponding principal bundle has a richer structure [5] and the Einstein field equations follow from a Lagrangian which is linear rather than quadratic in curvature, as is the case for other gauge theories [6]. Fibre bundles constitute a mathematical framework convenient for the description of gauge fields in any case; they become necessary if one wishes to consider the geometry of topologically non-trivial fields such as those due to magnetic poles [7–9] and instantons [10–14]. A Higgs field has a natural definition as a section of an associated bundle. Wu and Yang have shown that a similar definition is applicable to wave-functions of particles moving in the field of a magnetic pole [15]. The notion of spontaneous symmetry breaking has also a clear interpretation in this framework [16].

This paper contains a short summary of the fundamental notions, underlying any gauge theory, in the language of fibre bundles. The reader may consult standard books [17, 18] and articles [5, 19] for the precise definitions and properties of bundles with connections. Magnetic poles and simple instantons are discussed at some length, as examples of physical situations requiring non-trivial bundles. Considerable attention is paid to gravity and the differences distinguishing it from other gauge theories.

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A smooth principal bundle includes, in its definition, the following list of differentiable manifolds and smooth maps:

a total (bundle) space P,
a Lie group G,
a base space M,
a projection $\pi : P \to M$,
a map $\delta : P \times G \to P$ defining the action of G

on P; if a $a, b \in G$ and $\varepsilon \in G$ is the unit element, then

$$\delta(a) \circ \delta(b) = \delta(ba) \quad \text{and} \quad \delta(\varepsilon) = \text{id}\,,$$

where $\delta(a)\, p = \delta(p, a)$, $p \in P$; moreover

$$\pi \circ \delta(a) = \pi\,.$$

A connection is given by a 1-form ω on P, with values in G', the Lie algebra of G. For any $v \in G'$, $\delta(\exp tv)$ is the one-parameter group of transformations of P generated by v; if $\Delta(v)$ is the vector field on P induced by $\delta(\exp tv)$, then

$$\omega(\Delta(v)) = v \quad \text{and} \quad \delta(a)^* \omega = \text{Ad}\,(a^{-1})\, \omega$$

where $\delta(a)^* \omega$ is the pull-back of ω by $\delta(a)$ and $\text{Ad}\,(a)$ is the automorphism of G' associated to $a \in G$ by the adjoint representation of G in its Lie algebra.

At any point $p \in P$ one defines the horizontal subspace by

$$\text{hor}_p\, P = \{u \in T_pP : \omega(u) = 0\}\,.$$

There is a direct sum decomposition

(1) $$T_pP = \text{hor}_p\, P \oplus \text{ver}_p\, P$$

where the vertical space $\text{ver}_p\, P$ coincides with the kernel of the map $T_p\pi$, tangent to π at p. Therefore, on the bundle with connection, a vector $u \in T_pP$ admits a decomposition,

$$u = \text{hor}\, u + \text{ver}\, u$$

corresponding to (1).

Let $\varrho : G \to GL(V)$ be a representation of G in a vector space V. A k-form of type ϱ on P is a k-form α on P, with values in V and equivariant under the action of G,

$$\delta(a)^* \alpha = \varrho(a^{-1})\, \alpha\,.$$

If α is such a form, then so is the form $\text{hor}\, \alpha$ defined by

$$\text{hor}\, \alpha(u_1, \ldots, u_k) = \alpha(\text{hor}\, u_1, \ldots, \text{hor}\, u_k)$$

where $u_1, \ldots, u_k \in T_pP$.

If α is a k-form of type ϱ, then its covariant exterior derivative

$$\text{D}\alpha = \text{hor}\, \text{d}\alpha$$

is a $(k + 1)$-form of the same type. For example,

$$\Omega = \text{D}\omega$$

is a 2-form of type Ad, called the curvature form. Explicitly,

(2) $$\Omega = \text{d}\omega + \tfrac{1}{2}[\omega, \omega]$$

where the commutator of two G'-valued forms is defined as follows: Let $\alpha = \alpha^i e_i$ and $\beta = \beta^i e_i$ be two such forms and let (e_i) be a frame (basis) in G', then

$$[\alpha, \beta] = \alpha^i \wedge \beta^j [e_i, e_j] = \alpha^i \wedge \beta^j c_{ij}^k e_k$$

where (c_{ij}^k) are the structure constants of G with respect to (e_i). There always holds the Bianchi identity

(3) $$\text{D}\Omega = 0\,.$$

If α is a k-form of type ϱ such that

$$\text{hor}\, \alpha = \alpha$$

then $\text{D}\alpha$ may be evaluated from the formula

(4) $$\text{D}\alpha^a = \text{d}\alpha^a + \varrho_{bi}^a \omega^i \wedge \alpha^b$$

where

$$\alpha = \alpha^a e_a\,, \quad \text{D}\alpha = (\text{D}\alpha^a)\, e_a\,, \quad \omega = \omega^i e_i\,,$$

(e_a) is a frame in V, and ϱ_{bi}^a is the ath component of the vector

$$\frac{\text{d}}{\text{d}t}\, \varrho(\exp te_i)\, e_b \Big|_{t=0}\,.$$

A ϱ-invariant metric on V is a bilinear symmetric map

$$h : V \times V \to \mathbf{R}$$

such that

$$h(\varrho(a)\, u, \varrho(a)\, v) = h(u, v) \quad \text{for any} \quad a \in G \quad \text{and} \quad u, v \in V\,.$$

If G is connected, then h is ϱ-invariant iff

$$h_{ac}\varrho_{bi}^c + h_{cb}\varrho_{ai}^c = 0\,,$$

where $h_{ab} = h(e_a, e_b)$. For example, the Killing metric on G' is Ad-invariant.

GAUGE AND HIGGS FIELDS

A gauge theory is defined by specifying (at least):

(i) a gauge group G,
(ii) the type of particles which are coupled to the gauge field, and
(iii) the precise form of the field equations.

In a pure gauge theory, other than gravitation, one neglects sources of the gauge field and assumes as the field equation

$$D * \Omega = 0,$$

where $* \Omega$ is the dual of Ω, computed with the help of a metric in the oriented base manifold M. This field equation may be obtained from a variational principle with a Lagrangian density (4-form) proportional to

$$k_{ij} * \Omega^i \wedge \Omega^j \tag{5}$$

where $\Omega = \Omega^i e_i$ and k is an Ad-invariant non-singular metric on G'.

A (generalized) Higgs field is a 0-form of type ϱ, i.e. a map $\alpha : P \to V$ such that

$$\alpha \circ \delta(a) = \varrho(a^{-1})\, \alpha .$$

The representation ϱ of the gauge group defines thus the type of particles under consideration. A standard Higgs field corresponds to $V = G'$ and $\varrho = \mathrm{Ad}$. Typically, the interaction between the gauge and the Higgs field is taken into account by supplementing (5) with a term of the form

$$h_{ab} * D\alpha^a \wedge D\alpha^b + U(|\alpha|)\, \eta \tag{6}$$

where $|\alpha|^2 = h_{ab}\alpha^a\alpha^b$, h is a ϱ-invariant metric and η is a horizontal volume element. One says that the coupling between α and ω is minimal since the interaction term (6) contains ω only through $D\alpha$.

In most cases, physicists work with potentials and fields expressed in a local gauge. This corresponds to taking a local section of the bundle,

$$f : N \to P\,, \quad \pi \circ f = \mathrm{id}\,, \quad N \subset M\,,$$

and defining the potential as the pull-back of ω by f,

$$A = f^* \omega .$$

If (x^μ) is a system of local coordinates in N, then

$$A = A^i_\mu \, dx^\mu \, e_i .$$

Similarly one has

$$F = f^*\Omega = \tfrac{1}{2} F^i_{\mu\nu} \, dx^\mu \wedge dx^\nu \, e_i$$

and equation (2) implies

$$F^i_{\mu\nu} = A^i_{\nu,\mu} - A^i_{\mu,\nu} + c^i_{jk} A^j_\mu A^k_\nu .$$

A change of the local section implies a change of the potential; the new potential is related to the old one by a well-known "gauge transformation of the second kind" (cf., for example, [20]). The pull-back of α by f is the physicists' Higgs field

$$\varphi = \alpha \circ f : N \to V .$$

According to (4) one has for the covariant derivative $f^* D\alpha = D\varphi = \nabla_\mu \varphi^a \, dx^\mu e_a$ the following explicit formula

$$\nabla_\mu \varphi^a = \varphi^a{}_{,\mu} + \varrho^a_{bi} A^i_\mu \varphi^b .$$

ELECTROMAGNETISM AND THE YANG-MILLS THEORY

The Maxwell and Yang-Mills theories are the two most important examples of gauge theories. Their structure groups consist of unit complex numbers and unit quaternions, respectively. This remark allows a "unified" treatment of these two theories. Let $K = \mathbf{C}$ (complex numbers) or $\mathbf{H}$ (quaternions). For $u \in K$, $\bar{u}$ denotes its conjugate. Consider the group

$$U(1, K) = \{u \in K : \bar{u}u = 1\} .$$

Then

$$U(1, \mathbf{C}) = U(1)$$

is the unitary group in one dimension, isomorphic to $SO(2)$, and

$$U(1, \mathbf{H}) = Sp(1)$$

is the (quaternionic) symplectic group, isomorphic to $SU(2)$. The latter isomorphism is obtained by mapping the quaternion units 1, i, j, k into the 2×2 matrices I, $(\sqrt{-1})\,\sigma_x$, $(\sqrt{-1})\,\sigma_y$, $(\sqrt{-1})\,\sigma_z$. The Lie algebra of $U(1, K)$ is isomorphic to the imaginary subspace of K,

$$\mathrm{Im}\, K = \{z \in K : \bar{z} + z = 0\} .$$

Therefore, in both Maxwell and Yang-Mills theories the connection and curvature forms are pure imaginary, $\bar{\omega} = -\omega$ and $\bar{\Omega} = -\Omega$.

There is a natural way of defining exterior and tensor products of K-valued forms. For example, if α is a K-valued 1-form, then

$$\bar{\alpha} \otimes \alpha = \tfrac{1}{2}(\bar{\alpha} \otimes \alpha + \alpha \otimes \bar{\alpha}) + \tfrac{1}{2}\bar{\alpha} \wedge \alpha ,$$

where the form in the bracket on the right is real and $\bar{\alpha} \wedge \alpha$ is pure imaginary. If

$$\omega = i\omega_1 + j\omega_2 + k\omega_3$$

is a 1-form with values in $\mathrm{Im}\, \mathbf{H}$, then

$$\omega \wedge \omega = 2i\omega_2 \wedge \omega_3 + 2j\omega_3 \wedge \omega_1 + 2k\omega_1 \wedge \omega_2 ,$$
$$\omega \wedge \omega \wedge \omega = -6\omega_1 \wedge \omega_2 \wedge \omega_3 ,$$
$$\omega \wedge \omega \wedge \omega \wedge \omega = 0 .$$

If $\bar{\Omega} = -\Omega$, then both $* \Omega \wedge \Omega$ and $\Omega \wedge \Omega$ are real-valued 4-forms. In this compact notation, the structure formula (2) reduces to

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega$$

and the Bianchi identity (3) becomes

$$\mathrm{d}\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0\,.$$

The space K may be considered as a real Euclidean space of 2 ($K = \mathbf{C}$) or 4 ($K = \mathbf{H}$) dimensions. In either case, its line-element is

$$\tfrac{1}{2}(\mathrm{d}\bar{z} \otimes \mathrm{d}z + \mathrm{d}z \otimes \mathrm{d}\bar{z})$$

and

$$*(\mathrm{d}\bar{z} \wedge \mathrm{d}z) = \begin{cases} 2\sqrt{-1} & \text{for} \quad K = \mathbf{C}\,, \\ -\mathrm{d}\bar{z} \wedge \mathrm{d}z & \text{for} \quad K = \mathbf{H}\,. \end{cases}$$

Clearly, in the quaternionic case, the 2-form $\mathrm{d}\bar{z} \wedge \mathrm{d}z$ corresponds to 't Hooft's η and $\mathrm{d}\bar{z} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}z/12$ is the volume element of $\mathbf{H}$.

MAGNETIC POLES AND SIMPLE INSTANTONS

Let P be the electromagnetic bundle over a four-dimensional space-time M. The structure group $U(1)$ being abelian, there is a globally defined electromagnetic field F on M, $\Omega = \pi^* F$. The field F is closed and, by the Weil lemma [21], its cohomology class $[F]$ is integral, i.e.

$$\frac{1}{2\pi\sqrt{-1}} \int_c F = n \in \mathbf{Z}$$

for any 2-cycle (closed surface) c. This condition, when applied to the field of a magnetic pole,

$$\mathbf{B} = g\mathbf{r}/r^3$$

with

$$F = (e\sqrt{-1}/\hbar c)\,(E_x c\,\mathrm{d}t \wedge \mathrm{d}x + \ldots - B_x\,\mathrm{d}y \wedge \mathrm{d}z - \ldots)$$

leads to the Dirac quantization rule for the strength g of the pole [22],

$$2eg/\hbar c = n\,.$$

Clearly, $[F] = 0$ if M has Euclidean topology. Therefore, if a magnetic pole is ever found, this will prove that either space-time has non-Euclidean topology, or the bundle picture of electromagnetism is incorrect, or both.

In order to construct the bundle and connection corresponding to a magnetic pole of lowest strength ($n = 1$), consider Minkowski spacetime $\mathbf{R}^4$ with the worldline of the pole removed. Since

$$\mathbf{R}^4 \setminus \{\text{line}\} \text{ is diffeomorphic to } \quad \mathbf{R}^2 \times S_2\,,$$

it is enough to consider $U(1)$-bundles over S_2, which are known. The simplest nontrivial among them, corresponding to the generator of $\pi_1(U(1)) = \mathbf{Z}$ [23] is given by the Hopf map $S_3 \to S_2$ [24, 25]. It turns out that the same computation will also yield the simplest instanton if one considers simultaneously the $SU(2)$-bundle $S_7 \to S_4$ [26]. Indeed, let $z_0, z_1 \in K$, then equation

$$\bar{z}_0 z_0 + \bar{z}_1 z_1 = 1$$

defines S_3 or S_7, depending on whether $K = \mathbf{C}$ or $\mathbf{H}$. The action of $u \in U(1,K)$ given by

$$(z_0, z_1)\,z = (z_0 u, z_1 u)$$

defines these two fibrations and

$$\omega = \tfrac{1}{2}(\bar{z}_0\,\mathrm{d}z_0 + \bar{z}_1\,\mathrm{d}z_1 - \mathrm{d}\bar{z}_0 z_0 - \mathrm{d}\bar{z}_1 z_1)$$

is a connection form on S_3 or S_7, corresponding to a magnetic pole ($K = \mathbf{C}$) or the BPST instanton ($K = \mathbf{H}$); cf. [10], [20] and [26] for details. The string singularities occurring in the components of the vector potential of a magnetic pole are due to the nontrivial character of the bundles for $n \neq 0$.

SPONTANEOUS SYMMETRY BREAKING

Consider a principal G-bundle P over M and a Higgs field whose range is an orbit W of G in V, i.e.

$$\alpha : P \to W \subset V$$

and W is such that for any pair of points, w_0, $w \in W$ there is an $a \in G$ so that $w = \varrho(a)\,w_0$. Let H be the isotropy group of w_0,

$$H = \{a \in G : \varrho(a)\,w_0 = w_0\}\,.$$

Then

$$Q = \{p \in P : \alpha(p) = w_0\}$$

is a subbundle of P over the same base M; its structure group is H[16, 18]. Conversely, given a reduction Q of the bundle P to a subgroup H of its structure group G, one defines a Higgs field $\alpha : P \to W = G/H$ by putting $\alpha(p) = H \in W$ for $p \in Q$.

A connection form ω on P, restricted to Q, defines an H-connection on Q iff it is H'-valued, i.e. iff

$$\mathrm{D}\alpha = 0\,. \tag{7}$$

As an example of spontaneous symmetry breaking, consider the 't Hooft-Polyakov solution [27, 28] of the Yang-Mills equations with a Higgs field of type Ad. For any fixed t and $r > 0$, the base may be identified with S_2. The $SO(3)$-bundle P over S_2 is trivial, $P = S_2 \times SO(3)$, and the (normalized) Higgs field $\alpha : P \to S_2 \subset \subset \mathbf{R}^3$ is $\alpha(\hat{r}, a) = a^{-1}\hat{r}$ for any $\hat{r} \in S_2$ and $a \in SO(3)$. The north pole $\hat{r}_0 = (0, 0, 1) \in \in S_2$ is unchanged by rotations around the z axis, thus $H = SO(2)$ and

$$Q = \{(\hat{r}, a) \in P : a^{-1}\hat{r} = \hat{r}_0\}$$

may be identified with $SO(3)$ by $Q \in (\hat{r}, a) \mapsto a \in SO(3)$. Clearly, $SO(3) \to S_2$ is non-trivial and carries an $SO(2)$-connection corresponding to a magnetic pole with $n = 2$ [26]. Therefore, by a spontaneous breaking of symmetry it is possible to obtain a non-trivial bundle Q even though P is trivial.

GRAVITATION

Gravitation is different from other gauge theories in several aspects. The origin of these differences may be traced to the soldering of the bundle of linear frames FM to the base manifold M[5]. For an n-dimensional manifold M, the soldering form $\theta : TFM \to \mathbf{R}^n$ is defined as follows: if $e = (e_\mu) \in FM$ and $u \in T_eFM$, then $\theta^\mu(u)$ is the μ^{th} component, with respect to e, of the vector $T_e\pi(u)$, obtained by projecting u on the base. Clearly, $\theta = (\theta^\mu)$, $\mu = 1, \ldots, n$, is a 1-form of type id,

$$\delta(a)^* \theta = a^{-1}\theta .$$

If $\omega = (\omega^\mu_\nu)$ is the form of a linear connection (= connection on FM), then the covariant exterior derivative of θ is the 2-form Θ of torsion,

$$\Theta^\mu = \mathrm{d}\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu . \tag{8}$$

A metric tensor g on M defines the map

$$(g_{\mu\nu}) : FM \to \mathscr{L}^2_s(\mathbf{R}^n, \mathbf{R})$$

given by

$$g_{\mu\nu}(e) = g(e_\mu, e_\nu) .$$

If M is four-dimensional, as it will be assumed from now on, one defines

$$\eta_{\mu\nu\varrho\sigma} : FM \to \mathbf{R}$$

by

$$\eta_{0123}(e) = |\det g_{\mu\nu}(e)|^{1/2} , \quad \eta_{\mu\nu\varrho\sigma} = \eta_{[\mu\nu\varrho\sigma]}$$

and also

$$\eta_{\mu\nu\varrho} = \theta^\sigma \eta_{\mu\nu\varrho\sigma} , \quad \eta_{\mu\nu} = \tfrac{1}{2}\theta^\varrho \wedge \eta_{\mu\nu\varrho} , \quad \eta_\mu = \tfrac{1}{3}\theta^\nu \wedge \eta_{\mu\nu} , \quad \eta = \tfrac{1}{4}\theta^\mu \wedge \eta_\mu .$$

In order to appreciate the differences between gravitation, understood as a theory based on a linear connection and a metric tensor, and a gauge theory on a principal bundle without soldering, consider the various invariant forms which may be constructed in each of the following four cases:

(i) A gauge theory on a bundle $P \to M$ with structure group $G \subset GL(4, \mathbf{R})$ based on a connection form $\omega = (\omega^\mu_\nu)$; no metric and no soldering. From the curvature form

$$\Omega^\mu_\nu = \mathrm{d}\omega^\mu_\nu + \omega^\mu_\varrho \wedge \omega^\varrho_\nu \tag{9}$$

one constructs two closed forms

$$\mathrm{Tr}\, \Omega = \Omega^\mu_\mu , \quad \mathrm{Tr}\, \Omega^2 = \Omega^\mu_\nu \wedge \Omega^\nu_\mu .$$

Upon integration on cycles, they both lead to quantities invariant under smooth deformations.

(ii) If one adds a metric g on M, then one can construct the dual $* \Omega^\mu_\nu$ and the conformally invariant Lagrangian density

$$* \Omega^\mu_\nu \wedge \Omega^\nu_\mu . \tag{10}$$

(iii) If one is given a connection form ω on FM, then in addition to curvature (9), one constructs torsion (8). There are no more invariant forms than in case (i).

(iv) Given ω on FM and g on M, in addition to the forms and invariants occurring in (i)–(iii) one can construct the η's and also

(11) the Einstein-Cartan Lagrangian $\eta_{\mu\nu} \wedge \Omega^{\mu\nu}$,

(12) the Euler form $\eta_{\mu\nu\varrho\sigma}\Omega^{\mu\nu} \wedge \Omega^{\varrho\sigma}$,

(13) the square of torsion, $g_{\mu\nu} * \Theta^\mu \wedge \Theta^\nu$,

(14) $g^{\mu\varrho}g^{\nu\sigma} * \mathrm{D}g_{\mu\nu} \wedge \mathrm{D}g_{\varrho\sigma}$, etc.

It is clear that gravitation, understood as a theory based on g and ω defined over FM (last case) is richer than other gauge theories. The metric tensor may be looked upon as a Higgs field which breaks the symmetry form $GL(4, \mathbf{R})$ down to the Lorentz group. As noted by NAMBU [19], equation (7) reduces in this case to the usual compatibility condition between a linear connection and a metric, assumed in both the Einstein and the Einstein-Cartan theories [30]. Moreover, the soldering form is also a kind of a Higgs field; it differs from other Higgs fields by being a 1-form (rather than a 0-form) and by being uniquely determined by M alone. If one takes these observations seriously, one may be led to consider a theory of gravitation based on a Lagrangian which is a sum of terms proportional to (10), (13) and (14). VON der HEYDE [31] proposed a theory based on a sum of (10) and (13), whereas SKINNER [32] considered (13) as a correction to (11).

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SOME REMARKS ON REFLECTION POSITIVITY*)

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The requirement of reflection positivity is investigated and its general applicability to different physical theories is pointed out. Its role is illustrated on an example from electrostatics and on several simple examples of field theories. Then, after presenting an abstract construction of the concept, the role of reflection positivity in classical lattice systems is discussed.

Reflection positivity has appeared in Euclidean quantum field theory and in lattice theory. It has been used in constructing the Hamiltonian in quantum field theory, the transfer operators in spin lattice systems, and in formalizing part of the Peierls argument in existence proofs for phase transitions.

What will be said below to this topic is not only incomplete but also highly subjective. Therefore I have to point out some very important aspects which are not considered here, and which have been handled much better than I can do in textbooks [1] and review articles [2] already: As a constructive tool RP is one of the Osterwalder and Schrader axioms [3]. Here RP reflects the positivity condition of the Wightman axioms as analytically continued to the Schwinger points. How is it possible to continue a positivity condition? For the complicated case of QFT this is well described in [1, 3], in GLASER [4], and other papers. Here, to see the flavour of the argument, let us notice one version of a theorem due to Fitz-Gerald: Assume $f(z, w)$ to be analytic in $|z| < 1$, $|w| < 1$, and choose $0 < \varepsilon < 1$. If then for every natural m and every choice of real $s_1, \ldots, s_m$, with $0 \leqq s_k \leqq \varepsilon$ the matrix $a_{ij} = f(s_i, s_j)$ turns out to be positive definite it follows the positive definiteness of the matrix $b_{jk} = f(z_j, \bar{z}_k)$, no matter how the complex numbers $z_1, z_2, \ldots$ have been chosen out of the interior of the unit circle [5]. Considering f along the imaginary axis and redefining $g(s_j, s_k) = f(is_j, is_k)$ one arrives at the positive definiteness of the matrix $g(s_j, -s_k)$ for real numbers s_j bounded by one. Though this example is widely oversimplified it shows how to obtain RP by analytic continuation.

Quite another way RP is entering in Nelson's approach, [7, 6], to Euclidean QFT. The centre of this theory is the famous generalizing the concept of Markov random processes to that of Markov random fields [1, 2]. RP shows up as a consequence of the Markov property. There are some indications, [8, 9], that with the aid of reflection positivity a "good" class of stochastic processes can be defined in which the Markovian behaviour is "weakened". The derivation of RP from Markov properties was presented by MACK [10] last year at Primorsko. His lecture further includes an application to lattice gauge theory found also in [11].

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