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On Gauge Transformations and Symmetries

by

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Summary. Gauge transformations are defined as automorphisms of a principal bundle preserving the absolute elements of a gauge theory. Pure gauge transformations correspond to vertical automorphisms. A symmetry of a gauge configuration is a gauge transformation preserving the corresponding connection form. It is shown that Coleman's generic non-Abelian plane-fronted waves have less symmetry than plane electromagnetic waves. In the theory of gravitation, because of soldering, the group of pure gauge transformations is trivial, and the full gauge group is isomorphic to Diff M, the group of all diffeomorphisms of the spacetime manifold M. An explicit formula is given for the natural lift of a vector field on M to the bundle LM of linear frames of M.

1. Introduction. Principal fibre bundles with connections provide a mathematical framework for classsical gauge theories such as electromagnetism and the Yang–Mills theory. There is also a "gauge ingredient" in the relativistic theory of gravitation. The bundle of frames is "soldered" to its base manifold; the soldering makes gravitation richer and rather different from other gauge theories [1, 2]. This note contains definitions of groups of gauge transformations and of symmetries of a gauge configuration given by a connection form in a principal bundle. According to these definitions, a generic, non-Abelian 'plane' wave in the Minkowski spacetime is less symmetric than a plane electromagnetic wave. The lack of full plane symmetry easily follows from my infinitesimal definition whereas Coleman attributes it to global properties [3]. In the last section, I summarize some of the special features of the gauge approach to gravitation which are due to soldering.

To a large extent, this paper follows the standard notation and terminology used in differential geometry [4] and applications of fibre bundle theory to physics [5–7]. All manifolds and maps are of class C^∞. A principal fibre bundle includes in its definition a projection π of the total space of the bundle P on the base M and an action of a Lie group G on P to the right. The action is free and transitive on the fibres of π. If $\delta: P\times G\to P$ is the map defining this action, then $\pi\circ\delta(a)=\pi$, $\delta(a)\circ\circ\delta(b)=\delta(ba)$, $\delta(e)=\mathrm{id}$, where $a, b\in G$, e is the unit of G and $\delta(p, a)=\delta(a)\,p=pa$

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for any $p \in P$. A connection on P is given by a one-form ω, defined on P and with values in G', the Lie algebra of G, such that

$$(\delta^* \omega)(p, a) = \mathrm{Ad}\,(a^{-1}) \circ \omega(p) + \widetilde{\omega}(a), \tag{1}$$

where

$$\delta^* \omega = \omega \circ T\delta$$

is the pullback of ω by δ, and $\widetilde{\omega}$ is the canonical form on G. Assuming that G is a group of matrices, one can write

$$a = (a^i_j), \quad \omega = (\omega^i_j), \quad \widetilde{\omega} = (\widetilde{\omega}^i_j),$$

and

$$\widetilde{\omega}^i_j(a) = (a^{-1})^i_k \, da^k_j .$$

Suppressing the indices, one may now write condition (1) as

$$(\delta^* \omega)(p, a) = a^{-1}\, \omega(p)\, a + a^{-1}\, da. \tag{2}$$

The curvature two-form is $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, where the bracket implies both the exterior and the Lie algebra product.

2. Automorphisms and gauge transformations. A diffeomorphism $u: P \to P$ is an automorphism of the principal bundle P, if there is a diffeomorphism $v = j(u): M \to M$ such that $\pi \circ u = v \circ \pi$ and $u(pa) = u(p)\, a$ for any $p \in P$ and $a \in G$. The set Aut P of all automorphisms forms a group under composition. An automorphism u is called vertical if $j(u) = \mathrm{id}$; the set $\mathrm{Aut}_M P$ of all vertical automorphisms of P is a normal subgroup of Aut P, and there is the exact sequence of homomorphisms of groups

$$0 \to \mathrm{Aut}_M P \xrightarrow{i} \mathrm{Aut}\, P \xrightarrow{j} \mathrm{Diff}\, M, \tag{3}$$

where i is the canonical injection. If $u \in \mathrm{Aut}_M P$, then both p and $u(p)$ lie in the same fibre. There exists, therefore, an element $U(p)$ of G such that $u(p) = pU(p)$ and

$$U(pa) = a^{-1}\, U(p)\, a \tag{4}$$

for any $p \in P$ and $a \in G$. Conversely, given a map $U: P \to G$ satisfying condition (4), there is a vertical automorphism $u = \delta \circ (\mathrm{id}, U)$. Putting $U = k(u)$, one has

$$k(u_1 \circ u_2) = k(u_1) \cdot k(u_2) \quad \text{for} \quad u_1, u_2 \in \mathrm{Aut}_M P,$$

where the product on the right is induced from G. The map k is a natural isomorphism of $\mathrm{Aut}_M P$ on the multiplicative group of maps, from P to G, subject to condition (4).

The group Aut P acts on (local) sections of π. If $s: N \to P$ is a section, $N \subset M$, then so is $s' = u \circ s \circ v^{-1}: v(N) \to P$, $v = j(u)$. If u is vertical, then s' is a section over N, and

$$s'(x) = s(x)\, U(s(x)) \quad \text{for} \quad x \in N.$$

In a physical theory of a gauge field, besides the dynamical variables, such as ω and the Higgs field, there may occur absolute elements, such as the metric tensor in special relativity [8]. By definition, the *gauge group* of such a theory is the subgroup $\mathscr{G}$ of Aut P, consisting of all automorphisms of P which preserve the absolute elements. The elements of $\mathscr{G}$ are called *gauge transformations*. A pure gauge transformation is a vertical element of $\mathscr{G}$. The *pure gauge group*

$$\mathscr{G}_0 = \mathscr{G} \cap \mathrm{Aut}_M P$$

is a normal subgroup of $\mathscr{G}$, and (3) induces the exact sequence

$$0 \to \mathscr{G}_0 \xrightarrow{i} \mathscr{G} \xrightarrow{j} \mathscr{G}/\mathscr{G}_0 \to 0, \tag{5}$$

where $\mathscr{G}/\mathscr{G}_0$ is identified with the group $j(\mathscr{G}) \subset \mathrm{Diff}\, M$.

For example, if M is the Minkowski space, then P is isomorphic to $M \times G$ and $\mathscr{G}$ is a semi-direct product of $\mathscr{G}_0$ by the Poincaré group $\mathscr{G}/\mathscr{G}_0$. For the Hopf bundle $S_3 \to S_2$, the gauge sequence (5) is $U(1) \xrightarrow{i} U(2) \to SO(3)$, where i is the diagonal embedding.

If ω is a connection form on P and $u \in \mathrm{Aut}\, P$, then the pullback $\omega' = u^* \omega$ is also a connection form on P. If $u \in \mathrm{Aut}_M P$, then

$$u^* \omega = (\mathrm{id} \times U)^* \circ \delta^* \omega,$$

and by virtue of (2)

$$\omega' = U^{-1} \omega U + U^{-1} dU. \tag{6}$$

The curvature form corresponding to $u^* \omega$ is

$$\Omega' = U^{-1} \Omega U. \tag{7}$$

In theoretical physics, one usually works with a "gauge potential" A which may be identified with a pullback of ω by a local section s,

$$A = s^* \omega.$$

Similarly, the "field strengths" are

$$F = s^* \Omega.$$

The gauge transformed potential

$$A' = s'^* \omega = s^* \omega'$$

may be interpreted as obtained either by a pullback of ω by the transformed section $s' = u \circ s$ or by a pullback of $\omega' = u^* \omega$ by the original section. Putting $S = U \circ s$, one obtains from (6) and (7) the physicist's formulae

$$A' = S^{-1} A S + S^{-1} dS, \qquad F' = S^{-1} F S.$$

3. Symmetries. Generally speaking, a symmetry of a physical configuration ω is a transformation preserving both ω and the absolute elements of the theory. If ω is a connection form on P, then a *symmetry* of ω is a gauge transformation u such that

$$u^* \omega = \omega. \tag{8}$$

A symmetry of ω preserves also the curvature form, i.e. condition (8) implies $u^* \Omega = \Omega$, but the converse is not true. A one-parameter group (u_t), $t \in \mathbf{R}$, of automorphisms of P induces a vector field Z on P which is projectable to M; its projection ξ is a vector field on M generating the one-parameter group (v_t), $v_t=j(u_t)$. The connection form is preserved by (u_t), if and only if its Lie derivative with respect to Z vanishes,

$$\mathscr{L}_Z\,\omega=0. \tag{9}$$

The corresponding condition for the curvature is

$$\mathscr{L}_Z\,\Omega=0. \tag{10}$$

Let $Y(p)$ be the value of $\omega(p)$ on $Z(p)$; the map

$$Y=\langle Z, \omega\rangle : P\to G'$$

is equivariant under the action of G,

$$Y\circ\delta(a)=\mathrm{Ad}(a^{-1})\circ Y,$$

and its covariant derivative,

$$DY=dY+[\omega, Y], \tag{11}$$

is a horizontal one-form of type Ad. By virtue of $\mathscr{L}_Z\,\omega=d(Z\lrcorner\,\omega)+Z\lrcorner\,d\omega$ and Eq. (11), one obtains

$$\mathscr{L}_Z\,\omega=Z\lrcorner\,\Omega+DY, \tag{12}$$

and, similarly,

$$\mathscr{L}_Z\,\Omega=D(Z\lrcorner\,\Omega)+[\Omega, Y].$$

A (local) section s of π may now be used to translate Eqs. (9) and (10) for infinitesimal symmetries into conditions containing the gauge potential A, field strengths F, the generator ξ of (v_t), and $\eta=Y\circ s$. Namely, (9) and (10) become

$$\xi\lrcorner\,F+D\eta=0, \tag{13}$$

and

$$D(\xi\lrcorner\,F)+[F, \eta]=0, \tag{14}$$

respectively. Here D denotes the "gauge derivative", e.g. $D\eta=d\eta+[A, \eta]$; explicitly

$$D\eta^i_j=d\eta^i_j+A^i_k\,\eta^k_j-\eta^i_k\,A^k_j,$$

or

$$D\eta^\alpha=d\eta^\alpha+c^\alpha_{\beta\gamma}\,A^\beta\,\eta^\gamma,$$

depending on the notation used for the components of elements of G'. In the latter notation, $A=A^\alpha E_\alpha$, $F=F^\alpha E_\alpha$, (E_α) is a basis in G',

$$[E_\alpha, E_\beta]=c^\gamma_{\alpha\beta}\,E_\gamma,$$

and (13) reduces to

$$F^\alpha_{\mu\nu}\,\xi^\mu+\nabla_\nu\,\eta^\alpha=0, \tag{51}$$

where

$$D\eta^\alpha = \nabla_\upsilon \eta^\alpha \, dx^\upsilon,$$

$$F^\alpha = \frac{1}{2} F^\alpha_{\mu\nu} \, dx^\mu \wedge dx^\nu \quad \text{and} \quad \xi = \xi^\mu \, \partial/\partial x^\mu.$$

4. A simple example. As an example, consider the following *plane-fronted wave* in the Minkowski space M [3]

(16) $$A = (ax + by) \, d\tau,$$

where a and b are arbitrary G'-valued functions of $\tau = t - z$. The field strengths

$$F = (a dx + b dy) \wedge d\tau$$

satisfy the sourcefree equation $D*F = 0$ because

$$*F = (a dy - b dx) \wedge d\tau \quad \text{and} \quad [A, *F] = 0.$$

Since here $P = M \times G$, ξ must be the generator of a Poincaré transformation. If $\xi = \partial/\partial x$, then $\xi \lrcorner F = a \, d\tau$, and $D(\xi \lrcorner F) = 0$. Eq. (14) implies $[F, \eta] = 0$, or

(17) $$[a, \eta] = 0 \quad \text{and} \quad [b, \eta] = 0,$$

and Eq. (13) reduces to

(18) $$d\eta + a \, d\tau = 0.$$

The last three equations are necessary and sufficient for the pair $(\xi = \partial/\partial x, \eta)$ to generate a symmetry of (16). If $a = 0$, then $\eta = 0$ is a solution: a wave linearly polarized along the y axis is invariant under translations in the x direction. However, a generic non-Abelian plane-fronted wave is not invariant under translations in the plane perpendicular to the direction of propagation. E. g., if $G = SO(3)$ and $[a, b] \neq 0$, then Eqs. (17) imply $\eta = 0$, in contradiction to (18).

5. Gravitation. In the theory of gravitation, ω is a *linear connection*, i.e. a connection on the bundle LM of linear frames of M. Since L is a functor for diffeomorphisms, there is a *lift*

$$L: \text{Diff}\, M \to \text{Aut}\, LM$$

such that $j \circ L = \text{id}$ and

$$u \in L(\text{Diff}\, M) \Leftrightarrow u^* \vartheta = \vartheta,$$

where $\vartheta = (\vartheta^\mu)$ is the canonical (soldering) form on LM. The group $\text{Aut}\, LM$ is an inessential extension of $\text{Diff}\, M$ by $\text{Aut}_M LM$. The soldering should be considered as an "absolute element" of the theory of gravitation. Therefore, in this case, $\mathcal{G}_0 = \{\text{id}\}$, and the gauge group $\mathcal{G}$ is isomorphic to $\text{Diff}\, M$.

The nature of the gauge group in the theory of gravitation is rather different from that in a Yang–Mills theory over the Minkowski space. In the latter case, $\mathcal{G}_0$ is "large" and $\mathcal{G}/\mathcal{G}_0$ "small" (finite-dimensional), in the former, $\mathcal{G}_0$ is trivial and $\mathcal{G} = \mathcal{G}/\mathcal{G}_0$ large.

As a result of soldering, and unlike in the Yang–Mills theory, the generator Z of a one-parameter group (u_t) of gauge transformations of LM is uniquely determined by the generator ξ of (v_t), $v_t=j(u_t)$. Indeed, let $p=(p_\mu)\in LM$ be a frame at a point $x=\pi(p)$ of an n-dimensional manifold M. Define

$$X=(X^\mu)\colon LM\to\mathbf{R}^n$$

by

$$X^\mu(p)=\langle\xi(x),p^\mu\rangle=\langle Z(p),\theta^\mu(p)\rangle.$$

Let $(P_\mu, M_v{}^\rho)$ be the basis of the module of vector fields on LM, dual to the basis of differential forms $(\theta^\mu, \omega^\nu_\rho)$, where (ω^ν_ρ) is a linear connection

$$\langle P_v,\theta^\mu\rangle=\delta^\mu_\nu,\qquad \langle M^p_\nu,\theta^\mu\rangle=0.$$

$$\langle P_\rho,\omega^\sigma_\nu\rangle=0,\qquad \langle M^\sigma_\rho,\omega^\mu_\nu\rangle=\delta^\mu_\rho\,\delta^\sigma_\nu\,.$$

The vector field Z may now be written as

$$Z=X^\mu P_\mu+Y^\mu_\nu M^\nu_\mu,\tag{19}$$

where

$$Y^\mu_\nu=\langle Z,\omega^\mu_\nu\rangle.$$

Since θ is invariant by (u_t), the Lie derivative of θ with respect to Z should vanish. This gives

$$Z\lrcorner\,\Theta^\mu+DX^\mu=Y^\mu_\nu\,\theta^\nu,$$

where

$$\Theta^\mu=d\theta^\mu+\omega^\pi_\nu\wedge\theta^\nu=\frac{1}{2}\,Q^\mu_{\nu\rho}\,\theta^\nu\wedge\theta^\rho$$

is the torsion two-form. Writing $DX^\mu=\theta^\nu\,\nabla_\nu X^\mu$, one obtains

$$Y^\mu_\nu=\nabla_\nu X^\mu+Q^\mu_{\rho\nu}\,X^\rho,$$

or

$$Y^\mu_\nu=\overline{\nabla}_\nu X^\mu,\tag{20}$$

where $\overline{\nabla}$ is the covariant derivative with respect to the transposed (associated) connection $\overline{\omega}$ [9],

$$\overline{\omega}^\mu_\nu=\omega^\mu_\nu+Q^\mu_{\nu\rho}\,\theta^\rho.$$

Formula (19), together with (20), is useful in computing the Lie derivatives and studying symmetry properties of geometric objects: For example, Eq. (12) gives directly the explicit expression for the Lie derivative of a linear connection,

$$R^\mu_{\nu\sigma\rho}\,X^\sigma+\nabla_\rho\,\overline{\nabla}_\nu\,X^\mu,$$

where $R^\mu_{\nu\rho\sigma}$ is the curvature tensor.

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А. Траутман, **О преобразованиях характеризации и симметриях**

Содержание. Преобразования характеризации определены в настоящей работе как автоморфизмы главного пучка, сохраняющие абсолютные элементы теории с характеризацией. Чистые преобразования характеризации соответствуют вертикальным автоморфизмам. Симметрией называется преобразование характеризации, сохраняющее форму конексии, соответствующую данной конфигурации поля характеризации. Показано, что найденные Колеманом общие, не абелевы плоские волны являются менее симметричными, нежели плоские электромагнитные волны. В теории гравитации, из-за появления канонической формы (форма сварки), группа чистых преобразований характеризации является тривиальной, а полная группа чистых преобразований характеризации является изоморфной группе Диффа M всех дифеиморфизмов пространства-времени M. Дано выражение в яеном виде для поднятия векторного поля из M к пучку M линейных реперов пространства M.