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LEOPOLD INFELD

His Life and Scientific Work

by E. Infeld, I. Białynicki-Birula, A. Trautman

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Motion and Radiation According to the Theory of General Relativity[1]

Andrzej Trautman

The problem of gravitational motion and radiation was a major topic in Leopold Infeld's scientific activity. He started working on it in 1936, in collaboration with Albert Einstein, and continued this line of research until his very last days. As a result, he published over 25 papers and a monograph **[91]**[2] on the problem of motion in the theory of general relativity. Einstein, Infeld and Hoffmann **[34]** have laid down the foundations of a "new approximation method", nowadays known as the EIH method. The method is admirably adapted to solving all problems related to the motion of slowly moving, gravitating bodies. The post-Newtonian equations of motion for the two-body problem, obtained by Einstein, Infeld and Hoffmann, were integrated in an accompanying paper by H. P. Robertson [1]. The EIH method has been improved and generalized **[37]**, and later modified **[51]** by the introduction of fictitious pole and dipole sources. Their role was to satisfy automatically the integrability conditions of the field equations. The equations of motion were obtained by setting, the additional sources equal to zero, at the end of the computation.

The connection between the gravitational field equations and the equations of motion had been recognized before the publication of the series of fundamental papers by Einstein and Infeld. Essentially, the connection is as follows: for a continuous medium with a simple structure (e.g., a perfect fluid) the equations of motion are equivalent to the (differential) conservation law of energy and momentum. Because of general invariance, the left side of the field equations,

$$G^{\alpha\beta} = -8\pi T^{\alpha\beta},$$

satisfies the (contracted) Bianchi identity,

$$G^{\alpha\beta}_{;\beta} = 0.$$

This implies $T^{\alpha\beta}_{;\beta} = 0$ and the equations of motion. For example, if matter consists

[1] Based in part on the author's Lectures on General Relativity, King's College, London, 1958 (unpublished) and Gravitational waves and radiation, a review presented at the International Conference on Relativistic Theories of Gravitation, London 1965 (unpublished).

[2] The boldface numbers correspond to Leopold Infeld's bibliography, pp. 165–179 of this volume. The lighface numbers in square brackets refer to the literature listed at the end of this essay.

of dust, $T^{\alpha\beta} = \varrho u^\alpha u^\beta$, then condition $T^{\alpha\beta}_{;\beta} = 0$ reduces to the geodetic equation

$$u^\alpha_{;\beta} u^\beta = 0$$

supplemented by the law of conservation of mass,

$$(\varrho u^\alpha)_{;\alpha} = 0.$$

This result may be generalized [2]: consider a classical field, other than $g_{\alpha\beta}$, interacting with pole particles and assume that the field equations are derivable from a Lorentz-invariant principle of least action. From arguments based on invariance alone, one obtains the identity

$$T^{\alpha\beta}_{;\beta} + M^\alpha + N^\alpha = 0,$$

where $T^{\alpha\beta}$ is the total energy-momentum tensor. M^α and N^α are linear homogeneous functions of the left sides of the equations of motion and of the field equations, respectively. Moreover, $M^\alpha = 0$ implies the equations of motion. In the theory of special relativity, one infers the conservation laws from $M^\alpha = 0 = N^\alpha$. In the general theory, where $T^{\alpha\beta}$ acts as a source of the gravitational field, $T^{\alpha\beta}_{;\beta} = 0$ must hold and, if $N^\alpha = 0$, then also $M^\alpha = 0$. It is not necessary to postulate separately a dynamical principle for the motion of particles in the theory of general relativity. A different derivation of the geodetic equation for test particles was given by Infeld and Schild **[52]**.

Einstein regarded the energy-momentum tensor as a temporary means for the description of matter and sought for a representation of nature in terms of purely geometrical fields. One of the provisional solutions was to treat particles as singularities in empty space-time. The paper by Einstein, Infeld and Hoffmann was motivated by a desire to show that the motion of *singularities* is also determined by the field equations and to work out an approximation method suited to the calculations of relativistic corrections to the Newtonian motion of celestial bodies. The equations of motion were obtained from the vanishing of some surface integrals surrounding singularities which expressed the integrability conditions for the approximate field equations.

The problem of motion was attacked also by V. A. Fock [3–5], and his students [6–9]. They used the same approximation method as Einstein and Infeld did, but the bodies were represented not by singularities but by a continuous energy-momentum tensor. Fock fixed the space-time coordinate system by the de Donder condition and obtained the equations of motion of the centre of inertia of a body by integrating the equations $gT^{\alpha\beta}_{;\beta} = 0$ over the three-dimensional region occupied by it. He obtained also some equations for the internal motion of rotating bodies. A similar approach was used by A. Papapetrou [10], who also gave a new derivation [11] of the Mathisson equation of motion of spinning particles [12]. S. Chandrasekhar and his school made substantial progress in general relativistic hydrodynamics [13, 14]. They simplified the formalism so as to make possible an actual derivation of the second post-Newtonian equations [15] and clarified the reaction of a fluid to the emission of gravitational radiation [16]. Chandrasekhar made a thorough study of the post-Newtonian effecs of general relativity on the equilibrium of uniformly rotating bodies [17], and on the stability of axisymmetric systems to axisymmetric perturbations [18].

A novel approach to the problem of motion was initiated by Infeld, who introduced an energy-momentum tensor involving Dirac δ-functions for the description of pole particles **[69]**. This produced a great simplification in the derivation of the post-Newtonian equations of motion which were obtained directly from $T^{\alpha\beta}_{;\beta} = 0$.

Einstein, Infeld and Hoffmann had assumed certain forms of series expansion of the metric tensor which, by analogy with electrodynamics, they interpreted as corresponding to the choice of the symmetric (half-advanced, half-retarded) Green's function. Infeld **[35]** wrote down the first terms in $g_{\alpha\beta}$ corresponding to the choice of a retarded Green's function and showed that they did not give any contribution to the equations of motion up to the 7th order (the Newtonian equations are of the 4th order and the post-Newtonian ones—found by EIH—of the 6th order). N. Hu [19] worked out the radiation terms in the next step and found "anti-damping"—the energy of a system of two bodies appeared to increase when the radiation was taken into account. The first radiation terms are functions of the time alone and several papers dealt with the problem whether they represent a true gravitational field or could be annihilated by a coordinate transformation **[58, 89]**. An answer to this question was given by J. N. Goldberg [20] and is summarized in a later section.

The theory of gravitational waves and radiation was the subject of much research done between 1955 and 1970; good reviews of that research may be found in [21] and [22]. Within the framework of the EIH method, a satisfactory account of the influence of radiation on the motion of bodies was given by Infeld and Michalska-Trautman **[101–104]** and by Chandrasekhar and Esposito [16].

In the subsequent parts of this essay, we present in elementary form some of the main issues raised in connection with the problem of motion treated by the EIH method.

The new approximation method

Consider the scalar wave equation,

(1) $$\Box\varphi \equiv \Delta\varphi - \varphi_{,00} = 0,$$

and introduce "time" instead of the coordinate $x^0 = ct$. If a solution $\varphi(x^0, x^k, c) = \varphi(ct, x^k, c)$ of the wave equation can be expanded into a power series in $1/c$,

(2) $$\varphi = \sum_{n=0}^{\infty} c^{-n} \underset{n}{\varphi}(t, x^k),$$

then the functions $\underset{n}{\varphi}$ satisfy

(3) $$\Delta\underset{0}{\varphi} = 0, \quad \Delta\underset{1}{\varphi} = 0, \quad \Delta\underset{2}{\varphi} = \ddot{\underset{0}{\varphi}}, \ldots, \quad \Delta\underset{k}{\varphi} = \ddot{\underset{k-2}{\varphi}}, \ldots$$

(the dots over the φ's stand for derivatives with respect to t). The structure of the system (3) is such that we can, if we wish, find solutions (2) containing only even or only odd terms. If we put $\underset{0}{\varphi} = 0$, $\underset{2n-1}{\varphi} = 0$ $(n = 1, 2, \ldots)$, start with the pole solution in the second order, $\underset{2}{\varphi} = a(t)/r$, and take the simple solutions $\underset{4}{\varphi} = \frac{1}{2}\ddot{a}r$, $\underset{6}{\varphi} = (4!)^{-1}r^3 d^4a/dt^4, \ldots$, then we obtain the standing wave solution:

$$2\varphi = a(t-r/c)+a(t+r/c).$$

A retarded solution can be obtained by introducing a first radiation term in the 3rd order:

$$\underset{0}{\varphi} = 0, \quad \underset{1}{\varphi} = 0, \quad \underset{2}{\varphi} = a/r, \quad \underset{3}{\varphi} = -\dot{a}, \quad \underset{4}{\varphi} = \tfrac{1}{2}\ddot{a}r,$$

$$\underset{5}{\varphi} = (3!)^{-1}\dddot{a}r^2, \ldots, \quad \varphi = a(t-r/c)/r.$$

It is important to note that $\underset{n}{\varphi} = 0(r^{n-3})$ for $r \to \infty$ and this is also a general property of solutions of the inhomogeneous wave equation with a spatially bounded source. If λ is the characteristic wavelength of the field, then one can safely stop after the few first terms of the series (2) only in the region where

(4) $$r \ll \lambda.$$

In other words, the new approximation method is not well suited for the description of a field in the wave zone. If we write Maxwell's equations in the form

(5) $$\Box A^{\alpha} = -4\pi j^{\alpha}, \quad A^{\alpha}_{,\alpha} = 0, \quad j^{\alpha}_{,\alpha} = 0,$$

and assume that j^0 is of order 2 and j^k of order 3, then the *retarded* solution of (5) can be expanded into a power series as follows (in future we shall put $c = 1$):

$$A^0(\mathbf{r}, t) = \int \underset{2}{j^0}(\mathbf{r}', t)R^{-1}dV' - \int \underset{2,0}{j^0}dV' + \frac{1}{2}\int \underset{2,00}{j^0} R dV' + \ldots$$

$$A^k(\mathbf{r}, t) = \int \underset{3}{j^k}(\mathbf{r}', t)R^{-1}dV' - \int \underset{3,0}{j^k} dV' + \frac{1}{2}\int \underset{3,00}{j^k} R dV' + \ldots$$

The conservation of charge implies $\underset{3}{A^0} = 0$ and the first radiation term appears in the 4th order $(\underset{4}{A^k})$. For large values of r and for $n \geqslant 3$ we have $\underset{n}{A^{\alpha}} = 0(r^{n-4})$.

In the linearized theory of gravitation the situation is similar but the radiation terms are shifted still further along the series. If we write $\mathfrak{g}^{\mu\lambda} = \sqrt{-g}\, g^{\mu\lambda} = \eta^{\mu\lambda} - \gamma^{\mu\lambda}$ and assume de Donder's conditions $\gamma^{\mu\lambda}_{,\lambda} = 0$, then the linearized Einstein equations become

(6) $$\Box\gamma^{\mu} = 16\pi T^{\mu\lambda}, \quad T^{\mu\lambda}_{,\lambda} = 0.$$

T^{00} can be assumed to be of order 2, T^{0k} of order 3 and T^{kl} of order 4. This corresponds to the EIH assumption that the mass is of 2nd order. Expanding into a power series the retarded solution

(7) $$\gamma^{\mu\lambda}(\mathbf{r}, t) = -4\int dV' T^{\mu\lambda}(\mathbf{r}', t-R)/R$$

of equation (6), we find that

$$(8) \qquad \underset{2,0}{T^{00}} + \underset{3,k}{T^{0k}} = 0 \quad \text{implies} \quad \underset{3}{\gamma^{00}} = 0, \text{ and}$$

$$\underset{3,0}{T^{k0}} + \underset{4,l}{T^{kl}} = 0 \quad \text{implies} \quad \underset{4}{\gamma^{0k}} = 0.$$

Thus $\underset{5}{\gamma^{ik}}$ is the first non-vanishing radiation term, and from (7) and (8):

$$(9) \qquad \underset{n}{\gamma^{\mu\lambda}} = 0(r^{n-5}) \quad \text{for} \quad n \geqslant 4.$$

In the theory of gravitation we have

$$(10) \qquad g_{\mu\lambda} = \sum_{n=0}^{\infty} \underset{n}{g_{\mu\lambda}},$$

where $\underset{0}{g_{\mu\lambda}} = \eta_{\mu\lambda}$ and $\underset{1}{g_{\mu\lambda}} = 0$. Expanding $R_{\mu\lambda}$ into a power series we obtain equations for $\underset{n}{g_{\mu\lambda}}$ which, in empty space-time, have the form

$$0 = \underset{n}{R_{\mu\lambda}} = \text{linear function of } \underset{n}{g_{\mu\lambda,ik}},\ \underset{n-1}{\dot{g}_{\mu\lambda,i}},\ \underset{n-2}{\ddot{g}_{\mu\lambda}} +$$

$$\text{nonlinear function of } \underset{n-2}{g_{\mu\lambda}}, \ldots, \underset{2}{g_{\mu\lambda}}.$$

Thus a solution for any $\underset{n}{g_{\mu\lambda}}$ will contain both terms of linear origin and terms of nonlinear origin. For example

$$\underset{4}{g_{00}} = \text{term coming from } \underset{2}{\ddot{g}} + \text{terms coming from } \underset{2}{g} \cdot \underset{2}{g}.$$

The first terms give rise to the same limitation as in electrodynamics: $r \ll \lambda$. If we apply the EIH method to a system of bodies whose masses are of order m, then the nonlinear terms in $\underset{4}{g_{00}}$ contain expressions like m^2/r^2 and we must have $r \gg m$. Further, if v is a characteristic velocity and l denotes a distance between the bodies we must have $r = l \ll \lambda$ or $v \ll 1$. Therefore, the applicability of the EIH method is limited by the following conditions:

$$m \ll r \ll \lambda, \quad v \ll 1.$$

The first of these inequalities, which is connected with the non linearity of Einstein's equations, is common to this and other approximation methods. The second and third limitations are due to the distinguished rôle played by time in the EIH method.

The linear part of $\underset{n}{g_{\mu\lambda}}$ can easily be calculated from (7). We may expect $\underset{n}{g_{\mu\lambda}}$ also to go like r^{n-5} $(n \geqslant 4)$, unless some nonlinear terms in $\underset{n}{g_{\mu\lambda}}$ cancel out the r^{n-5} terms in the linear part. In general, we cannot impose on the *expanded* metric the condition $\lim_{r\to\infty} \underset{n}{g_{\mu\lambda}} = 0$. However, this does not necessarily mean that the metric is curved at infinity.

Equations of motion

The basic idea of the EIH method can be explained by considering electrodynamics as a model theory; there, the conservation of charge is an equation of motion which follows from the field equations alone. Assuming that A_α has been expanded into a power series, we can write Maxwell's equations in the form:

$$A_{\underset{n}{0},ss} = \underset{n-1}{A_{s,s0}}, \tag{11a}$$

$$\underset{n+1}{A_{r,ss}} - \underset{n+1}{A_{s,rs}} = \underset{n-1}{A_{r,00}} - \underset{n}{A_{0,0r}}. \tag{11b}$$

If, as before, we put $\underset{0}{A_\alpha} = \underset{1}{A_\alpha} = 0$, then $\underset{2}{A_0}$ satisfies the Laplace equation and we may take $\underset{2}{A_0} = e(t)/r$, where $e(t)$ is an arbitrary function of time. Equations (11b), which in the present case become

$$\mathrm{rot}\,\mathrm{rot}\underset{3}{\mathbf{A}} = -\mathrm{grad}\underset{2}{\dot{A}_0}, \quad \mathbf{A} = (A_1, A_2, A_3), \tag{11c}$$

are not independent; the divergence of the left hand side of (11c) vanishes identically (strongly). The divergence of the right hand side also vanishes, by virtue of (11a). However, this is not sufficient to ensure the integrability of (11b) or (11c). The flux of $\mathrm{rot}\,\mathrm{rot}\underset{3}{\mathbf{A}}$ through a closed surface vanishes, and so also must the flux of $\mathrm{grad}\underset{2}{\dot{A}_0}$. The equation $\Delta\underset{2}{A_0} = 0$ tells us that the flux of $\mathrm{grad}\underset{2}{\dot{A}_0}$ does not depend on the shape of the surface (provided that we do not cross the singularity when deforming the surface). This means that the vanishing of the flux imposes a condition only on the singularity itself. We can calculate the flux of $-\mathrm{grad}\underset{2}{\dot{A}_0}$ through a sphere $r = \text{const}$; this turns out to be $4\pi\dot{e}$. Therefore e must be a constant.

The situation is analogous in Einstein's theory and can be presented in a concise form if one uses the superpotentials $\mathbf{U}_\mu^{\nu\lambda}$ [23]. The empty space field equations $\mathbf{G}_\mu^k = 0$ may be written:

$$\mathbf{U}^{sk}_{\mu,s} + \mathbf{U}^{0k}_{\mu,0} + \mathbf{t}^k_\mu = 0. \tag{12}$$

Contracting with n_k and integrating over a closed surface we obtain (since $\mathbf{U}_\mu^{ks}$ is skew in k and s!)

$$\frac{d}{dt}\oint \mathbf{U}^{0k}_\mu n_k\, dS + \oint \mathbf{t}^k_\mu n_k\, dS = 0. \quad \mu = 0, 1, 2, 3. \tag{13}$$

If we have an exact solution of the field equations, then (13) is identically satisfied and does not tell us anything. But if we use the EIH approximation method, and if we expand (12), then the conditions (13) written up to the l-th order will contain only known fields (of order $< l$) and will give nontrivial equations of motion (for $\mu = 1, 2, 3$). Equation (13) for $\mu = 0$ gives the conservation of energy.

Let us illustrate this by the simplest case, the Newtonian equations. From $\underset{2}{R_{00}} = 0$ we have

$$\Delta \underset{2}{g_{00}} = 0; \tag{14}$$

as a solution of this equation we may take

$$\underset{2}{g_{00}} = -\sum 2m/r, \tag{15}$$

where m denotes the mass of a body and r is the distance from it; the summation is to be carried out over all particles. $\underset{2}{R_{ik}} = 0$ gives the equation for $\underset{2}{g_{ik}}$; it appears that a possible solution is

(16) $$\underset{2}{g_{ik}} = \delta_{ik}\underset{2}{g_{00}}.$$

The lowest order fields are linear in the masses and therefore can also be evaluated from (7); g_{0k} is at least of order 3 and the problem of radiation does not appear before the 5th order. The knowledge of $\underset{2}{g_{00}}$ and $\underset{2}{g_{ik}}$ is sufficient for writing down the following surface integral

$$\frac{d}{dt}\oint \underset{2}{\mathrm{U}_0^{k0}} n_k dS = 0$$

(t_0^k is of order 5 at least). Evaluating this integral around each of the singularities, we get

$$m = \mathrm{const}.$$

Since $m = \mathrm{const}$, the field equations for $\underset{3}{g_{0k}}$

$$\underset{3}{g_{0k,ss}} - \underset{3}{g_{0s,ks}} = \underset{2}{g_{ks,0s}} - \underset{2}{g_{ss,0k}}$$

are now integrable and lead to

(17) $$\underset{3}{g_{0k}} = \sum 4m\dot{\xi}^k/r$$

where $\xi^k(t)$ are the coordinates of a particle, as yet arbitrary. The following surface integrals give the *Newtonian equations of motion*

$$\frac{d}{dt}\oint \underset{3}{\mathrm{U}_i^{k0}} n_k dS + \oint \underset{4}{\mathrm{t}_i^k} n_k dS = 0.$$

Infeld [**68**, **84**] developed a formalism in which particles are treated as singularities described by means of δ-functions. In this formalism it is necessary to define the value of singular functions on the world-lines of the particles. If $\varphi(t, x^k, \xi^s(t))$ is a function depending on a world-line ξ^s and singular on this world-line (e.g., $\varphi = |\mathbf{r}-\boldsymbol{\xi}(t)|^{-1}$) then Infeld puts

$$\tilde{\varphi}(t) = (\varphi - \text{part of } \varphi \text{ singular at } \mathbf{k} = \boldsymbol{\xi})_{|x^k=\xi^k(t)}.$$

For a regular function φ we can write:

(18) $$\tilde{\varphi} = \int \varphi(t, x^k, \xi^s)\delta_{(3)}(x^s - \xi^s)dV.$$

Infeld and Plebański [**79**, **83**, **91**] introduce "good" $\hat{\delta}$ functions which allow them to write an equation like this even for singular functions φ. The $\sim$ operation is not distributive in general but Infeld and Plebański assume it to be so when applied to functions occurring in their work: $\widetilde{\alpha\beta} = \tilde{\alpha}\tilde{\beta}$.

The energy-momentum tensor density of a system of pole particles is now written as

$$(19) \qquad \mathrm{T}^{\alpha\beta} = \sum \int_{-\infty}^{\infty} \mu^{\alpha\beta}\hat{\delta}_{(4)}(x^{\lambda}-\xi^{\lambda}(\tilde{s}))\,d\tilde{s} = \sum \mu^{\alpha\beta}\hat{\delta}_{(3)}(x^{k}-\xi^{k}(t))\,d\tilde{s}/dt,$$

where $\tilde{s}$ is defined by $d\tilde{s}^2 = \tilde{g}_{\alpha\beta}d\xi^{\alpha}d\xi^{\beta}$. By adapting an argument due to Mathisson it was shown by W. Tulczyjew that $\mu^{\alpha\beta} = m_0\xi'^{\alpha}\xi'^{\beta}$ ($\xi'^{\alpha} = d\xi^{\alpha}/d\tilde{s}$) and $m_0 = \text{const}$. He also generalized (19) so as to provide a description of bodies with internal rotation and structure [12, 24]. Equation (19) may be written in the form

$$\mathrm{T}^{\alpha\beta} = \sum m\dot{\xi}^{\alpha}\dot{\xi}^{\beta}\hat{\delta}_{(3)}(x^{k}-\xi^{k}), \qquad m = m_0 dt/ds, \qquad \dot{\xi}^{\alpha} = d\xi^{\alpha}/dt.$$

The equations of motion are obtained by integrating $\mathrm{T}^{\alpha\beta}_{;\beta} \equiv \mathrm{T}^{\alpha\beta}_{,\beta}+\mathrm{T}^{\mu\lambda}\begin{Bmatrix}\alpha\\ \mu\lambda\end{Bmatrix} = 0$ over the neighbourhood of one particle:

$$0 = \int \mathrm{T}^{\alpha\beta}_{;\beta}dV = \int\left[(m\dot{\xi}^{\alpha}\dot{\xi}^{\beta}\hat{\delta}_{(3)})_{,\beta}+m\dot{\xi}^{\mu}\dot{\xi}^{\lambda}\begin{Bmatrix}\alpha\\ \mu\lambda\end{Bmatrix}\hat{\delta}_{(3)}\right]dV =$$

$$= (m\dot{\xi}^{\alpha})^{\cdot}+m\widetilde{\begin{Bmatrix}\alpha\\ \mu\lambda\end{Bmatrix}}\dot{\xi}^{\mu}\dot{\xi}^{\lambda}.$$

It follows from this that $md\tilde{s}/dt = m_0 = \text{const}$ and that

$$(20) \qquad m_0\left(\frac{d^2\xi^{\alpha}}{d\tilde{s}^2}+\widetilde{\begin{Bmatrix}\alpha\\ \mu\lambda\end{Bmatrix}}\frac{d\xi^{\mu}}{d\tilde{s}}\frac{d\xi^{\nu}}{d\tilde{s}}\right) = 0.$$

According to Infeld, the equations of motion of heavy bodies have thus also the form of "geodetic" equations. One can eliminate ds from (20) and write the 3 equations of motion in the form

$$(21) \qquad \ddot{\xi}^{k}+\left(\widetilde{\begin{Bmatrix}k\\ \mu\lambda\end{Bmatrix}}-\widetilde{\begin{Bmatrix}0\\ \mu\lambda\end{Bmatrix}}\dot{\xi}^{k}\right)\dot{\xi}^{\mu}\dot{\xi}^{\lambda} = 0.$$

In this notation the Newtonian equations read $\ddot{\xi}^{k}+\underset{2}{\widetilde{\begin{Bmatrix}k\\ 00\end{Bmatrix}}} = 0$. It can be easily seen from (21) that if we know the equations of motion of the n-th order, then we will be able to write $(n+1)$th order equations if we calculate $\underset{n-3}{g_{ik}}$, $\underset{n-2}{g_{0k}}$ and $\underset{n-1}{g_{00}}$. However, it has been shown by Plebański and Bażański [25] that in a Lagrangian formalism [85], it is sufficient to know the explicit form of $\underset{2}{g_{00}}$ and $\underset{3}{g_{0k}}$ (and not necessarily $\underset{4}{g_{00}}$) in order to write down the post-Newtonian equations of motion.

The arbitrariness in the choice of coordinates

Let us perform the coordinate transformation

$$x^0 = \bar{x}^0 + \underset{n+1}{a^0}(\bar{x}^\mu),$$

(22a)
$$x^k = \bar{x}^k + \underset{n}{a^k}(\bar{x}^\mu).$$

The first terms to be affected by it are:

$$\underset{n+2}{\bar{g}_{00}} = \underset{n+2}{g_{00}} + 2\underset{n+1}{a_{0,0}},$$

(22b)
$$\underset{n+1}{\bar{g}_{0k}} = \underset{n+1}{g_{0k}} + \underset{n+1}{a_{0,k}} + \underset{n}{a_{k,0}},$$

$$\underset{n}{\bar{g}_{ik}} = \underset{n}{g_{ik}} + \underset{n}{a_{i,k}} + \underset{n}{a_{k,i}},$$

where $a_\alpha = \eta_{\alpha\beta} a^\beta$. It can easily be seen that if $(\underset{n}{g_{ik}}, \underset{n+1}{g_{0k}}, \underset{n+2}{g_{00}})$ is a solution of the field equations, then $(\underset{n}{\bar{g}_{ik}}, \underset{n+1}{\bar{g}_{0k}}, \underset{n+2}{\bar{g}_{00}})$ is also a solution of the same equations representing the same physical situation in a different coordinate system. The *form* of the equations of motion considered as functions of the ξ's obviously depends on the coordinate system used. Similarly, in the ordinary geodetic equation,

$$\xi''^\alpha + \Gamma^\alpha_{\mu\lambda}(\xi)\xi'^\mu\xi'^\lambda \equiv G^\alpha(\xi''^\mu, \xi'^\lambda, \xi^\varrho) = 0,$$

the form of the function G^α depends on the coordinate system. More precisely, the equations of motion of order $n+4$ ($n = 0, 2, \ldots$) depend on $\underset{n-1}{a^0}$ and $\underset{n}{a^k}$ (and also on coordinate changes of lower orders). The form of the Newtonian equations cannot be affected unless the Galilean character of $\underset{0}{g_{\alpha\beta}}$ is destroyed by the transformation. The post-Newtonian equations depend on the choice of $\underset{2}{a^k}$ in $\underset{2}{g_{ik}} = \delta_{ik}\underset{2}{g_{00}} + \underset{2}{a_{i,k}} + \underset{2}{a_{k,i}}$. The case $\underset{2}{a^k} = 0$ corresponds to the choice of harmonic coordinates in this approximation. It is possible to simplify the equations of motion of a given order, but only at the price of complicating the metric **[58, 61]**.

Radiation according to the EIH method

The structure of Einstein's equations is such that we can choose solutions of the form

$$g_{00} = 1 + \underset{2}{g_{00}} + \underset{4}{g_{00}} + \underset{6}{g_{00}} + \cdots$$

(23)
$$g_{0k} = \underset{3}{g_{0k}} + \underset{5}{g_{0k}} + \underset{7}{g_{0k}} + \cdots$$

$$g_{ik} = -\delta_{ik} + \underset{2}{g_{ik}} + \underset{4}{g_{ik}} + \underset{6}{g_{ik}} + \cdots$$

By analogy with the scalar wave equation and Maxwell's theory we can interpret solutions of the form (23) as representing standing wave fields (no secular losses of energy by radiation). It is only these solutions which were considered in the classical papers on the EIH method. In order to get solutions corresponding to "retarded" or "advanced" fields we must supplement the series (23) with the missing terms: odd in g_{00} and g_{ik} and even in g_{0k} ("radiation terms"). The first of these radiation terms satisfy linear *homogeneous* equations and we may expect them to be linear in the masses and hence their form can be derived from the linearized theory. The electromagnetic analogy suggests that the first radiation terms depend only on time and apparently can be removed by a coordinate transformation (22a); e.g., if $\underset{5}{g}_{00} = f(t)$ and $\underset{4}{a}_0 = -\frac{1}{2}\int f(t)dt$ then $\underset{5}{\bar{g}}_{00} = 0$ [58]. However, the entire field $(\underset{n}{g}_{ik}, \underset{n+1}{g}_{0k}, \underset{n+2}{g}_{00})$ can be annihilated by means of (22a) when the following conditions are satisfied:

$$(24)\qquad \begin{aligned} \underset{n+2}{g}_{00,ik} + \underset{n}{g}_{ik,00} - \underset{n+1}{g}_{i0,k0} - \underset{n+1}{g}_{k0,i0} &= 0, \\ \underset{n+1}{g}_{0m,ik} + \underset{n}{g}_{ik,0m} - \underset{n+1}{g}_{0i,km} - \underset{n}{g}_{km,0i} &= 0, \\ \underset{n}{g}_{im,kl} + \underset{n}{g}_{kl,im} - \underset{n}{g}_{il,km} - \underset{n}{g}_{km,il} &= 0. \end{aligned}$$

That is to say, equations (24) constitute a system of necessary and sufficient conditions for the existence of functions

$$\underset{n+1}{a}_0 \quad \text{and } \underset{n}{a}_k \text{ such that } \underset{n}{\bar{g}}_{ik} = \underset{n+1}{\bar{g}}_{0k} = \underset{n+2}{\bar{g}}_{00} = 0.$$

It was remarked by Goldberg [20] that starting with $\underset{n}{g}_{ik} = f_{ik}(t)$ we can choose solutions of the field equations in the $(n+1)$th and $(n+2)$th orders such that the conditions (24) will not be satisfied. However, it must be noted that since the solutions of the field equations are not unique, we can also start with the same $\underset{n}{g}_{ik}$ and obtain functions $\underset{n+1}{g}_{0k}$ and $\underset{n+2}{g}_{00}$ which can be annihilated. For example the field

$$\underset{n}{g}_{ik} = f_{ik}(t), \quad \underset{n+1}{g}_{0k} = \tfrac{1}{2}x^s\dot{f}_{sk}, \quad \underset{n+2}{g}_{00} = 0$$

is flat, but the field

$$\underset{n}{g}_{ik} = f_{ik}(t), \quad \underset{n+1}{g}_{0k} = 0, \quad \underset{n+2}{g}_{00} = -r^2\ddot{f}_{ss}/6$$

is empty and nonflat unless $\ddot{f}_{ik} = \frac{1}{3}\delta_{ik}\ddot{f}_{ss}$ (spherical symmetry), namely

$$\underset{n+2}{g}_{00,ik} + \underset{n}{g}_{ik,00} - \underset{n+1}{g}_{i0,k0} - \underset{n+1}{g}_{k0,i0} = \ddot{f}_{ik} - \tfrac{1}{3}\delta_{ik}\ddot{f}_{ss}.$$

The form of the linear part of the first radiation terms for a system of point particles can be obtained from (7). The linear part of $\underset{n}{g}_{\alpha\beta}$ is connected to $\underset{n}{\gamma}^{\alpha\beta}$ by the equation

$$(25)\qquad \underset{n}{g}_{\alpha\beta}^{linear} = \eta_{\alpha\mu}\eta_{\beta\lambda}(\underset{n}{\gamma}^{\mu\lambda} - \tfrac{1}{2}\eta^{\mu\lambda}\eta_{\pi\varrho}\underset{n}{\gamma}^{\pi\varrho}).$$

From (7) and (25) we have

$$(26a)\qquad \underset{3}{g}_{ik} = 0,$$

$$(26b)\qquad \underset{4}{g}_{0k} = -\underset{4}{\gamma}^{0k} = -\sum 4m\ddot{\xi}^k = 0.$$

The last equality holds by virtue of the Newtonian equations of motion and is to be read: $\sum m\ddot{\xi}^k$ is at least of order 6.

The linearized equation (6) is not accurate enough for the calculation of $\underset{5}{g}_{00}$ and of the next triple of radiative fields $(\underset{5}{g}_{ik}, \underset{6}{g}_{0k}, \underset{7}{g}_{00})$. In any case, since $\underset{5}{g}_{00}$ is a function of the time only, the triple $(\underset{3}{g}_{ik}, \underset{4}{g}_{0k}, \underset{5}{g}_{00})$ is trivial in the sense that it can be annihilated by a coordinate transformation and does not contribute to the equations of motion **[35]**. Correcting an earlier computation [26] and complementing the work of A. Peres [27], Infeld and Michalska-Trautman **[104]** evaluate the fields $\underset{5}{g}_{ik}, \underset{6}{g}_{0k}$ and $\underset{7}{g}_{00}$, find their contribution to the equations of motion of a system of particles and establish a complete agreement between the predictions of the linearized theory, the magnitude of the radiative damping force and the amount of energy emanating from the system in the form of gravitational waves (see also the work of J. Ryteń [28]). A little later, as a result of an independent investigation, Chandrasekhar and Esposito [16] publish a study of gravitational radiation in the framework of general relativistic hydrodynamics. The rates of dissipation of energy and angular momentum they obtain are also in agreement with the linearized theory of gravitational radiation.

A mechanical description of radiation damping

The equations of motion obtained by the EIH method are of a "mechanical" type: they contain quantities referring only to particles. Up to a certain order these equations may be derived from a Lagrangian which is invariant under space and time translations. This implies that to that order the total energy and momentum are conserved and the system does not radiate. A similar connection between radiation and motion is familiar from electrodynamics: for a system of interacting charges a mechanical Lagrangian exists giving their motion with a post-Coulombian accuracy [29]. In general, a conservative and more accurate electromagnetic Lagrangian does not exist; it can be introduced only when the interactions are assumed to be of the half-retarded, half-advanced type [30, 31]. In general relativity this problem was considered by Fock [5], Infeld **[85]**, Infeld and Plebański **[91]** and Plebański and Bażański [25].

Infeld and Michalska-Trautman **[101]** have analyzed in considerable detail the connection between radiation and the possibility of introducing a generalized mechanical Lagrangian for a particle. They propose a method which may be used in conjunction with the EIH technique to evaluate the energy and momentum radiated by a particle in the form of gravitational electromagnetic or other waves **[103]**. The following lines contain a brief description of the method.

To alleviate the formulae, a symbolic notation is used here: ψ denotes a field or a set of fields with indices suppressed; ψ' represents the set of first partial derivatives of ψ with respect to the coordinates x^μ; ξ stands for the three spatial coordinates of a particle. In many expressions a summation over the suppressed indices is taken for granted. It is assumed that there is given a privileged time coordinate t; e.g., it may be defined by an external static field. Let the action for the system consisting of a particle and the field be

$$W = \int_{t_1}^{t_2} dt \left(\Lambda + \int_V \mathscr{L}\, dV\right)$$

where

$$\Lambda = \Lambda(\xi, \dot{\xi}, \psi), \qquad \mathscr{L} = \mathscr{L}(\psi, \psi'),$$

V is a three-dimensional region which will later be assumed to coincide with the whole space $t = \text{const.}$; dV is an element of volume in V and $\dot{\xi} = d\xi/dt$. By varying W with respect to ξ and ψ one obtains the equations of motion

(27) $$\Omega = 0, \quad \text{where} \quad \Omega(\xi, \dot{\xi}, \ddot{\xi}, \psi, \psi') = \frac{\delta W}{\delta \xi},$$

and the field equations,

(28) $$\Phi = 0, \quad \text{where} \quad \Phi(\xi, \dot{\xi}, \psi, \psi', \psi'') = \frac{\delta W}{\delta \psi}.$$

The ψ's occurring in Ω should be evaluated at the point $x = \xi$. The equations (27) and (28) are coupled and should be solved simultaneously; in fact, in general relativity they cannot be solved otherwise. In special relativity, and also in general relativity when an appropriate approximation scheme is employed, it is possible to find a solution of the field equation $\Phi = 0$ corresponding to an arbitrary motion $\xi(t)$. Let

(29) $$\psi = \Psi(x, \xi, \dot{\xi})$$

be such a solution. Following Infeld, we assume here that ψ does not depend explicitly on t and is a rather simple function of the motion of the particle. These assumptions are fulfilled within the framework of the EIH method. Upon substituting (29) into (27) one obtains what is called an equation of motion of third kind [91]:

(30) $$\overline{\Omega} = 0.$$

Here and in the rest of this section a bar above a function of ψ will signify that ψ is to be replaced by the righthand side of (29). Clearly, $\overline{\Phi} \equiv 0$. The equations (30) contain only the coordinates of the particle and one may enquire as to whether they can be deduced from a "mechanical" Lagrangian.

Put

$$L(\xi, \dot{\xi}, \ddot{\xi}) = \overline{\Lambda} + \int_V \overline{\mathscr{L}}\, dV$$

and

$$\overline{W} = \int_{t_1}^{t_2} L\, dt.$$

Let $\delta\xi$ be a variation of the ξ's, vanishing at t_1 and t_2; the corresponding variation of the solution is

$$\delta\psi = \frac{\partial\Psi}{\partial\xi}\delta\xi + \frac{\partial\Psi}{\partial\dot{\xi}}\delta\dot{\xi}$$

and need not vanish on the boundary FV of V. A straightforward computation gives

$$\delta\overline{W} = \int_{t_1}^{t_2} dt\left(\overline{\Omega}\delta\xi + \int_V \overline{\Phi}\delta\psi dV + \int_{FV} \frac{\overline{\partial\mathscr{L}}}{\partial\psi'}\delta\psi dS\right)$$

where the notation is self-explanatory. One thus obtains a sufficient condition for the existence of a variational principle

$$\int_{FV} \frac{\overline{\partial\mathscr{L}}}{\partial\psi'}\delta\psi dS = 0 \Rightarrow (\delta\overline{W} = 0 \Leftrightarrow \overline{\Omega} = 0). \tag{31}$$

Since $\overline{\Omega}$ contains no derivatives of ξ of order higher than the second, if the surface integral (31) vanishes then the dependence of L on $\dddot{\xi}$ is inessential.

The quantity

$$E = -L + \frac{\partial L}{\partial\dot{\xi}}\dot{\xi} - \frac{\partial L}{\partial\ddot{\xi}}\ddot{\xi}$$

is conserved if (31) holds and may be interpreted as the total energy of the system. In general

$$\frac{dE}{dt} = -\oint_{FV} S_i^k \dot{\xi}^i n_k dS, \tag{32}$$

where

$$S_i^k = \frac{\overline{\partial\mathscr{L}}}{\partial\psi_{,k}}\frac{\partial\Psi}{\partial\xi^i} - \frac{d}{dt}\left(\frac{\overline{\partial\mathscr{L}}}{\partial\psi_{,k}}\frac{\partial\Psi}{\partial\dot{\xi}^i}\right).$$

The quantity

$$\oint_{FV} S_i^k n_k dS$$

may be shown to correspond to the damping force acting on the particle. For simple systems the amount of radiation computed on the basis of (32) is in agreement with that obtained by other methods. This method has been successfully employed by Infeld and Michalska-Trautman to evaluate the amount of electromagnetic and gravitational radiation **[101, 104]**.

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