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THEORETICAL PHYSICS

On the Conservation Theorems and Equations of Motion in Covariant Field Theories

by

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1. In this note we shall deal with a tensor field (or set of tensor fields) the components of which will be denoted by $\psi_A(x^\nu)$ $(A=1,...,N;$ $\nu=0,...,3)$. The field equations for the pure field case are supposed to be derivable from a variational principle

$$\delta I = 0, \tag{1}$$

where:

$$I = \int_\Omega \mathfrak{L} d_4 x, \qquad \mathfrak{L} = \sqrt{-g} L, \tag{2}$$

$$L = L(g_{\mu\nu}, g_{\mu\nu,\sigma}, \psi_A, \psi_{A,\nu}), \qquad g = \det(g_{\mu\nu}); \tag{3}$$

$g_{\mu\nu}(x^a)$ is the metric tensor of the space-time V_4 (in general non-Euclidean). We assume that L is an invariant with respect to general co-ordinate transformations in V_4.

The purpose of this paper is to obtain a conservation law for the ψ-field and to derive the equations of motion for a "cluster of particles" moving in such a field. The latter problem is closely related to a paper by Infeld [1] on the equations of motion in linear field theories.

2. We use the notation of the general relativity theory (summation convention, ordinary (covariant) differentiation denoted by a comma (semicolon), $D\varphi^A \equiv \varphi^A{}_{;\nu} dx^\nu$).

The field equations resulting from (1) are:

$$L^A \underset{\text{def.}}{=} \frac{1}{\sqrt{-g}} \left(\frac{\partial \mathfrak{L}}{\partial \psi_A} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathfrak{L}}{\partial \psi_{A,\nu}} \right) = 0. \tag{4}$$

We generalise (4), assuming that the sources of the field are described by a tensor field j^A; the equations in the presence of sources become $L^A = j^A$.

We define a symmetric energy-momentum density tensor:

$$T_{\alpha\beta} \underset{\text{def.}}{=} \frac{2}{\sqrt{-g}} \left(\frac{\partial \mathfrak{L}}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathfrak{L}}{\partial g^{\alpha\beta},_\nu} \right). \tag{5}$$

It is well known [2], [3] that the conservation laws for physical systems are intimately connected with the invariance properties of the Lagrangian function. The invariance under the group of general co-ordinate transformations in V_4 leads to four *identities* connecting $T_{\alpha\beta}$, L^A and the ψ's.

Let us take an infinitesimal co-ordinate transformation $x^\nu \to x^{\nu\prime} = x^\nu + \delta\zeta^\nu$, where $\delta\zeta^\nu(x^\alpha)$ is a vector field. We form the "*substantial*" *variation* of ψ_A:

$$\delta^*\psi_A \underset{\text{def.}}{=} \psi_{A'}('x^\nu) - \psi_A(x^\nu), \quad \text{where } 'x^{\nu'} = 'x^\nu + \delta\zeta^\nu = x^\nu, \; *) \tag{6}$$

and assume, following Belinfante [5] and Bergmann [6], that

$$\delta^*\psi_A = -\psi_{A,\nu}\delta\zeta^\nu + F_{A\mu}{}^{B\nu}\psi_B\delta\zeta^\mu{}_{,\nu}. \tag{7}$$

$F_{A\mu}{}^{B\nu}$ is a constant (numerical) tensor depending on the transformation properties of ψ_A and fulfilling certain commutation relations [6]. From (7), we obtain:

$$\psi_{A;\alpha} = \psi_{A,\alpha} + F_{A\mu}{}^{B\nu}\Gamma^\mu_{\alpha\nu}\psi_B. \tag{8}$$

The original region of integration Ω is mapped, by the transformation $x^\nu \to x^{\nu\prime}$, to a new region Ω'. We can now write:

$$\delta^* I \underset{\text{def.}}{=} \int_{\Omega'} \mathfrak{L}' d_4 x' - \int_{\Omega} \mathfrak{L} d_4 x = 0. \tag{9}$$

Further, evaluating this difference:

$$\delta^* I = \int_{\Omega} (\delta^*\mathfrak{L} + (\mathfrak{L}\delta\zeta^\nu),_\nu) d_4 x, \tag{10}$$

or explicitly:

$$\delta^* I = \int_{\Omega} \left(\frac{\partial \mathfrak{L}}{\partial g^{\alpha\beta}} \delta^* g^{\alpha\beta} + \frac{\partial \mathfrak{L}}{\partial g^{\alpha\beta}{}_{,\nu}} \delta^* g^{\alpha\beta}{}_{,\nu} + \frac{\partial \mathfrak{L}}{\partial \psi_A} \delta^*\psi_A + \frac{\partial \mathfrak{L}}{\partial \psi_{A,\nu}} \delta^*\psi_{A,\nu} + (\mathfrak{L}\delta\zeta^\nu)_{,\nu} \right) d_4 x. \tag{11}$$

3. The field $\delta\zeta^\nu$ being arbitrary, let us assume that $\delta\zeta^\nu$ and its first derivatives vanish on the boundary of Ω. Thus, bearing in mind that $\delta^*\varphi_{,\nu} = (\delta^*\varphi)_{,\nu}$, and integrating by parts we obtain from (11):

$$\delta^* I = \int_{\Omega} \sqrt{-g} \left(\tfrac{1}{2} T_{\alpha\beta} \delta^* g^{\alpha\beta} + L^A \delta^*\psi_A \right) d_4 x; \tag{12}$$

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*) More precisely, $\delta^*\psi_A$ denotes the principal part of this difference. $\delta^*\psi_A$ differs only in sign from the Lie differential treated in [4].

$\delta^* g^{\alpha\beta}$ can be easily computed:

(13) $$\delta^* g^{\alpha\beta} = \delta\zeta^{\alpha;\beta} + \delta\zeta^{\beta;\alpha}.$$

Substituting (7) and (13) in (12) and integrating by parts we have:

$$\delta^* I = -\int_\Omega \sqrt{-g}\,\delta\zeta^\alpha \left(T_{\alpha\beta}{}^{;\beta} + L^A \psi_{A;\alpha} + \frac{1}{\sqrt{-g}} (\sqrt{-g} L^A F_{A\alpha}{}^{B\beta} \psi_B)_{,\beta} \right) d_4 x = 0$$

and, by virtue of (8) and the arbitrariness of Ω and $\delta\zeta^\alpha$:

(14) $$(T^\beta{}_\alpha + L^A F_{A\alpha}{}^{B\beta} \psi_B)_{;\,\beta} + L^A \psi_{A;\alpha} \equiv 0.$$

This is a set of "strong" equations, holding independently, whether the field equations (4) are satisfied or not. When $L^A = 0$, we obtain from (14) a "weak" conservation law for the energy-momentum tensor: $T^{\ \beta}_{\alpha;\beta} = 0$.

4. Introducing the canonical (in general non-symmetric) energy-momentum density tensor $t_\alpha^{\ \beta} \underset{\text{def.}}{=} -\delta_\alpha^\beta L + \frac{\partial L}{\partial \psi_{A,\beta}} \psi_{A;\alpha}$, we can, by a method developed by Hilbert [7] or de Wet [8], obtain a general relation between $T_\alpha{}^\beta$ and $t_\alpha{}^\beta$. We give the formula in question for the case when L does not depend on $g_{\mu\nu,\sigma}$ (this assumption will be made throughout section **4**):

(15) $$T_\alpha{}^\beta \equiv t_\alpha{}^\beta - F_{A\alpha}{}^{B\beta} \left(\frac{\partial L}{\partial \psi_A} \psi_B + \frac{\partial L}{\partial \psi_{A,\nu}} \psi_{B;\nu} \right).$$

It is worth observing that in a non-Euclidean space-time, the canonical energy-momentum tensor is divergenceless even when (4) holds, viz.:

(16) $$t^{\ \beta}_{\alpha;\beta} \equiv -L^A \psi_{A;\alpha} - \frac{\partial L}{\partial \psi_{A,\beta}} \psi_B F_{A\mu}{}^{B\nu} R^\mu{}_{\nu\alpha\beta}.$$

When the field equations (4) are satisfied, we can write (15) in the form: $T_\alpha{}^\beta = t_\alpha{}^\beta + S_\alpha{}^{\beta\nu}{}_{;\nu}$, where

$$S_\alpha{}^{\beta\nu} \underset{\text{def.}}{=} -F_{A\alpha}{}^{B\beta} \psi_B \frac{\partial L}{\partial \psi_{A,\nu}} = -S_\alpha{}^{\nu\beta}.$$

The tensor $s^{\alpha\beta\nu} \underset{\text{def.}}{=} S^{\beta\alpha\nu} - S^{\alpha\beta\nu}$ corresponds — when the V_4 is flat — to the spin angular momentum density tensor [5].

5. The problem of the motion of bodies in the general relativity theory has been investigated by many authors. Einstein and Infeld represented the bodies as singularities of the field (see for example [9]). In this paper, we assume that matter is characterised by a scalar density of mass $\varrho(x^\nu)$ and a velocity field $u^\alpha = dx^\alpha/ds$ (perfect fluid without pressure).

We assume that the left hand side of the gravitational equations can be obtained by varying the integral

$$J = \int_{\Omega} K \sqrt{-g}\, d_4 x.$$

K is an invariant built from $g_{\mu\nu}$ and their derivatives. The gravitational field equations are

$$(17) \qquad K_{\alpha\beta} \underset{\text{def.}}{=} \frac{\delta J}{\delta g^{\alpha\beta}} = \varrho u_\alpha u_\beta + T_{\alpha\beta}.$$

For $K = \varkappa R$, we have $K_{\alpha\beta} = \varkappa(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R)$, and (17) becomes Einstein's equation. In a manner similar to that used in section 3, we get $K^{\alpha\beta}{}_{;\beta} = 0$.

Equation (17) can be integrated only when

$$(18) \qquad (\varrho u^\alpha u^\beta + T^{\alpha\beta})_{;\beta} = 0.$$

Assuming $L^A = j^A$, we obtain by virtue of (14), and $u^\alpha{}_{;\beta} u^\beta = Du^\alpha / ds$:

$$(19) \qquad \varrho \frac{Du_\alpha}{ds} + u_\alpha (\varrho u^\beta)_{;\beta} = j^A \psi_{A;\alpha} + F_{A\alpha}{}^{B\beta} (j^A \psi_B)_{;\beta}.$$

Transvecting (19) with u^α, we get the law of "conservation of mass":

$$(20) \qquad (\varrho u^\beta)_{;\beta} = u^\alpha (j^A \psi_{A;\alpha} + F_{A\alpha}{}^{B\beta} (j^A \psi_B)_{;\beta}).$$

From (19) and (20) we have the *equations of motion*:

$$(21) \qquad \varrho \frac{Du^\alpha}{ds} = (g^{\alpha\nu} - u^\alpha u^\nu) \left((j^A \psi_B)_{;\beta} F_{A\nu}{}^{B\beta} + j^A \psi_{A;\nu} \right).$$

The equations of motion appear as necessary *integrability conditions* for the gravitational field equations.

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