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Dedicated to André Lichnérowicz

A CLASSIFICATION OF SPACE-TIME STRUCTURES*

ANDRZEJ TRAUTMAN

Institute of Theoretical Physics, Warsaw University, Warsaw, Poland

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The paper contains a review of various bundles which may be associated to the bundle of linear frames and used to describe properties of space relevant to physics. Restrictions, extensions, prolongations and reductions are defined in terms of morphisms of principal bundles. It is shown that the holonomic prolongation of a G-structure exists iff the corresponding structure function vanishes. G-connections are related to restrictions of the bundle of second-order frames. It is shown that these restrictions may be used to classify theories of space-time and gravitation. A distinction is made between a projective connection and a geodetic structure. In the framework of the Einstein–Cartan theory, the projective connection of a space-time is compatible with its metric tensor iff the spin density is bivector-valued. As an example, we mention a new theory of gravitation and electromagnetism based on the Weyl–Cartan structure of space-time and on the Yang quadratic Lagrangian.

1. Introduction

The fundamental significance of a linear connection in the relativistic theory of space, time, and gravitation could have been recognized as early as in 1913 by Einstein and Grossmann [13]. Weyl was the first to distinguish the two basic elements of Riemannian geometry: the conformal structure of space and the projective geometry of paths, defined by the set of all geodesics [40]. In the space-time of general relativity, the former is determined by the propagation of light and the latter is related to freely falling particles. E. Cartan clarified the role played by the linear connection in the Newtonian theory [5], introduced the notion of torsion [6], and put forward a general theory of connections which soon became a classical mathematical subject [28], [20]. The role played by, and the uniqueness of, the connection in space-time may be looked upon as consequences of the principle of equivalence [1], [32]. These and other problems related to the physical interpretation of the Newtonian, path (geodetic), and conformal structures are presented in recent

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books [38], [26] and articles [33], [10], [11], [27]. The purpose of this paper is to give an overview of the fundamental differential-geometric notions which may be associated with a space-time and used to classify theories of gravitation founded on connections. The paper is based in part on the text of a lecture delivered by the author on July 3, 1973, at the Symposium on New Mathematical Methods in Physics held in Bonn [34] and in part on seminars given in May, 1975, at Princeton University, Yeshiva University, the University of Texas at Austin and at Dallas, the University of Chicago, and Brandeis University.

A geometric model for gravitation is determined by giving a four-dimensional differential manifold X, interpreted as the space-time, and

(i) a restriction E of the bundle LX of linear frames of X to a group $G \subset GL_4(\mathbf{R})$; this restriction defines a generalized metric structure on X;

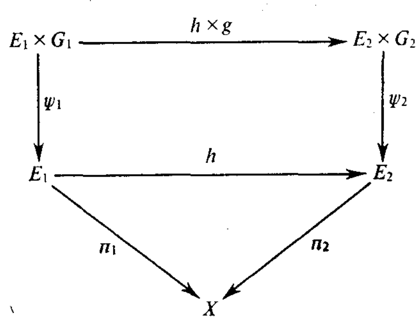
(ii) a connection on E, or, equivalently, a linear connection on LX compatible with the generalized metric structure;

(iii) a system of "equations of motion" which further restrict the geometric elements of X and relate them to the spatio-temporal distribution of matter.

This paper deals only with the "kinematic" part of the model, i.e. with its properties defined by (i) and (ii). It will be shown that conditions (i) and (ii) may be replaced by a single one, requiring that the bundle KX of aholonomic second-order frames of X be restricted to the group G.

2. Morphisms of principal bundles

Let $\varkappa_i = (E_i, X, G_i, \pi_i)$, $i = 1, 2$, be a pair of principal differential bundles over the base X and let ψ_i: $E_i \times G_i \to E_i$ be the map defining the action of G_i in E_i ([3], [9], [35], [31]). If (h, g) is a morphism of $\varkappa_1$ into $\varkappa_2$, then g: $G_1 \to G_2$ is a morphism of Lie groups and the diagram



is commutative. We accept the following terminology, which is in agreement with [9], but at variance with that of [20] and [35]:

If both h and g are injective immersions, then

$\varkappa_1$ is said to be a *restriction* of $\varkappa_2$ relative to (h, g), or, by abuse of language, E_1 is called a restriction of E_2 to G_1;

$\varkappa_2$ is said to be an *extension* of $\varkappa_1$ relative to (h, g), or, simply, E_2 is called an extension of E_1 to G_2.

If both h and g are surjective submersions, then

$\varkappa_1$ is said to be a *prolongation* of $\varkappa_2$ relative to (h, g) and E_1 is called a prolongation of E_2 to G_1;

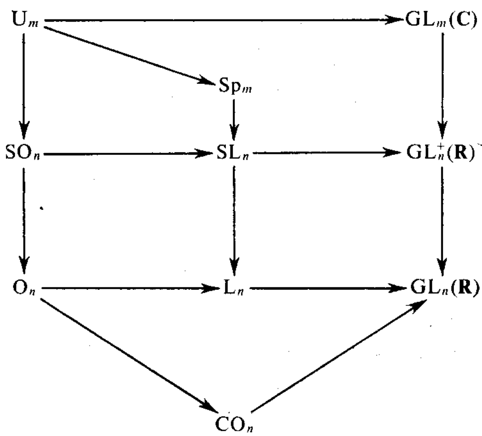
$\varkappa_2$ is said to be a *reduction* of $\varkappa_1$ relative to (h, g) and E_2 is obtained from E_1 by reducing the structure group to G_2.

3. *G*-structures

Let $(LX, X, \mathrm{GL}_n(\mathbf{R}), \pi)$ be the bundle of linear frames of a real, n-dimensional differential manifold X. A linear frame $r \in LX$ may be identified with the linear isomorphism

$$r\colon \mathbf{R}^n \to T_{\pi(r)}X.$$

A restriction of LX to $G \subset \mathrm{GL}_n(\mathbf{R})$ defines a G-structure on X [31]. The following diagram contains some of the most important subgroups of $\mathrm{GL}_n(\mathbf{R})$ for $n = 2m$ [18]:



Let ${}^t a$ be the transpose of the matrix a and let $j = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$, where $\mathbf{1}$ is the unit $m \times m$ matrix. The definitions of the subgroups occurring in the diagram and the names of the corresponding G-structures on an n-dimensional manifold are summarized in Table I. If $ja = aj$, then $a = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A and B are $m \times m$ real matrices. The map $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$ is an isomorphism of groups and $\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A+iB)|^2$. The condition ${}^t aa = I$ is then equivalent to $({}^t A - i\,{}^t B)(A+iB) = 1$. The adverb "almost" has to do with the lack of integrability of the corresponding structure. It is seldom used in the context of conformal structure, but we adjoined it here for the sake of uniformity.

TABLE I

Symbol of the group G	Defining equations	Name of the group	Name of the corresponding G-structure
$GL_n(\mathbf{R})$	$\det a \neq 0$	general linear (in n variables over $\mathbf{R}$)	
$GL_n^+(\mathbf{R})$	$\det a > 0$	restricted linear	oriented space
L_n	$(\det a)^2 = 1$	unimodular	space with a volume element
SL_n	$\det a = 1$	special linear	oriented space with a volume element
CO_n	${}^t aa = \mu I,\ \mu \in \mathbf{R}$	homothetic	(almost) conformal structure
O_n	${}^t aa = I$	orthogonal	(almost) Euclidean structure
SO_n	${}^t aa = I$ $\det a = 1$	special orthogonal	oriented, (almost) Euclidean structure
$GL_m(\mathbf{C})$	$ja = aj,$ $\det a \neq 0$	general linear (in m variables over $\mathbf{C}$)	almost complex structure
Sp_m	${}^t aja = j$	symplectic	almost symplectic structure
U_m	$ja = aj,$ ${}^t aa = I$	unitary	almost Hermitian structure

In general, there are obstacles hindering the introduction of a G-structure on a manifold [30]. Roughly speaking, it is hard to restrict LX to a "small" group, unless certain topological conditions are satisfied. If the manifold X admits a G-structure, then it also admits any H-structure, where $G \subset H$. As a rule, the G-structure, if it exists, is not unique.

4. Bundles of projective and affine frames

Given a G-structure on an n-dimensional manifold X and a homomorphism of Lie groups $g: G \to H$, there is a canonically defined principal bundle (F, H, X, π_F); if $E \subset LX$ is the total space of the G-structure, then (cf. [3], 6.6.1)

$$F = (E \times H)/G.$$

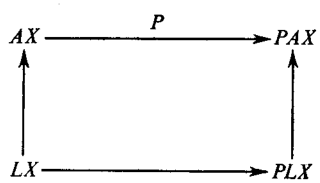
Clearly, F is the total space of a bundle associated to E. If g is the inclusion homomorphism of G into a larger subgroup H of $GL_n(\mathbf{R})$, then F is a restriction of LX, i.e., F is the total space of a H-structure. If g is surjective, then $F = E/\ker g$. An important example of a construction of the latter type is provided by the definition of the *bundle of projective frames* of X. Let

$$P: GL_n(\mathbf{R}) \to PGL_n(\mathbf{R})$$

be the canonical map onto the projective general linear group [4]. In this case, ker$P$ is the multiplicative group of the reals. The total space PLX of the bundle associated to LX by P consists of projective frames. The bundle of projective frames is a reduction of LX to $PGL_n(\mathbf{R})$. The natural action of $PGL_n(\mathbf{R})$ in the projective space $\mathbf{P}_{n-1} = P(\mathbf{R}^n)$ defines a bundle associated to PLX, namely the projective tangent bundle PTX. The latter may also be obtained from the tangent bundle by removing the zero section and identifying, at any point of the manifold X, all parallel vectors. In other words, PTX is the *bundle of* (tangent) *directions* of X.

Another possibility is to consider the projective extension of the tangent bundle. The fibre over $x \in X$ of the projective extension is the n-dimensional projective space $P(\mathbf{R} \times \times T_x X)$. Here the typical fibre is $\mathbf{P}_n$, and the corresponding principal bundle PAX has $PGL_{n+1}(\mathbf{R})$ as the structure group [7], [19].

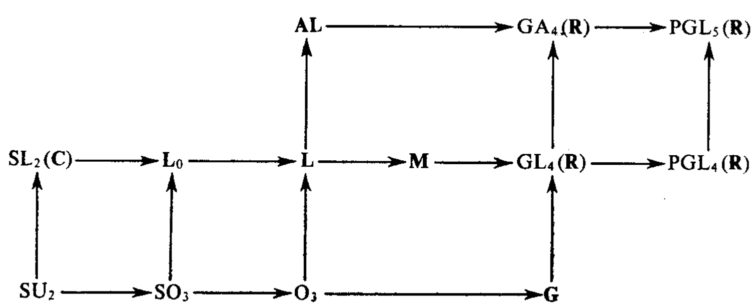
Another important principal bundle which may always be associated with a manifold X is the bundle of affine frames, AX. It may be obtained from LX by the canonical injection of $GL_n(\mathbf{R})$ into the general affine group, $GA_n(\mathbf{R})$, considered as a semi-direct product of $GL_n(\mathbf{R})$ by $\mathbf{R}^n$ with respect to the natural action of $GL_n(\mathbf{R})$ in $\mathbf{R}^n$. There are natural transformations among these bundles. In the diagram



the vertical arrows are natural injections and P maps the affine frame (r_0, r), where $r_0 \in T_x X$ and $r = (r_1, \ldots, r_n) \in L_x X$, into the extended projective frame $\big(p(1, r_0), p(0, r_1), \ldots \ldots, p(0, r_n)\big)$. Here p is the canonical map of $\mathbf{R} \times T_x X - \{0, 0\}$ onto $P(\mathbf{R} \times T_x X)$.

5. Fibre bundles associated to space-time

Lichnérowicz [23], [24] was among the first to emphasize the importance of the Lorentz bundle in the theory of relativity, whereas Künzle [22] defined the Galilei bundle which underlies Newtonian physics. The following diagram contains some of the groups relevant for theories of space-time:



Most of the symbols occurring above have a generally accepted meaning. $\mathbf{L}$ is the Lorentz group, $\mathbf{L}_0$ is the connected component of the identity in $\mathbf{L}$. The Maxwell group $\mathbf{M}$ is the group of "Lorentz homotheties": $a \in \mathbf{M}$ iff a is the product of a Lorentz transformation and a dilatation. $\mathbf{AL}$ is the inhomogeneous Lorentz group ("Poincaré group"). The orthochronous Galilei group $\mathbf{G}$ is defined as the group of all invertible 4×4 matrices a which satisfy

$$ah = ha \quad \text{and} \quad \tau a = \tau,$$

where $h = \mathrm{diag}(0,1,1,1) \in \mathbf{R}^4 \otimes \mathbf{R}^4$ and $\tau = (1,0,0,0) \in (\mathbf{R}^4)^*$ is the one-form of absolute time.

A restriction of LX to $\mathbf{M}$ defines the conformal structure of space-time (the *Maxwell bundle*). The *Lorentz* and *Galilei bundles* are defined as restrictions of LX to $\mathbf{L}$ and $\mathbf{G}$, respectively. An O_3-structure on a four-dimensional space-time X is assumed in the theories of electromagnetism based on the notion of an aether.

6. Prolongations

Let us now consider examples of structures which may be associated to a manifold X by introducing prolongations of G-structures on X. Given a surjective morphism $g\colon H \to G$ of Lie groups, the corresponding prolongation need not exist and, if it does, it is not unique, in general. For example, the introduction of a *spin structure* on a manifold may be regarded as a restriction of its bundle of frames to SO_n, followed by a prolongation corresponding to the canonical map $\mathrm{Spin}_n \to \mathrm{SO}_n$. In the theory of relativity, one usually considers the spin structure corresponding to $\mathrm{SL}_2(\mathbf{C}) \to \mathbf{L}_0$ [25], [15], [8].

There is one type of prolongations which always exist and are canonically attached to the bundle of linear frames of any manifold: these are the *differential prolongations* due to Ehresmann [12]. They may be conveniently described in terms of a functor (the "lifting functor") from the category of local diffeomorphisms to the category of principal bundles [21].

For the purposes of this article, it suffices to consider only the *first* order differential prolongations of the bundle of linear frames. (They are also said to be *second* order prolongations of X.) For this reason, we use an *ad hoc* notation which is explained below. If $s\colon U \to LX$ is a local section of π and $x \in U \subset X$, then $j_x(s)$ is the jet of order one of s at x (cf. [3], 12.1.2). This jet is completely characterized by the vector space $J_x(s) = T_x s(T_x X) \subset T_{s(x)} LX$. The space $V_{s(x)} = \ker T_{s(x)}\pi$ of vertical vectors is complementary to $J_{s(x)}$,

$$T_{s(x)} LX = J_x(s) \oplus V_{s(x)}.$$

Let $\alpha_x(s)\colon T_{s(x)} LX \to \mathscr{L}(\mathbf{R}^n)$ be the linear form associating to a vector $u \in T_{s(x)} LX$ that element of the Lie algebra $\mathscr{L}(\mathbf{R}^n)$ of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbf{R})$ which, by the action of the group, induces the vertical component of u. Clearly, the set KX of all first-order jets of local sections of π is in a bijective and natural correspondence with the set of all forms α which have the generic properties implied by the previous definition. We may identify $j_x(s)$ with the corresponding form and write $\alpha_x(s) \in KX$. There is a natural map $\sigma\colon KX \to LX$ defined by $\sigma(\alpha_x(s)) = s(x)$, and a natural structure on KX which makes (KX, LX, σ) into a bundle. Moreover, this is a principal bundle and its structure group may be determined as follows. Let θ be the canonical, $\mathbf{R}^n$-valued horizontal one-form on LX, $\theta_r\colon T_rLX \to \mathbf{R}^n$. If $\alpha, \alpha' \in KX$ and $\sigma(\alpha) = \sigma(\alpha') = r$, then $\alpha' - \alpha$ is horizontal. There thus exists a linear map

$$b\colon \mathbf{R}^n \to \mathscr{L}(\mathbf{R}^n)$$

such that

$$\alpha' = \alpha + b \circ \theta.$$

Therefore, the structure group of σ is the additive group $\mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n))$ of linear maps of $\mathbf{R}^n$ into $\mathscr{L}(\mathbf{R}^n)$. Moreover, $\pi \circ \sigma\colon KX \to X$ is a principal bundle with structure group GK_n, a semi-direct product of GL_n by $\mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n))$ relative to the action of GL_n given by $b \mapsto \mathrm{Ad}_a \circ b \circ a^{-1}$, where $\mathrm{Ad}_a(c) = a \circ c \circ a^{-1}$, $a \in \mathrm{GL}_n$, $b \in \mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n))$, and $c \in \mathscr{L}(\mathbf{R}^n)$.

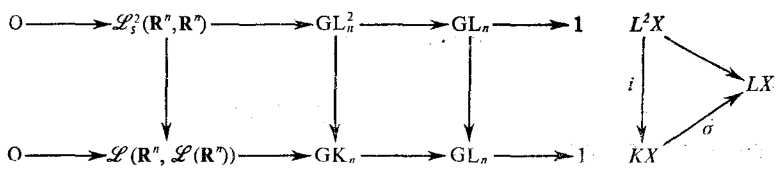
A local section $s\colon U \to LX$ is said to be *holonomic* if it is induced by a local coordinate map $\xi\colon U \to \mathbf{R}^n$, $\langle s_i(x), (d\xi^j)_x\rangle = \delta^j_i$, for any $x \in U$. The form $\alpha = (\alpha^i_j) \in KX$ corresponds to a holonomic section iff [42]

$$d\theta^i_r + \alpha^i_j \wedge \theta^j_r = 0 \tag{1}$$

where $r = \sigma(\alpha)$. The set L^2X of all first-order jets of local holonomic sections of π is a subbundle of KX. Its structure group $GL^2_n \subset GK_n$ corresponds to the natural inclusion

$$\mathscr{L}^2_s(\mathbf{R}^n, \mathbf{R}^n) \subset \mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n)),$$

where $\mathscr{L}^2_s(\mathbf{R}^n, \mathbf{R}^n)$ is the Abelian group of bilinear symmetric maps of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}^n$. In other words, if $(a, b) \in \mathrm{GL}^2_n$, then $a = (a^i_j) \in \mathrm{GL}_n$ and $b = (b^i_{jk})$ with $b^i_{jk} = b^i_{kj}$, where $i, j, k = 1, \ldots, n$. Of course, the symmetry of b may be related to that of the second derivatives of coordinate transformations. (Ehresmann and his school write $H^2(X)$ and $\overline{H}^2(X)$ for what is denoted here by L^2X and KX, respectively.) The following diagrams



summarize the most important relations among the differential prolongations of the lowest order and their structure groups. The rows of the first diagram are exact and i is the natural injection.

It is often convenient to consider the groups GL^2_n and GK_n as subgroups of $\mathrm{GL}(\mathbf{R}^n \oplus \mathscr{L}(\mathbf{R}^n))$. The map

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ b \circ a & \mathrm{Ad}_a \end{pmatrix}$$

defines a monomorphism of GK_n into $\mathrm{GL}(\mathbf{R}^n \oplus \mathscr{L}(\mathbf{R}^n))$.

Let $E \subset LX$ be a G-structure; there is canonically associated to it a restriction $\bar{E}$ of KX to $\bar{G} \subset \mathrm{GK}_n$ called the *(first differential) prolongation* of E. The total space $\bar{E} \subset KX$ and its structure group are defined as follows. If $s\colon U \to E \subset LX$ is a local section around $x \in U$, then the element of KX associated to $j_x(s)$ defines a linear map,

$$\alpha\colon T_{s(x)}E \to G', \tag{2}$$

where G' is the Lie algebra of G. The set $\bar{E}$ of all such α's is a principal bundle over E with $\mathscr{L}(\mathbf{R}^n, G') \subset \mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n))$ as the structure group. Moreover, $\bar{E} \to X$ is a principal bundle with

$$\bar{G} = G \times \mathscr{L}(\mathbf{R}^n, G') \subset \mathrm{GK}_n$$

as the structure group. The intersection

$$E^2 = \bar{E} \cap L^2X$$

may—but need not—be a principal bundle, restriction of L^2X to

$$G^2 = G \times \mathscr{L}(\mathbf{R}^n, G') \cap \mathscr{L}^2_s(\mathbf{R}^n, \mathbf{R}^n).$$

For example, if E is a $\{I\}$-structure defined by a global section

$$s\colon X \to LX,$$

then E^2 is a principal bundle with $\{I\}$ as the structure group iff s is holonomic. In general, if E^2 is a principal bundle, then it is called the *(first) holonomic prolongation* of E. Clearly, E^2 is a holonomic prolongation iff each fibre of $\bar{E} \to X$ contains a holonomic element, i.e. a map (2) subject to (1).

For any $\alpha \in \bar{E}$, the left side of (1) is a horizontal two-form,

$$d\theta^i_r + \alpha^i_j \wedge \theta^j_r = \tfrac{1}{2}\, \bar{c}^i_{jk}(\alpha)\theta^j_r \wedge \theta^k_r.$$

If $b = (b^i_{jk}) \in \mathscr{L}(R^n, G')$, $\alpha' = \alpha + b \circ \theta_r$, then

$$\bar{c}^i_{jk}(\alpha') = \bar{c}^i_{jk}(\alpha) + b^i_{jk} - b^i_{jk},$$

or

$$\bar{c}(\alpha') = \bar{c}(\alpha) + Ab \tag{3}$$

where

$$A\colon \mathscr{L}(\mathbf{R}^n, G') \to \mathscr{L}(\textstyle\bigwedge^2 \mathbf{R}^n, \mathbf{R}^n)$$

is the antisymmetrizing map,

$$Ab(v_1, v_2) = b(v_1)v_2 - b(v_2)v_1, \quad v_i \in \mathbf{R}^n,$$

and

$$\bar{c}\colon \bar{E} \to \mathscr{L}(\textstyle\bigwedge^2 \mathbf{R}^n, \mathbf{R}^n).$$

Let k be the canonical map of $\mathscr{L}(\bigwedge^2 \mathbf{R}^n, \mathbf{R}^n)$ on the cokernel of A,

$$k\colon \mathscr{L}(\textstyle\bigwedge^2 \mathbf{R}^n, \mathbf{R}^n) \to \mathscr{L}(\bigwedge^2 \mathbf{R}^n, \mathbf{R}^n)/\mathrm{Im}(A) = \mathrm{Coker}(A).$$

It follows from (3) that $k \circ \bar{c}$ factors through $\sigma\colon \bar{E} \to E$,

$$k \circ \bar{c} = c \circ \sigma.$$

The map

$$c\colon E \to \mathrm{Coker}(A)$$

is the first-order structure function of E [31]. Since $\bar{c}(\alpha) = 0$ iff α is holonomic, we have

THEOREM 1. *A G-structure $E \subset LX$ has a holonomic prolongation iff its structure function vanishes.*

7. Connections

A linear connection on X may be given by a connection form which, considered as a map

$$\omega\colon LX \to KX,$$

is a section of σ and defines a restriction of KX to the general linear group $\mathrm{GL}_n \subset \mathrm{GK}_n$. According to (1), the linear connection has no torsion iff the map ω factors through $i\colon L^2X \to LX$. In other words, a *torsionless* connection defines a restriction of the bundle L^2X of *holonomic* second-order frames to the group $\mathrm{GL}_n \subset \mathrm{GL}_n^2$ [24]. There exists a similar characterization of linear connections without curvature. It may be formulated in terms of differential prolongations of X of the third order.

A connection on a G-structure $E \subset LX$ is called a *G-connection* on X. Its connection form determines a map

$$\omega\colon E \to KX$$

which, together with the canonical injection $g\colon G \to \mathrm{GK}_n$, $g(a) = (a, 0)$, defines a restriction $\omega(E)$ of KX to G. Conversely, any such restriction leads, by projection on LX, to a G-structure with a connection. This may be summarized in

THEOREM 2. *There is a bijective and natural correspondence between the set of all G-connections on X and the set of all restrictions of KX to $G \subset \mathrm{GL}_n$.*

In addition to connections on G-structures, one has to consider connections on principal bundles which are obtained from the bundle of linear frames by prolonging it or by forming a principal associated bundle in the sense of § 6.6.1 of [3]. A connection on AX, PLX, and PAX is called *affine*, *projective*, and *extended projective*, respectively.

A linear connection on X induces, in a natural manner, unique affine, projective, and extended projective connections on the same manifold. If $\omega = (\omega^i_j)$ is the connection form of a linear connection and λ is any invariant one-form on LX, then the linear connection corresponding to

$$\omega'^i{}_j = \omega^i{}_j + \delta^i{}_j \lambda \tag{4}$$

induces the same projective connection as ω does [14]. Two linear connections give rise to the same projective connection iff they are related by (4). In other words, a projective connection may be defined as an equivalence class of linear connections, two linear connections being considered as equivalent if their forms are related by (4). As the name implies, a projective connection serves to establish parallel transport of directions from one tangent space to another. Let (Ω^i_j) and (Θ^i) be, respectively, the curvature and torsion two-forms of a linear connection. The forms

$$\Omega^i{}_j - \frac{1}{n}\,\delta^i{}_j\,\Omega^k{}_k \quad \text{and} \quad \theta^i \wedge \Theta^j + \theta^j \wedge \Theta^i \tag{5}$$

are invariant under the projective transformations. They vanish iff there exists a form λ such that ω' is a (locally) flat linear connection [2].

A projective connection should be distinguished from what is sometimes called a "projective structure" on a manifold [10], [11], [27]. The latter structure, which I prefer to call the *geodetic structure* or the geometry of paths [39], is determined by the set of all unparametrized geodesics on a manifold.

The projective connection and the geodetic structure may be defined as restrictions of KX and L^2X, respectively. The corresponding groups are of the form $\mathrm{GL}_n \times H$, where H should be replaced by the additive group

$$H = \{b \in \mathscr{L}(\mathbf{R}^n, \mathscr{L}(\mathbf{R}^n))|\ b^i{}_{jk} = \delta^i{}_j c_k\}$$

in the first case, and by

$$H_s = \{b \in \mathscr{L}^2_s(\mathbf{R}^n, \mathbf{R}^n)|\ b^i{}_{jk} = \delta^i_j c_k + \delta^i_k c_j\}$$

in the second.

8. Compatibility and the Einstein–Cartan theory

A projective connection induced by a G-connection is said to be *compatible* with the underlying G-structure. In particular, a projective connection induced by (ω^i_j) is compatible with the O_n-structure defined by the metric (g_{ij}) iff

$$Dg_{ij} = \mu g_{ij} \tag{6}$$

for some one-form μ. If this is the case, one also says that the projective connection and the metric are compatible with each other. A projective connection compatible with an O_n-structure is also compatible with the CO_n-structure induced by (g_{ij}). Among the linear connections which induce such a projective connection there is exactly one compatible with the O_n-structure. It is clear that the last three propositions remain valid if O_n is replaced by the Lorentz group $\mathbf{L}$ and CO_n by the Maxwell group $\mathbf{M}$. If (6) holds, then the angles between the directions undergoing projective parallel transport remain unchanged: such is the geometric significance of compatibility.

The physical significance of compatibility between a projective connection and the metric structure is manifested in the Einstein–Cartan theory of space-time (cf. [36], [17], [37] and the references given there). Let X be a four-dimensional manifold with a metric tensor, considered as a map

$$(g_{ij}) \colon LX \to (\mathbf{R}^4)^* \otimes (\mathbf{R}^4)^*, \quad g_{ij} = g_{ji},$$

equivariant under the action of GL_4. For any $r \in LX$, put $\eta_{1234}(r) = |\det g_{ij}(r)|^{1/2}$ and define η_{ijkl}: $LX \to \mathbf{R}$ for $i, j, k, l \in \{1, 2, 3, 4\}$ by the requirement $\eta_{ijkl} = \eta_{[ijkl]}$. The "pseudotensor" (η_{ijkl}) may be used to define a collection of forms on LX,

$$\eta_{ijk} = \theta^l \eta_{ijkl}, \quad \eta_{ij} = \tfrac{1}{2}\theta^k \wedge \eta_{ijk}, \quad \eta_i = \tfrac{1}{3}\theta^j \wedge \eta_{ij}, \quad \eta = \tfrac{1}{4}\theta^i \wedge \eta_i.$$

The metric tensor may be used to raise and lower indices in the usual manner.

Assume now that X has a projective connection, i.e., an equivalence class of linear connections, the equivalence being defined by (4). The four-form ϱ on LX defined by

$$8\pi\varrho = \tfrac{1}{2}\eta_k{}^l \wedge \Omega^k{}_l$$

is invariant under the "projective transformation" (4): it depends only on g and the projective connection. Given a local section s: $U \to LX$ of π, i.e., a field of linear frames on U, one can compute the corresponding action integral $\int_U s^*\varrho$. This integral corresponds to "pure gravitation". By supplementing it with an integral describing "ponderable matter" and by varying the sum with respect to the components of the metric and of the connection, one arrives at the system of equations

$$\tfrac{1}{2}\eta_{ijk} \wedge \Omega^{jk} = -8\pi t_i, \tag{7}$$

$$D\eta_j{}^k = 8\pi s_j{}^k. \tag{8}$$

The three-forms occurring on the right sides of equations (7) and (8) describe the sources of the gravitational field. They may be interpreted as the density of energy-momentum and of spin, respectively. If both (t_i) and (s_{ij}) are invariant under projective transformations, then the system of equations (7) and (8) also enjoys this property. It is easy to prove

THEOREM 3. *The projective connection and the metric which are subject to equations* (7) *and* (8) *are compatible with each other if and only if*

$$s_{ij} + s_{ji} = 0. \tag{9}$$

Equation (9) means that the density of spin is described by a three-form with values in the Lie algebra of the Lorentz group rather than that of the general linear group. If (9) holds, then there is exactly one linear connection which is metric,

$$Dg_{ij} = 0, \tag{10}$$

and induces the projective connection under consideration. Equations (7)–(10) constitute the basis of the Einstein–Cartan theory of space, time, and gravitation. Of course, if the bundle KX of a four-dimensional manifold is restricted to $\mathbf{L}$, then X is thereby endowed with a metric and a linear connection which are compatible with each other.

9. The classification

A restriction F of KX or L^2X to a group G defines a certain geometry on the manifold X. For several groups, the geometry may be interpreted as providing the "kinematic" part of a theory of space-time and gravitation. As a rule, the restriction must be further constrained by "dynamic" conditions, expressed by "equations of motion", such as the Einstein–Cartan equations. If $G \subset \mathrm{GL}_n$, then, by projection onto LX, the restriction defines a G-structure E on X. Its prolongation $\overline{E}$ contains F, and $G \subset \overline{G}$. In general, the restriction of KX or L^2X to G endows X with a richer structure than a restriction of LX to G or a restriction of KX to $\overline{G}$.

Table II summarizes some of the geometries used in theories of space-time. Any entry such as "Einstein structure" corresponds, in fact, to many theories, depending on the form of the field equations. The Nordström theory and also special relativity belong, in the sense described here, to the class of Einstein structures. Theories of the Weyl–Cartan and Newton–Cartan type have not been developed so far. Recently, Harnad and Pettitt [16] have considered the holonomic prolongation of an **M**-structure which is closely related to, but more general than, the Weyl structure.

TABLE II

Restriction of KX corresponds to	Restriction of to the group		Restriction of L^2X corresponds to
projective connection	$\mathrm{GL}_n \times H$	$\mathrm{GL}_n \times H_s$	geodetic structure
linear connection	GL_n		symmetric linear connection
Weyl–Cartan structure	**M**		Weyl structure
Einstein–Cartan structure	**L**		Einstein structure
Newton–Cartan structure	**G**		Newton structure

An interesting possibility of constructing a new theory based on the Weyl–Cartan structure has been recently suggested by C. N. Yang (private communication and [41]). If the "linear" Lagrangian of the Einstein–Cartan theory is replaced by the quadratic expression

$$*\Omega^i{}_j \wedge \Omega^j_i, \tag{11}$$

where

$$*\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\eta^{kl}$$

is the dual of $\Omega^i_j = \frac{1}{2} R^i_{jkl}\theta^k \wedge \theta^l$, then instead of the Cartan equation (8), which is algebraic with respect to the torsion tensor, one obtains the differential equation

$$D*\Omega^i{}_j \sim s^i{}_j. \tag{12}$$

Since expression (11) is conformally invariant, it is natural to begin with a Weyl–Cartan structure. In this case,

$$\omega^i{}_j = \gamma^i{}_j + \delta^i{}_j\varphi$$

where $\gamma_{ij}+\gamma_{ji} = 0$ and φ is a one-form associated with the electromagnetic potential. From $\Omega^k_k = 4d\varphi$ it follows that the trace of equation (12) coincides, in form, with the Maxwell equation. Since the Lie algebra of the Maxwell group is a direct sum, $\mathbf{M}' = \mathbf{R}\oplus\oplus\mathbf{L}'$, there is hardly any "unification" of electromagnetism with gravitation in this theory [29].

REFERENCES

[1] Anderson, J. L.: *Principles of relativity physics*, Academic Press, New York, 1967.

[2] Bortolotti, E.: Ann. of Math. **32** (1931), 361.

[3] Bourbaki, N.: *Variétés différentielles et analytiques*, Fasc. de résultats, Hermann, Paris, §§ 1–7, 1967; 1971, § 8–15.

[4] —: *Éléments de mathématique, Algèbre*, Ch. II, Hermann, Paris, 1962.

[5] Cartan, É.: Ec. Norm. **40** (1923), 325; **41** (1924), 1; **42** (1925), 17.

[6] —: Comptes Rend. (Paris) **174** (1922), 593.

[7] —: Bull. Soc. Math. France **52** (1924), 205.

[8] Crumeyrolle, A.: Ann. Inst. H. Poincaré **11** (1969), 19; **14** (1971), 309.

[9] Dieudonné, J.: *Éléments d'analyse*, Gauthier-Villars, Paris, 1970, vol. 3; 1971, vol. 4.

[10] Ehlers, J., F. A. E. Pirani, and A. Schild: *The geometry of free fall and light propagation*, article in: *General relativity*, ed. by L. O'Raifeartaigh, Clarendon Press, Oxford, 1972.

[11] Ehlers, J., and A. Schild: Commun. math. Phys. **32** (1973), 119.

[12] Ehresmann, C.: Comptes Rend. (Paris), **233** (1951), 598, 777, 1081; **234** (1951), 1028, 1424; **239** (1954), 1762; **240** (1955), 397; **240** (1955), 1755.

[13] Einstein, A., and M. Grossmann: Z. Math. und Phys. **62** (1913), 225.

[14] Friesecke, H.: Math. Annalen **94** (1925), 101.

[15] Geroch, R.: J. Math. Phys., Part I: **9** (1968), 1739; Part II **11** (1970), 343.

[16] Harnad, J. P., and R. B. Pettitt: *Gauge theories for space-time* symmetries, Department of Physics, Carleton University (Preprint), 1976.

[17] Hehl, F. W., P. von der Heyde, and G. D. Kerlick: Phys. Rev. **D10** (1974), 1066.

[18] Kobayashi, S.: *Transformation groups in differential geometry*, Springer, Berlin, 1972.

[19] —: *Canonical forms on frame bundles of higher order contact*, in: *Differential Geometry*, Proc. of Symposia in Pure Math., vol. III, AMS, Providence, 1961.

[20] —, and K. Nomizu: *Foundations of differential geometry*, Interscience, New York, vol. I, 1963; vol. II, 1969.

[21] Krupka, D., and A. Trautman: Bull. Acad. Polon. Sci., sér. sci. math. astr. et phys. **22** (1974), 207.

[22] Künzle, H. P.: Ann. Inst. H. Poincaré **17** (1972), 337.

[23] Lichnérowicz, A.: *Théoriés relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris, 1955.

[24] —: J. Gen. Relativity Gravitation **1** (1971), 235.

[25] Milnor, J.: Enseign. Math. **9** (1964), 198; Topology **3** (1965), 223.

[26] Misner, C. W., K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973.

[27] Pirani, F. A. E.: Symposia Mathematica **XII** (1973), 67.

[28] Schouten, J. A.: *Ricci calculus*, 2nd ed., Springer, Berlin, 1954.

[29] Sciama, D. W.: J. Math. Phys. **2** (1961), 472.

[30] Steenrod, N.: *The topology of fibre bundles*, Princeton Univ. Press, Princeton, 1951.

[31] —: *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, 1964.

[32] Trautman, A.: *Theory of gravitation*, article in: *The Physicist's Conception of Nature*, ed. by J. Mehra, Reidel, Dordrecht, 1973.

[33] Trautman, A: *Comparison of Newtonian and relativistic theories of space-time*, article in: *Perspectives in Geometry and Relativity*, ed. by B. Hoffmann, Indiana Univ. Press, Bloomington, 1966.

[34] —: *Infinitesimal connections in physics*, in: *The Proceedings of the Symposium on New Mathematical Methods in Physics and Problems in General Relativity*, ed. by K. Bleuler and A. Reetz (mimeographed), Bonn, 1973.

[35] —: Reports Math. Phys. **1** (1970), 29.

[36] —: Symposia Math. **XII** (1973), 139.

[37] —: Ann. N. Y. Acad. Sciences **262** (1975), 241.

[38] —, F. A. E. Pirani, and H. Bondi: *Lectures on general relativity*, Prentice-Hall, Englewood Cliffs, 1965.

[39] Veblen, O., and T. Y. Thomas: Trans. A. M. S. 25 (1923), 551.

[40] Weyl, H.: *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, Göttinger Nachr. (1921), 99.

[41] Yang, C. N.: Phys. Rev. Lettr. **35** (1974), 445.

[42] Yuen Ping Chen, Cahiers de Topol. et Géom. Diff. **12** (1971), 333; Comptes Rend. (Paris) **270** (1970), 1589.