ENERGY AND PHYSICS

Proceedings of the Third General Conference of the European Physical Society

9-12 September 1975
Bucharest, Romania

Published by European Physical Society
PO-Box 39, CH-1213, Petit-Lancy 2, Switzerland

Energy, Gravitation and Cosmology

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Introduction

The purpose of this short article, which is intended for non-specialists, is to review the relationship between energy and gravitation and to give basic information on cosmological models of the Universe.

I do not dwell upon the concept of energy which has been analyzed in other lectures at this Conference. For our purposes, it suffices to remember that energy has its roots in Newtonian physics. Total energy is conserved, kinetic energy is part of the total energy, different kinds of energy can be transformed one into another. In relativistic physics, energy becomes a component of the energy-momentum vector. Moreover, energy is related to mass by the celebrated Einstein formula. In order to retain the conservation laws in the presence of electromagnetic radiation, energy and momentum have to be ascribed to the electromagnetic field. According to the theory of special relativity, the densities of energy and momentum, together with the stresses, form a second-rank tensor field $T^{\alpha\beta}$, where $\alpha, \beta = 0, 1, 2, 3$. Its divergence vanishes,

$$T^{\alpha\beta}{}_{,\beta} = 0, \tag{1}$$

if all relevant kinds of matter are taken into account. Total energy and momentum are obtained from this tensor by a suitable integration.

The *Newtonian theory of gravitation* may be briefly summarized as follows: there is a gravitational potential φ, subject to the Poisson equation,

$$\Delta\varphi = 4\pi G\rho, \tag{2}$$

and a reference system $(t, x, y, z) = (t, \vec{r})$ such that the free fall of a particle of inertial mass m_I and gravitational mass m_G is governed by the equation of motion

$$m_I \ddot{\vec{r}} + m_G \operatorname{grad} \varphi = 0. \tag{3}$$

Since the inertial and gravitational masses are known to be equal,

$$m_I = m_G,$$

equation (3) is invariant under the replacement of $\vec{r}$ and φ, respectively, by

(4a) $$\vec{r}' = \vec{r} - \vec{a}(t)$$
and

(4b) $$\varphi' = \varphi + \ddot{\vec{a}}(t)\,\vec{r} + b(t),$$

where the vector $\vec{a}$ and the scalar b depend arbitrarily on time. Eqs. (4) are closely related to the Einstein elevator thought experiment: by a suitable choice of $\vec{a}$ the 'gravitational force' grad φ may be reduced to zero along the trajectory of any freely falling particle. For an isolated system, the gravitational potential φ may be fixed by the boundary condition $\lim_{r\to\infty} \varphi = 0$; this implies $\ddot{\vec{a}} = 0$ and reduces (4a) to Galilei transformations, $\vec{r}' = \vec{r} - \vec{V}t - \vec{r}_0$. The requirement $\lim \varphi = 0$ is non-local and cannot be imposed in the case of cosmology. As a result, even in the Newtonian approximation, we have the *principle of equivalence:* gravitational forces and inertial translatory forces are locally indistinguishable. Locally, 'inertial' frames are defined up to transformations (4a); the arbitrariness of the functions $\vec{a}$ (t) is the Newtonian residue of the *general invariance* underlying Einstein's relativistic theory of gravitation [1].

Energy as a source of gravity

Any relativistic theory of gravitation should be based on a set of field equations which reduce to (2) in the limit of small velocities and weak fields. One can guess the form of the equations by considering the relativistic generalization of the Newtonian density of mass ρ. The three simplest possibilities are summarized in Table I. In the scalar theory, the Newtonian ρ is replaced by the density of rest mass, whereas a tensor theory is obtained if the density of energy is considered as the main source of the gravitational field. Vector theories are unacceptable because, like electrodynamics, they predict repulsion between charges of the same sign. In the scalar case, the electromagnetic field does not directly contribute to the sources of gravity.

Table I

Relativistic theories of gravitation classified according to the nature of the sources

Nature of the source	Nature of the gravit. potential	Functional form of the source for		Theory due to	light deflection
		a dust	an e.m. field		
density of rest mass	scalar	ρ	e.m.$T^{\alpha}_{\alpha} = 0$	Nordström	no
current of mass	vector	ρu^{α}	(predicts gravitational repulsion instead of attraction)		
energy-momentum tensor	tensor	$\rho u^{\alpha} u^{\beta}$	e.m.$T^{\alpha\beta}$	Einstein	yes

By the same token, the rays of light are not deflected in the gravitational field. Since light is known to be deflected, scalar theories, such as the Nordström theory [2], have to be ruled out and we are left with tensor theories.

In the theory of special relativity, there is in space-time a set of inertial frames characterized by the simple form of the Minkowski metric tensor ${}_0g$ relative to these frames, $({}_0g_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1)$. According to the argument of the preceding section, global inertial frames cannot be properly defined in the presence of gravitational interactions and the flat metric tensor ${}_0g$ must be replaced by a more general, curved or Riemannian metric g. On the other hand, a (symmetric) tensor is required to serve as the potential of the gravitational field. By a stroke of genius, Einstein made the *fundamental assumption* of general relativity:

the gravitational potential coincides with the metric tensor field g determining the geometry of space-time.

If it is further assumed that the field equations should (i) contain nothing besides g and the sources, (ii) be of second differential order with respect to g, and (iii) reduce to eq. (2) in the Newtonian limit, then their form is uniquely determined,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi\, G c^{-4}\, T_{\alpha\beta} \tag{5}$$

where $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$, $R = g^{\mu\gamma} R_{\mu\gamma}$, and $R^{\mu}{}_{\alpha\beta\gamma}$ is the Riemann curvature tensor. The Einstein field equations (5) are sometimes modified by the inclusion of a cosmological term [3] which will not be considered here. The ordinary conservation law of special relativity, eq. (1) is now replaced by the covariant law

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{6}$$

where the semicolon denotes covariant differentiation. Unless there is a symmetry of space-time, eq. (6) does not lead by itself to any globally conserved quantity: the energy and momentum of the sources, described by the tensor T, must be supplemented by quantities related to the gravitational field.

In the Newtonian or special relativistic description of physical phenomena, the local distribution of energy is hardly ever of any significance. In the theory of the electromagnetic field, one is sometimes even encouraged to use a 'canonical', asymmetric and not gauge-dependent energy-momentum tensor because "only the integrated, total energy and momentum play a rôle and they are gauge-invariant anyhow". The situation changes drastically when gravitational interactions are taken into account: since the densities and currents of energy and of momentum of matter are sources of the gravitational field, their local distribution acquires a direct physical meaning.

Gravitational energy

The significance of the *local* distribution of energy is restricted, however, to particles and fields other than the gravitational field. Surprisingly enough, gravitation provides a unique definition of the energy-momentum tensor of matter T, but fails to do so with respect to itself. Occasionally, people regard this to be a defect of the

theory which should be remedied by modifying its structure. Personally, I believe the non-localizability of gravitational energy to be a fundamental physical aspect of general relativity which results from the principle of equivalence. Its roots may already be seen at the Newtonian level: because of (4b), 'gravitational force' is locally an ill-defined concept. It is even more so at the relativistic level. Because of the importance of this issue, I formulate in several ways the argument against a careless use of the notion of gravitational force or field strength. (i) In a freely falling rocket one does not feel any gravitational forces; if the rocket is sufficiently large, one can notice the tidal forces, depending, however, on the *second* derivatives of φ.
(ii) Elementary particle theoreticians favour the following approach to gravity: consider a theory of gravitons, defined as particles of mass zero and spin 2. The corresponding potentials $h_{\alpha\beta}$ are subject to gauge transformations

$$h'_{\alpha\beta} = h_{\alpha\beta} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha}$$

where ξ is an arbitrary vector field. Unlike in electrodynamics, there is no gauge-invariant combination of the first derivatives of h. The simplest gauge-invariant form

$$\frac{1}{2}\left(h_{\alpha\beta,\mu\nu} - h_{\alpha\mu,\beta\nu} + h_{\mu\nu,\alpha\beta} - h_{\beta\nu,\alpha\mu}\right) \tag{7}$$

depends on the *second* derivatives of h.
(iii) For a mathematician, general relativity is based on (pseudo) Riemannian geometry. He knows that the Christoffel symbols,

$$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}),$$

presumed to play the role of gravitational field strengths, can be reduced to zero along any curve by a suitable choice of the coordinates. The mathematician will also say that the simplest tensor which can be constructed from g by differentiation is the curvature tensor, that it contains *second* derivatives of the metric and bears a close relation to (7) and to tidal forces.

Since the gravitational field strengths are not well defined, it is not surprising that gravitational energy cannot be localized. In the important case of an isolated, gravitating system it is possible, however, to define the total energy-momentum content of the system. This can be done in several, essentially equivalent manners. For the purposes of this lecture it is enough to present briefly the original approach of Einstein [4]. Omitting indices and using a symbolic notation which is easy to decipher, the field equations (5) may be written as

$$\text{curl } U - t = T \tag{8}$$

where

$$U = \text{curl } g$$

is a 'superpotential' and

$$t = (\text{grad } g)^2 \tag{9}$$

is the 'pseudotensor' of energy and momentum of the gravitational field. The total energy (and momentum) may be obtained from eq. (8) by integration,

$$E = \int_{\text{volume}} (T + t) = \oint_{\text{surface}} U. \tag{10}$$

For an isolated system, the surface integral occurring in (10) may be evaluated well outside of the system, where the field is weak. Its value is, therefore, unaffected by the freedom of reference frames.

An effective energy-momentum tensor may be defined for gravitational waves provided that their wavelength λ is much smaller than the radius of curvature of the background space and that one does not insist on localization of the gravitational energy in regions with an extension smaller than several wavelengths [5—7]. Let the metric g be split into a 'background' part g_0 and a wave-like part h, $g = g_0 + h$. This implies a corresponding splitting of U and t. By evaluating the average value of both sides of eq. (8) over many wavelengths and noting that $\langle \text{grad h} \rangle = 0$, one obtains

$$\text{curl } U_0 - t_0 = T + \langle (\text{grad h})^2 \rangle$$

where $\langle (\text{grad h})^2 \rangle$ is the effective energy-momentum tensor of gravitational waves; it acts as a source of the background metric on the same footing as T. An important feature of the effective energy is that it is positive-definite. This results automatically from the structure of Einstein's equations. In connection with this, it has been conjectured, and shown in many important cases [5, 8—10], that the total energy E of an isolated system is never negative, provided that the energy-momentum tensor of the system satisfies an appropriate energy condition, say

$$T_{\alpha\beta}\, u^\alpha u^\beta \geqslant 0 \text{ for any time-like vector } u. \tag{11}$$

Cosmology

Surprisingly enough, the same simple laws of gravitation determine not only the fall of an apple on the ground, the motion of the Moon around the Earth but also the overall motion in the Universe. However much doubt we may have about particular cosmological models, it is encouraging that they correctly account for the two fundamental properties of the observed Universe: its expansion and the existence of a hot stage in the past.

The basic equation which determines the evolution of the Universe may be derived in the Newtonian theory on the basis of the law of conservation of energy. Consider a model of a Universe filled with a dust of density $\rho(t)$. Let R(t) be the radius of a sphere comoving with the dust: the mass contained in the sphere is constant,

$$M = \frac{4}{3}\pi\rho(t)\, R(t)^3 = \text{const.} \tag{12}$$

The total energy of an element of the dust with unit mass is

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \varepsilon = \text{const.} \tag{13}$$

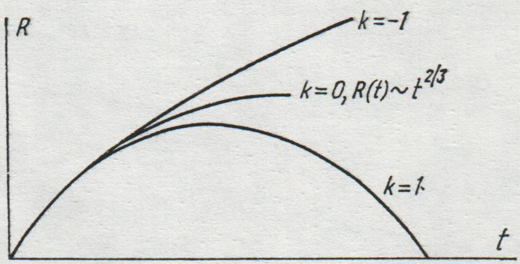
This is the fundamental Friedmann [11] equation which, in the Einstein theory, must be supplemented by a relation between ε and the curvature k/R^2 of space,

$$\varepsilon = -\frac{1}{2} kc^2.$$

The line-element is then

$$ds^2 = R(t)^2\, dl_k^2 - c^2\, dt^2,$$

where dl_k^2 describes a three-dimensional space of constant curvature $k = \pm 1$ or 0. The solution of eq. (13), represented on the figure, contains 'singularities': each of the functions R vanishes for at least one value of t.
These singularities may be thought of as corresponding to a dense and hot stage in the evolution of the Universe. The microwave background radiation which permeates the Universe with a black body distribution corresponding to a temperature of 2.7°K is surely a relic of that hot era. Unfortunately, the mathematical singularities in the geometry implied by the Friedmann equation are more obstinate than one would like them to be. Unlike in the theory of shock waves, they cannot be simply removed by a more realistic description of the sources producing the gravitational field or by consideration of less symmetric models. The nature of the singularities occurring in cosmology and in local collapse has been the subject of studies



of the Soviet [12—15] and British [16—18] schools. Hawking and Penrose show that singularities in cosmology follow necessarily from Einstein's equations under very reasonable assumptions which include an 'energy condition' of the type (11). A possibility of preventing the cosmological singularity by a direct influence of spin on the geometry has been described at this Conference by Kopczyński [19].

An intriguing question is whether the Universe is closed ($k = 1$) or open ($k = 0$ or -1)? If we believe in eq. (13), the question can be in principle answered by measuring the Hubble time,

$$\tau = (R/\dot{R})_{\text{now}},$$

and the present density ρ_{now} of matter in the Universe. The Universe is closed if ρ_{now} exceeds the critical density

$$\rho_c = \frac{3}{8\pi G\tau^2}.$$

Alternatively, the Universe is closed if the deceleration parameter

$$q = -(R\ddot{R}/\dot{R}^2)_{now} = \frac{1}{2}\,\rho_{now}/\rho_c$$

is larger than $\frac{1}{2}$. Many theoreticians favour the closed model for aesthetic reasons. A recent analysis [20] of all observational evidence yields the value

$$q = 0.03 \pm 0.01$$

thus supporting the open model. (See, however, [26] where a different view is expressed).

The expression (10) for the total energy is not applicable in cosmology. If one insists on using it to evaluate the energy content of a closed Universe, one obtains $E = 0$.

The *large numbers* that can be formed from cosmological and atomic quantities, and the coincidences between them, have received much attention since they were noticed for the first time by Weyl, Eddington and Dirac [21] (see also, 22, 23, IV, VI and VIII). One such large number is the inverse of the *gravitational fine structure constant*,

$$\gamma = Gm^2/\hbar c \cong 6\times 10^{-39},$$

where m is the proton mass. Following an idea of Chandrasekhar [24], we introduce

$$N_p = \gamma^{-p/2}.$$

The fundamental coincidence,

$$c\tau \sim \frac{\hbar}{mc} N_2 \qquad (14)$$

has led Dirac to conjecture that G is inversely proportional to the present age of the Universe (which is of the same order as τ). In Table II, which is adapted from [22] and [24] we summarize some of the definitions (D), theoretical results (T) and observations (O) involving the numbers N_p.

Personally, I am in favour of the explanation of (14) due to Carter (VI p. 126): τ is such as to allow for the development of life. This is possible during a specific period in the evolution of the Universe. I feel, however, that γ is as much in need of an explanation as the electromagnetic fine structure constant.

Table II

p	mass mN_p	length $N_p \hbar/mc$	density $N_p m^4c^3/\hbar^3$
-2		grav. radius of proton (D)	Universe (O)
-1		Planck (D)	
0	proton (D)	Compton (D)	nuclear (D)
$\frac{2}{3}$		R_{min} of observable Universe in a model with spin and torsion [25](T)	
1	Planck (D)	radius of neutron star (T)	
2		Hubble radius $c\tau$ (O) grav. Bohr radius (T)	ρ_{max} in a model with spin and torsion[25] (T)
3	Chandrasekhar limit (T)		
$\frac{7}{2}$	galaxy [24] (O)		
4	observable Universe (O)		Planck (D)

Acknowledgement

I am grateful to S. Chandrasekhar and J. A. Wheeler for suggestions and remarks which influenced this paper.

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