36.

RECENT ADVANCES IN THE EINSTEIN-CARTAN THEORY OF GRAVITY*

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Introduction

The Einstein-Cartan theory of space-time is motivated by the desire to provide a simple description of the influence of spin on gravitational phenomena. This is achieved by taking as a model of space-time a four-dimensional differential manifold endowed with a metric tensor and a linear connection and by relating certain combinations of the curvature and torsion tensors to the densities of energy-momentum and of spin, respectively. The essential idea that underlies this theory was advanced by Cartan[1] as early as 1923. Much later, and independently of Cartan, an equivalent theory was formulated by Sciama[2] and Kibble,[3] whereas Weyl[4] considered the special case of the Dirac equation in a curved space-time with torsion. In recent years, the Einstein-Cartan theory has been developed by Hehl *et al.*[5-10] and by the Warsaw group.[11-20] The review paper by Hehl[9] gives a historic account of the theory and a tensorial formulation its equations, whereas Reference 15 contains an exposition based on Cartan's tensor-valued differential forms. Kopczyński[13] and Tafel[14] discovered a class of nonsingular homogeneous solutions of the Einstein-Cartan field equation. The credibility of the interpretation of these solutions as models of the universe[16] was questioned by Stewart and Hájiček.[21]

The present paper, which is based in part on the lecture by the author at this Conference, contains a short account of some of the work performed recently in Warsaw on the Einstein-Cartan theory. It should be read in conjunction with other review articles.[15,22,23]

Vector Fields on the Lorentz Bundle

Consider a Lorentz bundle $\pi: E \to X$, that is, a reduction of the bundle of linear frames of a four-dimensional manifold X to the Lorentz group; the reduction is defined by a metric tensor field g on X, and the elements of E are linear frames ("tetrads") orthonormal relative to g.[24-26] The components of the metric are

$$g_{11} = g_{22} = g_{33} = -g_{44} = -1, \qquad g_{ij} = 0 \quad \text{for } i \neq j.$$

A linear (metric) connection defines on E 10 fundamental vector fields (P_i, S_{jk}),

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$S_{jk} + S_{kj} = 0$, i, j, k = 1, 2, 3, 4, which are linearly independent at each point of E. The vector fields S_{ij} generate Lorentz transformations of the tetrads, whereas the Ps correspond to infinitesimal parallel transfers of the frames. More precisely, if (θ^i) and (ω^{jk}) are, respectively, the canonic and the connection 1-forms on E, P and S are defined by the duality relations

$$\theta^i(P_j) = \delta_j^{\ i}, \qquad \theta^i(S_{jk}) = 0, \qquad \omega^{jk}(P_i) = 0,$$
$$\omega^{jk}(S_{lm}) = \delta_l^{\ j}\delta_m^{\ k} - \delta_m^{\ j}\delta_l^{\ k}.$$

The equations of structure

$$\Theta^i = d\theta^i + \omega^i_{\ j} \wedge \theta^j, \qquad \Omega^{jk} = d\omega^{jk} + \omega^j_{\ l} \wedge \omega^{lk} \tag{1}$$

are equivalent to the commutation relationships

$$[P_i, P_j] + Q^k_{\ ij} P_k + \tfrac{1}{2} R^{kl}_{\ \ ij} S_{kl} = 0, \tag{2a}$$
$$[P_i, S_{jk}] = g_{ij} P_k - g_{ik} P_j, \tag{2b}$$
$$[S_{ij}, S_{kl}] = g_{jk} S_{il} - g_{ik} S_{jl} + g_{il} S_{jk} - g_{jl} S_{ik}, \tag{2c}$$

where $Q^i_{\ jk}$ and $R^i_{\ jkl}$ are the torsion and the curvature tensors, respectively,

$$\Theta^i = \tfrac{1}{2} Q^i_{\ jk} \theta^j \wedge \theta^k, \qquad \Omega^j_{\ k} = \tfrac{1}{2} R^j_{\ klm} \theta^l \wedge \theta^m. \tag{3}$$

If both torsion and curvature vanish, the system of Equations 2 becomes identical in form with the set of commutation relations among the standard generators of the inhomogeneous Lorentz group. The analogy between torsion and curvature with regard to their relation to translations and rotations is a strong argument in favor of the Einstein-Cartan theory.[15,22] The last two terms on the left-hand side of Equation 2a are similar in form to the Mathisson-Papapetrou force,[27,28] which occurs in the equation of motion of a spinning particle when the equation is generalized[11] to a space with torsion.

The commutation relationships (Equations 2) define a 10-dimensional Lie algebra if, and only if, (X,g) is a space of constant curvature without torsion.

The Variational Principle

From now on, we restrict our attention to a four-dimensional differential manifold X endowed with a metric g and a linear connection. Locally, we can always introduce a field **e** of frames on X that may, but need not, be orthonormal relative to g. To alleviate the notation, the frame dual to **e** will be denoted by the same symbol as the canonical 1-form on E considered in the preceding section. This abuse of notation is justified by the fact that for an orthonormal **e**, the dual frame is equal to the pull-back of the canonic 1-form by the section of E that corresponds to **e**. In the same spirit, $(\omega_j^{\ i})$ will denote the 1-forms of the connection relative to the coframe (θ^i). Equations 1 and 3 may now be interpreted as defining the 2-forms of torsion and curvature referred to (θ^i). The completely antisymmetric "pseudotensor" (η_{ijkl}), $\eta_{1234} = |\det g|^{1/2}$, is used to define a collection of forms on X,

$$\eta_{ijk} = \theta^l \eta_{ijkl}, \qquad \eta_{ij} = \tfrac{1}{2}\theta^k \wedge \eta_{ijk}, \qquad \text{etc.}$$

The metric tensor allows one to raise and lower indices in the usual manner. The covariant exterior derivative is denoted by D; thus, for example,

$$D\Omega^i_{\ j} = 0 \qquad \text{and} \qquad D\Theta^i = \Omega^i_{\ j} \wedge \theta^j$$

are the Bianchi identities. D coincides with the exterior derivative for scalar-valued forms, and $D\phi_A = \theta^i \nabla_i \phi_A$ for a tensor field (ϕ_A).

The principle of least action for a relativistic theory of gravity coupled to a tensor field (ϕ_A) may be written as

$$\delta \int \Lambda = 0,$$

where Λ is a pseudo-4-form constructed from θ^i, $\omega^j_{\ k}$, g_{lm}, ϕ_A, and $D\phi_A$. If these fields are varied independently of one another,

$$\delta\Lambda = \tfrac{1}{2}E^{ij}\delta g_{ij} + \delta\theta^i \wedge \mathrm{e}_i + \tfrac{1}{2}\delta\omega^i_{\ j} \wedge c^j_{\ i} + \delta\phi_A \wedge L^A + \text{an exact form.}$$

If the variations are induced by an infinitesimal change of the frames,

$$\delta\theta^i = -\alpha^i_{\ j}\theta^j,$$

then

$$\delta\omega^i_{\ j} = D\alpha^i_{\ j}, \qquad \delta g_{ij} = \alpha_{ij} + \alpha_{ji}$$

and

$$\delta\Lambda = 0,$$

which results in the equality

$$E^j_{\ i} = \theta^j \wedge \mathrm{e}_i + \tfrac{1}{2} D c^j_{\ i}, \tag{4}$$

provided that the field equation $L^A = 0$ is satisfied. This shows that it is immaterial whether one chooses $(\theta^i, \omega^j_{\ k})$ or $(g_{ij}, \omega^k_{\ l})$ as the set of independent gravitational variables for a principle of least action of the Palatini type.

Because any pseudo-4-form on X that is intrinsic and linear homogeneous in (Ω, Θ) is proportional to

$$8\pi K = \tfrac{1}{2}\eta_i^{\ j} \wedge \Omega^i_{\ j},$$

we take

$$\Lambda = K + L,$$

with

$$L = L(g, \theta, \phi, D\phi),$$

and arrive at the following set of Einstein-Cartan equations:

$$\tfrac{1}{2}\eta_{ijk} \wedge \Omega^{jk} = -8\pi t_i, \tag{5}$$
$$D\eta_k^{\ l} = 8\pi s_k^{\ l},$$
$$L^A = 0, \tag{6}$$

where (t_i) and $(s_i{}^j)$ are vector and tensor-valued 3-forms that describe the sources of the gravitational field; these forms are obtained by varying L relative to (θ^i) and $(\omega^k{}_l)$, respectively. Because K is invariant under the projective or λ-transformations,

$$\omega^i{}_j \to \omega^i{}_j + \delta^i{}_j \lambda,$$

Equation 6 cannot be solved unless $s_i{}^i = 0$. If the trace of s does not vanish, the situation can be rectified by replacing L with

$$\bar{L} = L + \mu^{ijkl} \wedge D\eta_{ijkl},$$

where μ is treated as a Lagrange multiplier to be determined from the requirement $\bar{s}_i{}^i = 0$. By varying the action integral that corresponds to $K + \bar{L}$ with respect to μ, we obtain $D\eta_{ijkl} = 0$. This may be interpreted by saying that X admits a covariantly constant unit volume. Clearly, the equation of motion of the field ϕ is unaffected by the replacement $L \to \bar{L}$.

A straightforward algebraic computation leads to the following theorem on compatibility: if the Cartan equation

$$D\eta_k{}^l = 8\pi s_k{}^l$$

is satisfied, then

$$s_{ij} + s_{ji} = 0 \quad \text{is equivalent to} \quad Dg_{ij} = \kappa g_{ij},$$

and

$$\left.\begin{aligned} D\eta_{ijkl} &= 0 \\ s_{ij} + s_{ji} &= 0 \end{aligned}\right\} \Longleftrightarrow Dg_{ij} = 0.$$

The skew symmetry of s_{ij} is in agreement with its physical interpretation as the density of intrinsic angular momentum. If s computed from the Lagrangian fails to have this property, one can again save the situation by having recourse to a new term with a Lagrange multiplier.[29]

Summary

It is pointed out that the commutation relations among the fundamental vector fields defined on a Lorentz bundle with a connection generalize the corresponding relations for the Lie algebra of the Poincaré group. It is apparent from these commutators that there is an analogy between torsion and curvature with regard to their relation to translations and rotations in the tangent spaces of the manifold. A detailed analysis of the variational principle that underlies the Einstein-Cartan theory leads to a theorem on the compatibility of the metric and affine structures in space-time.

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