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THEORETICAL PHYSICS

The Principle of Equivalence for Spin *)

by

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Summary. It is pointed out that the motion of spin of a classical particle moving in a gravitational field satisfies a principle similar to the principle of equivalence of inertial mass and gravitational charge. The torsion of space-time may be measured by observing the precession of spin of a particle. The precession is predicted by the Einstein—Cartan theory of gravitation.

1. Introduction

According to relativistic quantum mechanics, mass and spin are the two fundamental characteristics of an elementary system (particle). In Einstein's theory of general relativity, mass — but not spin — plays a direct dynamical role: the density of energy-momentum is the source of curvature. By intorducing torsion and relating it to the density of intrinsic angular momentum, the Einstein-Cartan theory restores the analogy between mass and spin [1—3]. The similarity between mass and spin extends to the principle of equivalence, at least in its "weak form" [2, 4]. According to this principle, the world-line of a spinless test particle, moving under the influence of gravitational fields only, depends on its initial position and velocity, but not on its mass. Similarly, the motion of spin depends on the initial data, but not on the magnitude of the spin of the particle: if S_{ij} is a solution of the equation of motion of spin [5—8], then so is λS_{ij}, λ=const.

In the Einstein—Cartan theory, space-time is assumed to be a four-dimensional differential manifold X with a metric g and a linear connection ω. The metric has a normal hyperbolic signature and is compatible with the linear connection, $Dg_{ij}=0$. Since *torsion has no* (direct) *influence on the propagation of light nor on the motion of spinless test particles* [3, 9, 10] the metric tensor can be determined from measurements in the same manner as in Einstein's theory [11]. In this paper we show how the precession of spin, which follows from the equation of motion, may be used — at least in principle — to determine the torsion of space-time. A different method of measuring torsion has been proposed by Hehl [12].

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The notation of this paper follows closely that of [3] and [7]. In particular, if (θ^i) is a field of co-frames, then

$$\omega^i{}_j=\gamma^i{}_j-\tfrac{1}{2}\,(Q^{i}_{.jk}-Q_{j.k}^{\ i}-Q_{k.j}^{\ i})\,\theta^k,$$

where γ is the Riemannian connection defined by g and $Q^i{}_{jk}$ is the torsion tensor corresponding to ω. The covariant derivative in the direction of the unit vector field (u^i) tangent to a time-like world-line l is denoted by a dot or a stroke depending on whether it corresponds to ω or γ. E. g., if S^i is a vector field defined along l, then

(1) $$\dot{S}_i=S'_i-\tfrac{1}{2}\,(Q_{ijk}-Q_{jik}-Q_{kij})\,S^j u^k.$$

The proper time measured along l will be denoted by s.

We assume that an observer in a space-time (with torsion) uses photons to determine a non-rotating local frame of reference. By the classical argument of the 'bouncing photon', this is equivalent to constructing a Fermi frame along the world-line l of the observer, the Fermi propagation being defined relative to the Riemannian connection γ [13, 14]. In the sequel, all tensor quantities will be described by their components with respect to such a Fermi frame (e_i). The vectors e_α $(\alpha=1,2,3)$ of the Fermi triad are orthogonal to the vector $e_4=u$ tangent to l and satisfy

$$e'_\alpha=-(u'\,|\,e_\alpha)\,u.$$

The Fermi triad is assumed to be orthonormal,

$$(e_\alpha\,|\,e_\beta)=-\delta_{\alpha\beta}.$$

The round brackets occurring in the last two equations denote the scalar product of the vectors they enclose.

2. The precession of spin

Consider a spinning test particle moving along a world-line l. The tensor S_{ij} describing the spin of the particle is assumed to satisfy the standard subsidiary condition [5]

(2) $$S_{ij}\,u^j=0.$$

By virtue of this condition, the space-like vector of spin,

$$S_i=\tfrac{1}{2}\,\eta_{ijkl}\,S^{jk}\,u^l,$$

is sufficient to reconstruct the spin tensor,

$$S_{ij}=\eta_{ijkl}\,u^k\,S^l.$$

The equation of motion of a point test particle with spin may be obtained from the corresponding equations for a continuous medium by a method outlined by Weyssenhoff and Raabe [15]. Essentially, the method boils down to replacing in the equations the particle derivative defined for a fluid by the covariant derivative in the direction of the world-line of the point particle [7, 16]. As a result, in the notation used in this paper, the equation of motion of spin for a point particle has the same form as the corresponding equation for a continuous medium, in the notation of [7]. It reads

$$\dot{S}_{ij}=(u_i\,\dot{S}_{kj}+u_j\,\dot{S}_{ik})\,u^k$$

and, taken together with (2), may be interpreted to mean that the tensor S_{ij} is Fermi propagated along l *relative to the asymmetric connection* ω. The same is true of the spin vector [17],

(3) $$\dot{S}_i+u_i\,\dot{u}^k\,S_k=0.$$

Because od $S_i\,u^i=0$, the vector of spin has a vanishing timelike component relative to the Fermi frame (e_i) defined in the Introduction. We write

$$\boldsymbol{S}=(S_1,S_2,S_3),\qquad \boldsymbol{K}=(K_1,K_2,K_3),$$

where

$$K_1=\tfrac{1}{2}\,(Q_{234}-Q_{423}-Q_{324})\qquad \text{etc.},$$

and we use (1) to replace the equation of motion (3) by the equivalent equation

(4) $$\frac{d\boldsymbol{S}}{ds}=\boldsymbol{K}\times\boldsymbol{S}.$$

Instantaneously, the vector of spin $\boldsymbol{S}$ precesses around the vector $\boldsymbol{K}$. In agreement with the principle of equivalence, the angular velocity of precession is independent of the spin of the test particle. In this respect, the influence of torsion on spin differs from that of a magnetic field on the behaviour of charged spinning particles.

3. Measurements of torsion

According to the equation of motion (4), the vector $\boldsymbol{K}$ may be mesured by observing the precession of spin of particles moving along l. From the knowledge of $\boldsymbol{K}$ one can obtain the values of some components of the torsion tensor. More information may be obtained by observing spinning particles moving with different velocities.

The relation

$$K_{ijk}=\tfrac{1}{2}\,(Q_{jik}+Q_{kij}-Q_{ijk})$$

defining the "contortion tensor" [9] may be solved with respect to Q,

$$Q_{ijk}=K_{ikj}-K_{ijk},$$

so that the measurement of torsion reduces to that of the tensor K. Let

$$h^l_j=\delta^l_j-u^l\,u_j$$

be the projector on the hyperplane orthogonal to u. Clearly, the antisymmetric tensor

$$K_{ij}\,(Q,u)=K_{klm}\,h^k_i\,h^l_j\,u^m$$

bears the same relation to $\boldsymbol{K}$ as S_{ij} does to $\boldsymbol{S}$. The torsion tensor may be split into three irreducible parts [1]

$$Q={}^VQ+{}^AQ+{}^TQ,$$

where

$$^{V}Q^{i}{}_{jk}=\frac{1}{3}\left(\delta^{i}_{j}\,Q^{l}{}_{lk}-\delta^{i}_{k}\,Q^{l}{}_{lj}\right)$$

and

$$^{A}Q_{ijk}=Q_{[ijk]}\,.$$

For any Q and u we have

$$K_{ij}\,(^{V}Q,\,u)=0\,.$$

This shows that the vector part of torsion cannot be determined on the basis of Eq. (4). It is easy to see that torsion may be easily measured if it is either purely antisymmetrical or due to a spinning fluid of the Weyssenhoff type.

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В. Адамович, А. Траутман, **Принцип эквивадентности для спина**

Содержание. В настоящей работе доказывается, что движение спина классической частицы движущейся в гравитационном поле осучествляет принцип похож на принцип эквивалентности инертной и тяжелой масс. На основании наблюдений процессии спина частицы возможно определение кручения пространства-времени. Прецессию предсказывается теорией Эйнштейна—Картана.